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# Theory of the Propagation of Shock Waves*

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A new theory of propagation of one-dimensional shock waves is described. The partial differential equations of hydrodynamics and the Hugoniot relation between pressure and particle velocity are used to provide three relations between the four partial derivatives of pressure and particle velocity, with respect to time and distance from the source, at the shock front. An approximate fourth relation is set up by imposing a similarity restraint on the shape of the energy-time curve of the shock wave and by utilizing the second law of thermodynamics to determine, at an arbitrary distance, the distribution of the initial energy input between dissipated energy residual in the fluid already traversed by the shock wave and energy available for further propagation. The four relations are used to formulate a pair of ordinary differential equations for peak pressure and shock wave energy as functions of distance from the source. The theory takes proper account of the finite entropy increment of the fluid produced by the passage of the shock and permits the use of the exact Hugoniot curve of the fluid in the numerical integration of the basic equations.

## INTRODUCTION

IN discussing earlier work by Poisson,[1] Stokes[2] appears to have first suggested the possibility of the propagation at greater than acoustic velocity of discontinuous pressure waves. Earnshaw[3] and Riemann[4] have discussed the laws of propagation of waves of finite amplitude and the building up of the discontinuity. The conditions for a wave of permanent type have been investigated by Rankine[5] and Hugoniot[6] who have provided statements of the conditions for the continuity of mass, momentum, and energy across the moving discontinuity. Rayleigh[7] has shown how to solve the hydrodynamic equations for plane shock waves when the pressure and density are connected by the adiabatic law.

An exact solution for the cases of spherical and cylindrical shock waves cannot be given, due to the spherical and cylindrical divergence terms of the equation of motion. A straight forward attack on the mathematical problem may be based upon the numerical integration of the partial differential equations of hydrodynamics. However, the labor involved in this approach is so great as to limit it to special applications. The need for a more flexible and rapid theoretical method, based if necessary on well-defined approximations, is therefore clear. Such a method has been previously developed for shock waves in water from spherical charges of explosive.[8] Underwater explosion waves are simpler to treat than blast waves in air, since the relatively small entropy increment produced at the shock front permits the use of the approximation of adiabatic flow. Failure of this approximation in air has until now prevented the formulation of an adequate theory of blast waves in air.

A theory of underwater shock waves has recently been presented by Osborne and Taylor.[9] Their theory is based upon the acoustic approximation and is therefore strictly valid only for small excess pressures at large distances from the source.

In the present communication, we describe a a theory of the propagation of one-dimensional —that is, plane, cylindrical, and spherical— shock waves which is valid both in air and water. The theory takes account of the finite entropy increment in the fluid resulting from the passage of the shock wave. It also permits the use of the

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[1] M. Poisson, J. de l'école polyt. **7**, 319 (1808).
[2] G. G. Stokes, Phil. Mag. **33**, 349 (1848).
[3] S. Earnshaw, Phil. Trans. **A150**, 133 (1850).
[4] B. Riemann, Nachr. Ges. Wiss. Göttingen **8**, 43 (1860).
[5] W. J. M. Rankine, Phil. Trans. **A160**, 277 (1870).
[6] H. Hugoniot, J. de l'école polyt. **57**, 3 (1887); **58**, 1 (1888).
[7] Lord Rayleigh, Proc. Roy. Soc. **A84**, 247 (1910). This paper gives a full discussion of earlier work.
[8] J. G. Kirkwood and H. A. Bethe (1941).
[9] F. M. Osborne and A. H. Taylor, Phys. Rev. **70**, 322 (1946).

exact Hugoniot curve for the fluid and in this respect is superior to previous treatments of the shock wave in air which have been based upon ideal gas adiabatics with constant heat capacity,[10] an approximation which fails badly near the explosive source.

### THE PROPAGATION EQUATIONS

In discussing the propagation of one-dimensional shock waves, it is convenient to write the equations of hydrodynamics in the form,

$$\frac{\rho r^\alpha}{\rho_0 R^\alpha}\frac{\partial u}{\partial R}+\frac{\alpha u}{r}=-\frac{1}{\rho c^2}\frac{\partial p}{\partial t},$$
$$\frac{R^\alpha}{r^\alpha}\frac{\partial u}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial R},\tag{1}$$
$$u=\left(\frac{\partial r}{\partial t}\right)_R,$$

where $u$ is the particle velocity, $p$ the pressure in excess of the pressure $p_0$ of the undisturbed fluid, $\rho$ the density, $\rho_0$ the density of the undisturbed fluid, $t$ the time, and $r$ the Euler coordinate at time $t$ of an element of fluid with Lagrange coordinate $R$. The Euler sound velocity $c$ is equal to $[(\partial p/\partial\rho)_S]^{\frac{1}{2}}$. The coefficient $\alpha$ takes on the values,

$\alpha=0$ for a plane wave,
$\alpha=1$ for a cylindrical wave,
$\alpha=2$ for a spherical wave.

Equations (1) are supplemented by the equation of state of the fluid and the entropy transport equation $\partial S/\partial t=0$, the latter of which we shall not explicitly use. Equations (1) are of a hybrid form in that we use the Lagrange coordinates $R$ and $t$ as independent variables but retain the Euler equation of continuity. Equations (1) are to be solved subject to initial conditions specified on a curve in the $R$, $t$-plane and to the Rankine[5]-Hugoniot[6] conditions at the shock front,

$$p=\rho_0 uU,$$
$$\rho(U-u)=\rho_0 U,\tag{2}$$
$$\Delta H=(p/2)(1/\rho_0+1/\rho),$$

where $\Delta H$ is the specific enthalpy increment experienced by the fluid in traversing the shock front and $U$ is the velocity of the shock front. Equations (2) constitute supernumerary boundary conditions which are compatible with the differential equations and specified initial conditions only if the shock front follows an implicitly prescribed curve $R_0(t)$ in the $R$, $t$-plane.

We denote a derivative in which the shock front is stationary by

$$\frac{d}{dR}=\frac{\partial}{\partial R}+\frac{1}{U}\frac{\partial}{\partial t}.\tag{3}$$

If the operator $d/dR$ is applied to the first of Eqs. (2) and if Eqs. (1) are specialized for the shock front, $r=R$, three relations are obtained between the four partial derivatives $\partial p/\partial t$, $\partial p/\partial R$, $\partial u/\partial t$, $\partial u/\partial R$.

$$\frac{\rho}{\rho_0}\frac{\partial u}{\partial R}+\frac{1}{\rho c^2}\frac{\partial p}{\partial t}+\frac{\alpha u}{R}=0,$$
$$\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial R}=0,\tag{4}$$
$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial R}-\frac{g}{\rho_0}\frac{\partial p}{\partial R}-\frac{g}{\rho_0 U}\frac{\partial p}{\partial t}=0,$$

where

$$g=\rho_0 U\frac{du}{dp}=1-\frac{p}{U}\frac{dU}{dp}.$$

All of the coefficients in Eqs. (4) can be expressed as functions of pressure alone by means of the Hugoniot conditions, Eqs. (2), and the equation of state of the fluid. If Eqs. (4) can be supplemented by a fourth relation between the partial derivatives, it would be possible to solve for each of the four derivatives as a function of $p$ and $R$, and with the aid of Eqs. (3) to formulate an ordinary differential equation,

$$\frac{dp}{dR}=\frac{\partial p}{\partial R}+\frac{1}{U}\frac{\partial p}{\partial t}=F(p,R),\tag{5}$$

for the peak pressure $p$ of the shock wave as a function of the distance $R$, and in addition to obtain the initial slope, $-1/\theta$, of the Euler

[10] G. I. Taylor, Proc. Roy. Soc. A186, 273 (1946).

pressure-time curve of the wave,

$$\frac{1}{\theta}=\frac{1}{p}\frac{\partial p}{\partial t}-\frac{a}{p}\frac{\rho}{\rho_0}\frac{\partial p}{\partial R}. \tag{6}$$

The physical basis for the supplementary relation to be established lies in the fact that the non-acoustical decay of waves of finite amplitude is closely associated with the entropy increment experienced by the fluid in passing through the shock front and the accompanying dissipation of energy. As a shock wave passes through a fluid, it leaves in its path a residual internal energy increment in each element of fluid determined by the entropy increment produced in it by the passage of the shock front. As a result, the energy propagated ahead by the shock wave decreases with the distance it has traveled from the source.

The adiabatic work $w_0$ per unit area of initial generating surface done on the fluid exterior to the surface is given by

$$w_0a_0{}^\alpha=\int_{a_0}^{R}\rho_0r_0{}^\alpha E[p(r_0)]dr_0+\int_{t_0(R)}^{\infty}r^\alpha u'(p'+p_0)dt, \tag{7}$$

where $u'$ and $p'$ denote particle velocity and excess pressure behind the shock front (the unprimed quantities being reserved henceforth for quantities at the shock front), $t_0(R)$ is the time of arrival of the shock front at $R$, $E(p)$ is the specific energy increment of the fluid at pressure $p_0$ and for an entropy increment corresponding to shock front pressure $p$, and $a_0$ is the Lagrange coordinate of the generating surface of the shock wave. Now,

$$\int_{t_0(R)}^{\infty}p_0r^\alpha u'dt=p_0\Delta V+p_0\int_{a_0}^{R}\left(\frac{\rho_0}{\rho}-1\right)r_0{}^\alpha dr_0, \tag{8}$$

where $\rho$ is the final density of the fluid and $\Delta V$ is the volume swept out by the generating surface per unit area of initial generating surface. Combining Eqs. (7) and (8) and introducing $h(p)=E+p_0\Delta(1/\rho)$, the specific enthalpy increment of an element of fluid traversed by a shock wave of peak pressure $p$ after return to pressure $p_0$ along its new adiabatic, one obtains

$$w_0a_0{}^\alpha=p_0\Delta V+\int_{a_0}^{R}\rho_0r_0{}^\alpha h[p(r_0)]dr_0+\int_{t_0(R)}^{\infty}r^\alpha u'p'dt. \tag{9}$$

The time integral may be assumed to vanish at $R=\infty$. If one subtracts from Eq. (9) the expression obtained from that equation at $R=\infty$, there results the relation,

$$D(R)=\int_{t_0(R)}^{\infty}r^\alpha u'p'dt, \tag{10}$$

where

$$D(R)=\int_{R}^{\infty}\rho_0r_0{}^\alpha h[p(r_0)]dr_0. \tag{11}$$

The energy of the shock wave at the point $R$ is by definition the work done on the fluid exterior to $R$. Thus, the shock wave energy at $R$ per unit area of initial generating surface is given by the quantity $a_0{}^{-\alpha}D(R)$.

Our dissipation assumption breaks down if the first shock wave can be overtaken by second shocks built up in its rear. This will not be the case if the pressure-time curve is initially monotone, decreasing with asymptotic value $p_0$. If the excess pressure $p'$ has a negative phase, a second shock will develop in the negative part of the pressure-time curve but cannot overtake the initial positive shock. In this case, our theory will apply to the positive phase if the time integrals of Eqs. (7) to (10) are extended not to infinity but to the time at which the excess pressure in the positive phase vanishes. The general theory of shock waves is not sufficiently developed to permit one to say that there is proof for these statements, but they can nevertheless be accepted with some assurance as plausible.

The energy-time integral can be expressed in reduced form,

$$D(R)=R^\alpha pu\mu\nu,$$

$$\frac{1}{\mu}=-\left(\frac{\partial\log p'u'r^\alpha}{\partial t}\right)_{t=t_0(R)}=-\frac{1}{p}\frac{\partial p}{\partial t}-\frac{1}{u}\frac{\partial u}{\partial t}-\frac{\alpha u}{R}, \tag{12}$$

$$\nu=\int_0^\infty f(R,\tau)d\tau,\quad \tau=\frac{t-t_0(R)}{\mu},$$

$$f(R,\tau)=r^\alpha p'u'/R^\alpha pu.$$

The function $f(R, \tau)$ is the energy-time integrand, normalized by its peak value $R^{\alpha}pu$ at the shock front, expressed as a function of $R$ and a reduced time $\tau$ which normalizes its initial slope to $-1$ if $\mu$ does not vanish. We also assume $f$ to be a monotone decreasing function of $\tau$. Elimination of $\mu$ between the first two of Eqs. (12) yields the desired fourth relation supplementing Eqs. (4) between the partial derivatives at the shock front. This set is exact, involving integrals of Eqs. (1) for the knowledge of the reduced energy-time function $f(R, \tau)$. However, if $f(a_0, \tau)$ is initially a monotone decreasing function of $\tau$, $f(R, \tau)$ will remain so, and in fact will at large $R$ become asymptotically a quadratic function of $\tau$ corresponding to the linear form of the pressure-time curve that has been shown[8] to be stable at large distances. This means that $\nu$ is a very slowly varying function of $R$, for which sufficiently accurate estimates for many purposes can be made without explicit integration of the hydrodynamic equations. The assignment of a value independent of $R$ to $\nu$ is equivalent to imposing a similarity restraint on the energy-time curve. This type of approximation is equivalent in principle to that underlying the Rayleigh-Ritz method, although we do not include a variational procedure to carry the result to any desired degree of approximation.

The initial pressure-time and energy-time curves of an explosion wave are rapidly decreasing. An expansion of the logarithm of the function in a Taylor series in the time, the well-known peak approximation, is appropriate for an initial estimate of $\nu$. This corresponds to an exponential $f(\tau)$,

$$f(\tau) = e^{-\tau},$$

and results in the value, $\nu = 1$. For the asymptotic quadratic energy-time curve, corresponding to a linear pressure-time curve of the positive phase of the wave,

$$\begin{aligned} f(\tau) &= (1-\tau/2)^2, \quad && \tau \leqslant 2, \\ f(\tau) &= 0, && \tau > 2, \end{aligned}$$

which leads to the value, $\nu = \frac{2}{3}$. As a convenient empirical interpolation formula between the two extreme values, we have employed the relation,

$$\nu = 1 - \tfrac{1}{3} \exp - [p' p_0] \tag{13}$$

in a series of calculations[11] of the peak pressure distance curves of shock waves from explosive sources.

Elimination of $\mu$ between the first two of Eqs. (12) and combination with Eq. (4) yields a set of four equations which may be solved for the four partial derivatives, and an ordinary differential equation for the peak pressure $p$ as a function of the distance $R$ may be formulated with the aid of Eq. (5). A second ordinary differential equation relating $D$ to $R$ may be obtained by differentiation of Eq. (11). The resulting expressions may be written in the form,

$$\begin{aligned} \frac{dD}{dR} &= -R^{\alpha} L(p), \\ \frac{dp}{dR} &= -\nu \frac{R^{\alpha} p^3}{D} M(p) - \frac{\alpha p}{2R} N(p), \end{aligned} \tag{14}$$

where

$$L(p) = \rho_0 h(p),$$

$$M(p) = \frac{1}{\rho_0 U^2} \frac{G}{2(1+g) - G},$$

$$N(p) = \frac{4(\rho_0/\rho) + 2(1-\rho_0/\rho) G}{2(1+g) - G},$$

$$G = 1 - (\rho_0 U/\rho c)^2, \quad g = 1 - \frac{p}{U} \frac{dU}{dp}.$$

The functions $L(p)$, $M(p)$, and $N(p)$ can be evaluated as functions of the pressure by means of the equation of state of the fluid and the Hugoniot relations, Eqs. (2). It may be remarked that Eqs. (14) are independent of any assumption regarding the equation of state of the fluid and that they take proper account of the finite entropy increment of the fluid produced by the passage of the shock.

The functions $L(p)$, $M(p)$, and $N(p)$ are most conveniently expressed as functions of the pressure in tabular form, and they may be evaluated by numerical methods from a tabular presentation of the exact Hugoniot curves for the fluid. Equations (14) can then be integrated numerically by the use of standard methods.[12]

[11] J. G. Kirkwood and S. R. Brinkley, Jr. (1945).
[12] See, for example, J. B. Scarborough, *Numerical Mathematical Analysis* (Johns Hopkins Press, Baltimore, 1930), pp. 218 ff.

## THE ASYMPTOTIC PROPAGATION EQUATIONS

It is of interest to examine the asymptotic form of the solutions of Eqs. (14) for small excess pressure $p$. If the exterior medium is air, the ideal adiabatic equation of state, $p=p_0[(\rho/\rho_0)^\gamma-1]$, may be employed in the limit of small excess pressure, where $\gamma$ is the ratio of the heat capacities of $p_0$ and at the temperature of the undisturbed fluid. Employing the acoustic approximation in the evaluation of the limiting shock wave velocity, it is easy to show that

$$\begin{aligned}\mathrm{Lim}(p\to 0)L(p)&=\frac{\gamma+1}{12\gamma^2}\frac{p^3}{p_0^2},\\ \mathrm{Lim}(p\to 0)M(p)&=\frac{\gamma+1}{8\gamma^2}\frac{p}{p_0^2},\\ \mathrm{Lim}(p\to 0)N(p)&=1,\\ \mathrm{Lim}(p\to 0)\nu&=\tfrac{2}{3},\end{aligned}\tag{15}$$

and the asymptotic equations are

$$\begin{aligned}\frac{dD}{dR}&=-\frac{\gamma+1}{12\gamma^2}\frac{R^\alpha p^3}{p_0^2},\\ \frac{dp}{dR}+\frac{\alpha p}{2R}&=-\frac{\gamma+1}{12\gamma^2}\frac{R^\alpha p^4}{Dp_0^2}.\end{aligned}\tag{16}$$

Equations (16) have the solutions,

$$\begin{aligned}Rp&=P_1(\log R/R_1)^{-\frac{1}{2}},\\ D&=[(\gamma+1)/6\gamma^2p_0^2]P_1^2Rp,\end{aligned}\tag{17a}$$

for the spherical wave, $\alpha=2$,

$$\begin{aligned}\sqrt{R}p&=P_1[2(\sqrt{R}-\sqrt{R_1})]^{-\frac{1}{2}}\\ D&=[(\gamma+1)/6\gamma^2p_0^2]P_1^2\sqrt{R}p,\end{aligned}\tag{17b}$$

for the cylindrical wave, $\alpha=1$, and

$$\begin{aligned}p&=P_1(R-R_1)^{-\frac{1}{2}},\\ D&=[(\gamma+1)/6\gamma^2p_0^2]P_1^2p,\end{aligned}\tag{17c}$$

for the plane wave, $\alpha=0$, where $P_1$ and $R_1$ are constants. Equations (15) to (17) are valid when the exterior medium is water if, in these expressions, the pressure $p_0$ is replaced by the characteristic pressure $B$ of the Tait[13] equation of state, which may be written for use along an adiabatic in the form $p=B[(\rho/\rho_0)^\gamma-1]$, and $\gamma$ is interpreted not as the heat capacity ratio but as the exponent of $\rho/\rho_0$ in the Tait equation. The asymptotic form for the peak pressure of the shock wave with spherical symmetry is in agreement with the results of the theories of Kirkwood and Bethe[8] and with the limiting theory of Osborne and Taylor.[9] The pressure, in this case, decays non-acoustically as the slowly varying factor $(\log R/R_1)^{-\frac{1}{2}}$.

[13] R. E. Gibson, J. Am. Chem. Soc. **56**, 4 (1934); **57**, 284 (1935). See also A. Wohl, Zeits. f. physik. Chemie **99**, 234 (1921), and H. Carl, *ibid.* **101**, 238 (1922).

## THE IMPULSE

The impulse $I$ delivered by the shock wave at a point of fixed Euler coordinate $r$ is

$$I=\int_{t_0(R)}^{\infty}p'dt\tag{18}$$

along a path of constant $r$. If the excess pressure $p'$ has a negative phase, the positive impulse is obtained if the time integral is extended not to infinity but to the time at which the excess pressure vanishes. The pressure-time integral can be expressed in reduced form in a manner analogous to the reduction of Eq. (10).

$$\begin{aligned}&I=\nu^*p\theta,\\ &\frac{1}{\theta}=-\left(\frac{\partial\log p'}{\partial t}\right)_{r,\,t=t_0(R)},\\ &\nu^*=\int_0^\infty(p'/p)d\tau^*,\quad \tau^*=\frac{t-t_0(R)}{\theta}.\end{aligned}\tag{19}$$

The initial slope, $-1/\theta$, of the Euler pressure-time curve is expressed in terms of the Lagrange partial derivatives at the shock front by Eq. (6), and these partial derivatives were obtained as functions of peak pressure by the solution of Eqs. (4) and (12). The following expression is obtained for $\theta$:

$$-\frac{1}{\theta}=\frac{U}{G}\left\{\frac{\alpha}{R}+\left[\frac{\rho}{\rho_0}(1+g)+\left(1-\frac{\rho}{\rho_0}\right)G\right]\frac{1}{p}\frac{dp}{dR}\right\}.\tag{20}$$

For an exponential pressure-time curve, consistent with the peak approximation and the exponential energy-time curve, the reduced pressure-time integral $\nu^*$ is equal to 1. For the asymptotic linear pressure-time curve, consistent with the asymptotic quadratic energy-time curve, we have $\nu^*=\frac{1}{2}$ for the positive phase of the wave. As an empirical interpolation formula between the two values, we have employed the relation,[11]

$$\nu^*=1-\tfrac{1}{2}\exp-[p/p_0]^{\frac{1}{2}}. \tag{21}$$

We have found that Eq. (21) leads to satisfactory agreement between calculated and experimental values of the positive impulse when used in conjunction with Eq. (13) for the reduced energy-time integral. A more detailed analysis of the relation between $\nu$ and $\nu^*$ does not seem justified in view of the empirical nature of Eq. (13).

## INITIAL CONDITIONS FOR SHOCK WAVES FROM EXPLOSIVE SOURCES

The two constants of integration may be determined from the thermodynamic properties of the explosive and those of its products, either in the Chapman-Fouget detonation state or in the instantaneous detonation state corresponding to adiabatic isometric conversion of the entire explosive charge to its decomposition products. Since the instantaneous detonation state may be expected to give the better average representation of the behavior of the detonation products in the generation of a shock wave in an exterior medium, we shall employ it in the discussion of the initial conditions for the shock wave.

If at the initial instant of time, a charge of explosive has been converted to products at a uniform pressure $p_e$, a shock wave will advance into the exterior medium and a rarefaction wave will recede into the detonation products. The initial excess pressure $p_1$, and particle velocity $u_1$, continuous at the boundary, are related by

$$\begin{aligned}\sigma^*(p_1)+u_1=0,\quad u=p/\rho_0 U(p),\\ \sigma^*(p)=(s^*)\int_{p_e}^{p+p_0}\frac{dp}{\rho^* c^*},\end{aligned} \tag{22}$$

where $\sigma^*$ is the Riemann[4] function in the explosion products and $U$, the shock velocity in the exterior medium, is determined as a function of the pressure by the Hugoniot conditions, Eqs. (2). The asterisked quantities refer to the explosion products. The first of Eqs. (22) expresses the fact that the Riemann $r$-function[4] initially vanishes in the receding rarefaction wave. In order to solve Eqs. (22), it is necessary to use the Hugoniot tables for the exterior medium, the energy of explosion, and the heat capacities and equilibrium constants for the substances making up the explosion products; and to evaluate $\sigma^*$ it is necessary to use an equation of state for those products.

In the development of the propagation equations, the rate of energy delivery has been approximated by an exponential function of time. For shock waves in air, it may be assumed that the integral of this exponential function is equal to the total energy of explosion, since experimental evidence suggests that there is little residual energy available for second shocks. For shock waves in water, experimental evidence suggests that approximately one-half of the energy of explosion is delivered to the first shock. The initial value of the quantity $D$ is readily estimated from these considerations, and the disadvantages of the approximate nature of this procedure are minimized by the circumstance that except in the immediate vicinity of the explosive charge, the shock wave parameters are not very sensitive to the initial energy.