One readily finds that

$$A=r/(1-\omega^2r^2)^{\frac{1}{2}},\quad B=(1-\omega^2r^2)^{\frac{1}{2}},\quad C=-\omega r^2/(1-\omega^2r^2)^{\frac{1}{2}}. \quad (46)$$

This is essentially the same system as the one used by Hill.[2] From the form of Eq. (45) it follows that

$$dl^2=dx'^2+dy'^2+dz'^2, \quad (47)$$

or, on the basis of the transformation equations,

$$dl^2=dr^2+r^2(1-\omega^2r^2)^{-1}d\theta^2+dz^2, \quad (48)$$

which is the result obtained by Berenda and shows that the spatial geometry on the surface of a rotating disk ($z$=constant) is non-Euclidean.

It is also interesting to note that the time interval between two events $dt'$ is given by

$$dt'=(1-\omega^2r^2)^{\frac{1}{2}}dt-\omega r^2(1-\omega^2r^2)^{-\frac{1}{2}}d\theta. \quad (49)$$

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# Note on Magnetic Energy

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## 1.

IN a recent paper[1] under this same title E. A. Guggenheim discusses certain results from a paper[2] published by me recently and claims that they are special cases of more general ones obtained by him on a previous occasion.[3] As the results quoted from my paper are themselves special cases of quite general ones contained therein, and his note generally misrepresents the scope and purpose of my work, perhaps I may be permitted a reiteration of the main outlines of my arguments and the opportunity to discuss the relation they bear to the methods employed by him.

The following symbols are used:

- $B$ magnetic induction,
- $H$ magnetic force intensity,
- $I_i$ induced magnetization intensity,
- $I_p$ permanent magnetization intensity,
- $i_s$ linear electric current,
- $N_s$ magnetic flux through $i_s$,
- $\mu$ permeability,
- $\mathcal{L}$ Lagrangian function,
- $\mathcal{H}$ Hamiltonian function.

Guggenheim bases his discussion not on Maxwell's theory in its original form but on the modification given to it by Cohn.[4] The essence of this form of the theory, like that proposed by Hertz which it follows closely, is that it incorporates the induced polarizations and the aether, whatever this may be, into a single transmitting medium whose elastic quality is summed up in the characteristic constant, the permeability $\mu$. This hypothesis of a single medium, excluding as it does the possibility of a displacement of the polarized medium from one position of the field to another, proves however to be a fatal handicap in a theory which has eventually to be extended to cover electromagnetic phenomena in moving media. And it was precisely for this reason that Larmor and Lorentz were forced back to the views held by Kelvin and Maxwell that the only really satisfactory treatment of these affairs interprets them in terms of a separate universal transmitting medium with its own stress on which is superposed the polarized media with their reacting mechanical forces. This implies that it is absolutely essential to distinguish between the parts of the field vectors and energy which belong to the aether and remain with it and the parts which belong to the matter

[1] E. A. Guggenheim, Phys. Rev. **68**, 273 (1945).
[2] G. H. Livens, Phil. Mag. **36**, 1 (1945). Cf. also Proc. Roy. Soc. **A93**, 200 (1916), and Phil. Trans. Roy. Soc. **A220**, 207 (1919).
[3] E. A. Guggenheim, Proc. Roy. Soc. **A155**, 49 (1936).
[4] E. Cohn, *Das elektromagnetische Feld* (1900). My knowledge of this book is derived from this earlier edition.

and are carried along with it in its motion through the field.[5]

However, so far as the present discussion of the quasi-static energy relations of the field are concerned there is no essential difference between the different forms of the theory, except in the matter of interpretation. The differences, however, begin to appear when these energy formulae are applied to determine the forces on the magnetic media by applying the principle of virtual work, and it has long been known that the theory in the Helmholtz-Hertz-Cohn form leads, when properly interpreted, to quite untenable results.[6]

## 2.

My own discussions in this subject are based on the generalizations of Maxwell's theory formulated by Larmor and Lorentz, and so far as the mechanical relations of the magnetic field are concerned they start from an assumption which is really the basis of all electrodynamic theories from Maxwell onwards. This, in its most general form, asserts that the magnetic part $\mathfrak{L}_m$ of the Lagrangian function of a system consisting of a series of linear currents $i_s$ enclosing fluxes $N_s$ and a distribution of permanent polarity $I_p$ in the presence of magnetizable media distributed over the field in any manner is such that

$$\begin{aligned}\delta\mathfrak{L}_m &= \textstyle\sum_s N_s\delta i_s + \int (F\delta I_p)dv \\ &= \frac{1}{4\pi}\int (B\delta H)dv + \int (F\delta I_p)dv \cdots .\end{aligned} \quad (1)$$

This formula is interpreted for the moment in terms of a theory which regards $F$ as the intensity of the magnetic force acting on the polar elements. The usual assumption is that $F \equiv H$, but there are grounds for thinking that $F \equiv B$ is a better choice. The integrands of all space integrals are understood to relate simply to the local conditions at a point, and their variation produced in any process of building up the system from infinite dispersion of its elements.

Formula (1) is of the utmost possible generality, especially in this virtual differential form, and it defines the complete electrodynamic relations of the quasi-stationary system, and this includes not only the relations of the currents and permanent magnets, but also those of the induced magnets as well. There is no implied necessity for any relations between the vectors involved in it, anything in the nature of a law of induction, that would make it the complete differential of a unique function $\mathfrak{L}_m$, although naturally the greatest mathematical interest attaches to the cases when such relations exist.

Guggenheim starts not from the Lagrangian function, but from the associated Hamiltonian or available (conserved) energy function which he writes in the integrated form

$$\mathfrak{H}_m = \sum \int_0^{N_s} i_s dN_s,$$

and then, assuming with Maxwell, that the velocities in the electrical coordinates are $i_s$ he derives the corresponding Lagrangian function $\mathfrak{L}_m$ in the form

$$\mathfrak{L}_m = \sum \int_{N_s=0}^{i_s} N_s di_s.$$

In both of these formulae the summations are taken not only over the ordinary currents of the system, but also over a whole series of other currents introduced to mask the effects of the permanent magnetism in the early stages of the build-up process represented in the integrals.

## 3.

The choice by Guggenheim of the Hamiltonian function as the more fundamental is based on the possibility of deducing it directly from Maxwell's equations. The argument used by him for this purpose depends however essentially on the concept of "zero states," and to define these he uses the following postulate, I quote his own words:[7]

"If, however, each permanent magnet is surrounded by suitable circuits it is theoretically possible for currents to flow in these circuits such

[5] This conclusion should be compared with Guggenheim's statement (in Proc. Roy. Soc. **A155**, 99 (1936)) that "as the ether and the matter occupy the same space, there is no means of distinguishing between the energy of the ether and the energy of the matter. We can therefore attach no physical meaning to this division of the energy into two parts."

[6] G. H. Livens, Phil. Mag. **32**, 162 (1916).

[7] E. A. Guggenheim, reference 3, page 55.

that magnetic induction due to them is everywhere equal and opposite to the magnetic induction due to the permanent magnetization."

This, of course, is true only when the permanent magnetism is distributed uniformly throughout the volume of each such magnet. If we try to solve Maxwell's equations for a steady magnetic system with currents *on the surfaces* of the permanent magnets and with $B=H+4\pi I=0$ everywhere, we soon realize that, in the general case, it is necessary to supplement the surface currents by a distribution of volume currents throughout each magnet with a density at each place proportional to curl $I$, and these volume currents vanish only when $I$ is constant throughout each magnet. In the general case the two sets of currents, surface and volume, constitute a distribution which is then the exact opposite of that which is the effective equivalent of the electron current whirls ultimately constituting the magnetism. The two systems would thus cancel out completely in every detail except as regards their magnetic forces, the definition of which for a current distribution being somewhat different to that for the equivalent polarization.[8] Properly interpreted therefore Guggenheim's "zero states" are identical with the usual concept of the zero fields of the infinitely dispersed system in the more conventional building up procedure, except that there is in the interior of the magnets a residual magnetic force $H_0'$ resulting from the different interpretations of this vector for the two cancelling systems.

The difficulty is, of course, the introduction of permanent magnetism and its associated energy into the simple form of the energy equation derived from Maxwell's theory. But if we are careful to interpret the energy equation properly this difficulty is soon overcome. The equation itself determines a formula for $W_m$, the part of the field energy associated with the magnetism, which is such that

$$\delta W_m=\frac{1}{4\pi}\int(H\delta B)dv.$$

But this formula represents the magnetic part of the corresponding Hamiltonian or conserved (available) energy only in so far as the magnetic connections of the system controlling the build-up of the polarizations and currents involve no dissipation other than that which occurs in the linear current circuits, and allowed for directly in the complete equation. But it is of the nature of a permanent magnet that any energy expended by the field in separating its poles in the general build-up of the field is immediately locked up in the rigid connections holding the magnetism permanent, so that it can no longer be reckoned as part of the available energy. In this case, therefore, the Hamiltonian or conserved energy function is diminished from the value $W_m$ by the amount expended by the field in building up the permanent magnetism. We have therefore

$$\delta\mathcal{H}_m=\frac{1}{4\pi}\int(H\delta B)dv-\int(F\delta I_p)dv, \qquad (2)$$

and as we have assumed a kinetic origin for the linear currents only this corresponds to a Lagrangian function $\mathfrak{L}_m$ which is such that

$$\delta\mathfrak{L}_m=\frac{1}{4\pi}\int(B\delta H)dv+\int(F\delta I_p)dv,$$

the formula (1) above.

## 4.

Guggenheim also converts his general formulae for $\mathfrak{L}_m$ and $\mathcal{H}_m$ into space integrals over the field obtaining results which are equivalent to

$$\mathfrak{L}_m=\frac{1}{4\pi}\int dv\int_{H_0'}^{H}(BdH'),$$
$$\mathcal{H}_m=\frac{1}{4\pi}\int dv\int_0^B H'dB, \qquad (3)$$

$H_0'$ being the value of $H'$, more clearly defined below, in the zero field when $H=B=0$ everywhere. These are apparently very different formulae from (1) and (2) above. But we must remember that in Guggenheim's treatment there are other currents involved in the build-up, so the

[8] Cf. Larmor, *Aether and Matter*, p. 106, "On the specification of a magnetic distribution in terms of a continuous distribution of currents." The same idea probably underlies Guggenheim's statement (Proc. Roy. Soc. **A88**, 100 (1936)) that "if we could imagine all matter removed leaving behind the molecular currents it is $B$, not $H$, which would remain unaltered." Actually, of course, it is the change in the specification of the magnetization as a distribution of poles to a distribution of currents which necessitates the change in $H$, and the presence, or otherwise, of the magnetically inert parts of the matter has nothing whatever to do with it.

magnetic field intensity which occurs in the integrands, and now called $H'$, is a vector which is circuital with respect not only to the specified currents of the system, but also to all these additional currents as well. In other words it is a very different vector from the $H$ employed in the usual treatment and used in our (1) and (2) above, although of course the upper limit of the two vectors, taken when the masking currents have disappeared, will be the same. The relation between the two vectors at any stage of the build-up is easily obtained if Guggenheim's method is properly interpreted to include the volume currents as well as the surface ones, and it shows that if $H$ is the usual vector which is circuital with respect only to the original currents of the system at this stage then the $H'$ of Guggenheim's formulae is

$$H' = H - 4\pi I_c$$

if the currents in this transition stage are just sufficient to cancel a distribution of magnetic polarity of intensity $I_c$. The formula for $\mathfrak{L}_m$ in (3) is therefore in the ordinary notation

$$\mathfrak{L}_m = \frac{1}{4\pi}\int dv \int_0^H (BdH) - \int dv \int_{I_{c0}}^0 (BdI_c), \quad (4)$$

where $I_{c0}$ is the value of $I_c$ which completely cancels $I_p$ together with any induced magnetization $I_{i0}$ in the zero state,

$$I_{c0} = I_p + I_{i0}.$$

Substituting then $I_q = I_p - I_c$, $dI_q = -dI_c$, in the second integral, it reduces to

$$\mathfrak{L}_m = \frac{1}{4\pi}\int dv \int_0^H BdH + \int dv \int_{-I_{i0}}^{I_p} (BdI_q)$$

which is the equivalent of our (1) with $F = B$ apart from a trivial constant which in fact measures the quasi-elastic energy associated with any induced magnetization which may be present in the zero field.

We have also

$$\mathfrak{H}_m = \frac{1}{4\pi}\int dv \int_0^B (H'dB)$$

$$= \frac{1}{4\pi}\int (HB)dv - \frac{1}{4\pi}\int dv \int_{H_0'}^H (BdH')$$

$$= \frac{1}{4\pi}\int dv \int_0^B (HdB) - \int dv \int_{-I_{i0}}^{I_p} (BdI_q)$$

which is the equivalent of formula (2).

Guggenheim considers two special cases of these formulae corresponding to two possible laws of induction. As $H$ is zero in the initial field we have

$$H_0' = -4\pi I_{c0} = -4\pi(I_p + I_{i0}).$$

If then we write $I_{i0} = (\mu - 1)H_0'/4\pi$, we have

$$H_0' = -4\pi I_p/\mu, \quad I_{i0} = -[(\mu-1)/\mu]I_p;$$

while if $I_{i0} = (\mu' - 1)B_0/4\pi = 0$, we have

$$H_0' = -4\pi I_p.$$

In this latter case the formulae (3) are actually identical with (1) and (2) interpreted of course with $F = B$, the lower limit of the second integral being zero.

## 5.

Guggenheim seems quite unaware of the difference between the $H'$ of his formulae—he, of course, calls it $H$—and the more conventional $H$ employed by Cohn and myself, for he criticizes Cohn's choice of a zero lower limit for his $H$, when this is precisely the value which corresponds to his own choice of $-4\pi I_p$ or $-4\pi I_p/\mu$ as the lower limit of $H'$.

The real difference between Cohn's formulae and those given by Guggenheim lies really in another direction altogether, and not, as surmised by Guggenheim, in the choice of a lower limit for the integrals involved. Cohn's whole treatment is in fact interpreted not in terms of the usual induction vector $B$, as implied by Guggenheim's quotations, but of another vector $M$, variously called by Hertz the polarization or magnetization of the field, and which is related to $B$ by the equation

$$M = B - 4\pi I_p.$$

This makes Cohn's formula for $\mathfrak{H}_m$, *viz.*,

$$\mathfrak{H}_m = \frac{1}{4\pi}\int_0^M HdM$$

identical with our (2) with $F = H$. Guggenheim, thinking in terms of his own treatment seems inclined to write off the second integrals in $dI_p$ in all our formulae as having zero values when the configuration is kept constant. As he says in a letter to me "I need hardly point out that at constant configuration as $I_p$ is constant

$$\int H dI_p = 0,\text{"}$$

and this generally speaking seems to form the basis of his claim to have generalized the results given by Cohn. But when properly interpreted by the conventional build-up process employed by Cohn and myself these integrals do not necessarily vanish. They will vanish in Guggenheim's treatment but only because an equivalent non-vanishing integral

$$\int H dI_c$$

is included with them.

It is this unwillingness to recognize the conventional build-up procedure used by all previous writers, as a legitimate method of evaluating, at constant configuration, integrals of the kind here under review which is the cause of Guggenheim's difficulties with previous treatments. And as he interprets "constant configuration" to include also a constant specification of the distribution of permanent magnetism he is obliged to employ the device of covering the magnetism with a current shield or mask, which has then to be removed gradually during the build-up of the finite currents which form the rest of the generating system. As we now see the two processes, removing the mask and building up the magnetism from zero are in effect almost the exact, equivalents of each other both mathematically and physically and they lead to identical formulae for the available energy in the field in every case. One method cannot therefore be described as any more general than the other.

## 6.

The rest of the argument of my paper was concerned with an attempt to derive the simplest and most natural form of the expression for the general Lagrangian function, or rather of its virtual variation, expressed as an integral over the field and in a form which exhibits its non-integrability when it occurs, as arising simply and solely in the hysteresis effects associated then with the induction of the polarization.

In the usual case when $F=H$ a familiar argument soon shows that

$$\delta \mathfrak{L}_m = \frac{1}{8\pi}\delta\int \{2(BH) - H^2\} dv - \int (H\delta I_i) dv:$$

while when $F=B$ the same argument gives

$$\delta \mathfrak{L}_m = \frac{1}{8\pi}\delta\int B^2 dv - \int (B\delta I_i) dv.$$

These expressions are independent of any particular law of induction, or even of whether a law exists at all, and they apply to every type of medium, isotropic or crystalline. In each of them the second integral measures the energy stored in the quasi-elastic connections holding the induced magnetism against the action of the field. The first integral, represented in both cases by a complete differential, thus represents the purely magnetic part of the function, the part not definitely located in the latter. And the fact that in the second case this purely magnetic part of the energy assumes a very simple natural form in terms of the single vector $B$ provides one of the arguments in favor of the choice of this vector as the fundamental force vector of the theory.

This was the main conclusion of my paper, but being anxious naturally to show the relation between these general formulae and those given in the more familiar treatments, which invariably use linear laws of induction, a good deal of attention was devoted to the special form of the results when these linear laws are assumed. It was then shown that the formula for $\mathfrak{L}_m$ which, in the case when $F=H$, becomes

$$\mathfrak{L}_m = \frac{1}{8\pi}\int \{2(HB) - \mu H^2\} dv$$

is identical with that usually employed in the dynamical theory of currents, and that, while it apparently differs from the more familiar form when the field arises from permanent magnets it still, even in such cases, gives a total for the whole field which is identical with that given by the more usual formula.

When the law of induction is linear there is, of course, also an energy or Hamiltonian function, the special case of formula (2). This is shown to be represented in all cases by the familiar result

$$\mathcal{H}_m = \frac{1}{8\pi} \int \mu H^2 dv$$

so the theory provides a self-consistent set of simple formulae for determining all the details of the mechanical and electrical behavior of the system in all cases, whether the law of induction is linear or not.

Under the impression, as he says, that I have confined myself to what I call linear laws of induction, it is these special results from my paper which Guggenheim claims as particular examples of his own general formulae. The discovery as we now see is not a very surprising one, seeing that one form of the general formulae of my paper, to which, however, he makes no reference, is in fact identical with the general formulae given by him, when these are properly interpreted, however different they may be in appearance.

## 7.

There are other imperfections and misunderstandings involved in the rest of Guggenheim's work which will be dealt with in detail in a more comprehensive survey of the whole subject to be published elsewhere. Enough has now been said to justify the conclusion that his claim to have produced more general formulae which include those obtained by Cohn and myself as special cases is entirely without justification. His formulae when properly interpreted prove to be identical with formulae given by both of us many years previously so that they are certainly no more general than the results given by Cohn and, in so far as his discussion is framed on the restricted basis of Cohn's theory, his results have a far less degree of generality than the equivalent ones obtained by myself on the much wider basis of the Kelvin-Maxwell theory. The only advantage of Guggenheim's formulae is in their mathematical compactness, but, as his subsequent discussion shows, this may have its dangers when attempts are made to interpret the relevant parts of them for dynamical purposes.

There are two other small points raised in Guggenheim's note which can be disposed of quickly before closing this already over-long note.

The question of the compatibility of the two interpretations of the linear law of induction given in my paper, and discussed at some length by him, is entirely irrelevant. There is really no conceivable reason why two such alternative physical laws should be compatible among themselves, but it is hoped that eventually one of them will prove to be more consistent with experience than the other.

Then at the end of the note Guggenheim complains that I use the term "fundamental (aethereal) force vector" without explaining what I mean. In following, however, the example of Cohn and ignoring the physical processes involved in the induction of the polarizations, he thereby overlooks one of the most important functions of the magnetic force vector. The forces on the currents are determined in terms of the induction vector $B$ and, as the same vector can also, without ambiguity, be used to determine the forces on the permanent magnets, it is the only field vector which functions as a force intensity in this restricted theory. In the more detailed theory, however, a choice has to be made for the force vector which is effective in the polarization process and tradition, following the electrostatic analogy, has chosen $H$ for this purpose. My contention is that, if $B$ is chosen for this purpose as well, so that it becomes the one and only fundamental vector which functions as a force intensity then the whole of the energy and mechanical relations of the field, in the only really consistent form so far framed, assume their simplest and most natural form.