The Normal Reflection of Shock Waves

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The problem of the normal reflection of a shock wave is solved by an approximate analytical integration of the hydrodynamical equations. The solution given here leads to nearly the same numerical results as an exact method, based on a graphical integration of the hydrodynamical equations, which has been described by Chandrasekhar. A method of computing the complete pressure-time curve at the reflector is given and applied to reflection in a class of fluids obeying the Tait adiabatic equation of state. It is found that in compressible fluids (gases) the pressure on the reflector is prolonged and that the impulse delivered to it *exceeds* the value predicted by the acoustic theory. In slightly compressible media (liquids and solids), on the other hand, the blow is shorter and the impulse delivered to the reflector is *less* than one would expect from the acoustic approximation. The method given here is also applicable to the reflection of gravity waves on the surface of a liquid.

INTRODUCTION

WHEN a shock wave is reflected from a solid, the pressure on the reflector at first rises to a high peak and then decays to the value previously existing. For weak shocks, or sound waves, which strike a rigid reflector, this initial peak is of such a height that the pressure ratio (ζ_r) across the reflected shock front and the pressure ratio (ζ_i) across the incident shock front are equal,

$$\zeta_r = \zeta_i,$$

so that the overpressure at the reflector is twice the overpressure in the incident shock. In the limiting case of very strong and normally incident shocks, on the other hand,

$$\zeta_r = [(3\gamma-1)/(\gamma-1)]\zeta_i.$$

For a fluid whose adiabatic constant, γ, is 1.4, ζ_r/ζ_i is 8. The rise in pressure at the first moment of reflection and the flow patterns simultaneously present have been studied at all angles of incidence and the theory is able to describe in an approximate way both the regular and the irregular, or Mach, reflection. One is, however, rarely interested in the initial, or peak, pressure alone unless the period of the reflector is very short compared to the duration of the incident wave. Similarly in the collision of equal and opposite shocks, which is hydrodynamically equivalent to reflection from a rigid wall, it is not just the first instant of the collision but rather the subsequent history of the interaction which is interesting. It is the complete course of the reflection of one shock, or of the interaction of two shocks, which we wish to discuss here.

QUALITATIVE DESCRIPTION OF RESULTS

For the case of normal incidence alone has the problem been solved. The possibility of using a very simple numerical method based on the Riemann form of the equations of motion was indicated by Chandrasekhar[1] who considered reflection in air. Here we replace the numerical method by an analytical one and extend the work to other media besides air. All media covered by our calculations are subject to the Tait equation of state for isentropic changes.

$$(p+\pi)v^\gamma = k. \tag{1}$$

Here p is overpressure; v is specific volume; π, γ, and k, which are characteristic of the medium, are constants for isentropic processes but are functions of the entropy. When Eq. (1) is applied to ideal gases, γ is the adiabatic constant lying between 1 and 1.67. The same equation is also applicable to matter in the condensed state, if π is now interpreted as the internal pressure and γ is an empirical constant, not simply related to the specific heats: e.g., for water, $\pi = 3000$ atmos., $\gamma = 7.15$; for freshly detonated explosives which

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[1] S. Chandrasekhar, a report of limited circulation issued by the Ballistics Research Laboratory, Aberdeen, Maryland.

have not yet expanded, $\gamma=3$. It has also been pointed out that an analogy may be set up between the equations of gas dynamics and of gravity waves on the surface of a fluid and this analogy corresponds to the case of $\gamma=2$; our results are in fact applicable to gravity waves if pressure is interpreted as h^2, where h is the height of the gravity wave.[2]

Although the numerical method used by Chandrasekhar is rigorous, the analytical method employed here contains certain approximations and the two methods were, therefore, compared for an incident linear shock (i.e., the pressure falls linearly with time behind the peak) of 1.5 atmos. in air (the particular example computed by him) and an incident linear shock of 1800 atmos. in water. The pressure time curves at the wall are shown in Fig. 1. The two methods agree well and differ considerably from the acoustic approximation.

The results of our calculations can be most conveniently expressed in terms of $\int_0^T p dt$, the time integral of the pressure at the wall, where T is the total time required for reflection. This integral is the momentum delivered to the wall or the difference in momenta between the incident and reflected pulses, i.e.,

$$\int \rho_r u_r dx = \int \rho_i u_i dx + \int_0^T p dt, \qquad (2)$$

where ρ and u are density and material velocity, respectively, and where the subscripts i and r refer to incident and reflected pulses. The space integration is extended over the whole lengths of the incident and reflected shocks. In Fig. 2 the impulse $\int_0^T p dt$ is normalized by dividing by the acoustic impulse, i.e., $2\int p_i dt$, where p_i is the value the over-pressure would have at the position of the wall if the incident wave moved forward without change of shape and with the velocity of sound corresponding to zero over-pressure. The intensity of the incident shock is also normalized in Fig. 2: it is specified by $\zeta=(p+\pi)/\pi$. In Fig. 2 the normalized impulse is plotted as a function of γ; the most interesting physical result of our calculation—the difference of behavior shown by compressible and incompressible media—is then brought out: in the case of gases $(1\leqslant\gamma\leqslant1.67)$ the impulse is *greater* than the prediction of acoustic theory; but in the case of liquids and solids the impulse is *less* than the acoustic approximation. Non-linear theory happens to agree with the acoustic estimate in the neighborhood of $\gamma=2$; $\gamma=2$ also corresponds to the analogy between gravity waves and gas flow. The reason for this difference between gaseous and condensed matter is that the *duration* of the blow exceeds the acoustic approximation in the gaseous case, whereas the situation is reversed for solids and liquids, as one can see by the following qualitative consideration of the reflection process.

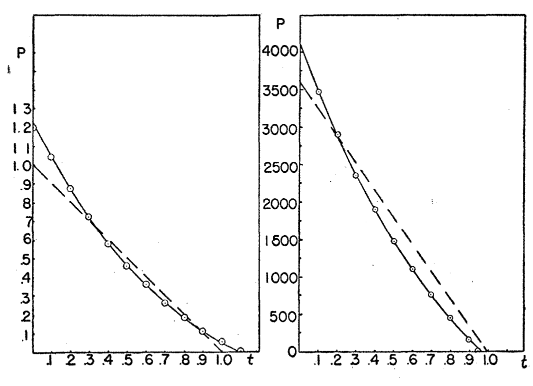

FIG. 1. Pressure-time curves at reflector. The curve at the left corresponds to reflection in air of an incident pulse whose peak overpressure is 0.5 atmos. The other curve describes reflection in water of an incident pulse whose peak overpressure is 1800 atmos. The ordinate is overpressure in atmos. and the unit of time is the duration of the incident pulse. ⊙ Computed by Chandrasekhar's method; — analytic approximation; – – acoustic approximation.

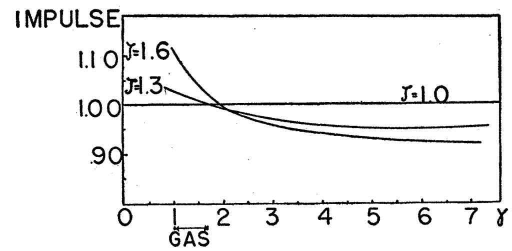

FIG. 2. Impulse given to reflector as function of γ. Ordinate is actual impulse given to rigid wall divided by approximate acoustic impulse. Abscissa is constant (γ) appearing in the equation of state. ζ is shock intensity. $\zeta=1.0$ corresponds to limiting case of sound wave.

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[2] T. von Karman, "Flow in Compressible Fluids," to be found in a collection of papers entitled *Fluid Mechanics and Statistical Methods in Engineering* (University of Pennsylvania Press, Philadelphia, 1941), page 25.

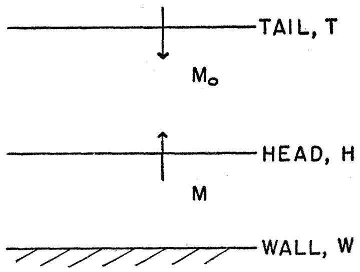

FIG. 3. Reflection in x space.

After the instant of impact the head, H, of the shock wave reverses direction and begins to travel toward its tail, T (Fig. 3). As soon as the direction of H has been reversed by the wall, the total disturbance may be regarded approximately as a (non-linear) composition of two waves of finite amplitude. One of these moves toward the wall and is bounded by T and W. The other moves away from the wall and is bounded by W and H. These two waves penetrate each other until T reaches the wall and thereby marks the end of the reflection. It is clear that the tail T moves toward the wall at speed, a, the velocity of sound corresponding to $p=0$, until it meets H. It is not so easy to guess the speed of H with respect to the wall, but it also turns out to be nearly a for the shock strengths of interest here. The reason is, roughly speaking, that the high pressure behind H produces a supersonic velocity not with respect to the wall but with respect to fluid rapidly moving *toward the wall*. The net effect is that H and T approach each other approximately with velocity a. Hence if the initial thickness of the pulse is L, the time until H and T meet is $L/2a$. After passing through the head the tail then moves with the velocity $c-u$, where c is the local velocity of sound and u is the material velocity of the fluid in the region M. Hence the total time of reflection is $L/2a+L/2\langle c-u\rangle_{\text{Av}}$ where $\langle c-u\rangle_{\text{Av}}$ is the average velocity of the tail in M. On the other hand the duration according to the acoustic approximation is L/a. Hence if u/c is so large that $\langle c-u\rangle_{\text{Av}}<a$, as in air, the pressure on the wall lasts for a longer time than acoustic theory indicates; but if u/c is so small that $\langle c-u\rangle_{\text{Av}}>a$, as in water, then we have the opposite situation. These qualitative considerations are supported by the detailed calculations to which we now turn.

ANALYTICAL METHOD

Consider a plane shock, S, impinging on a rigid wall, W, at normal incidence. Let the plane of the wall be $x=0$ and let the x axis be directed toward the oncoming shock. The time, t, is measured from the moment at which S strikes W. Since the problem is one-dimensional, there are only two independent variables, x and t, and it is convenient to speak in terms of the x, t plane shown in Fig. 4. Here D is the world line of the reflected shock front, and the world line of a typical element, initially at x_0, is shown with a discontinuity where it crosses D. M_0 and M are the regions between D and the x axis and t axis, respectively. The following boundary conditions are given on the axes and on D:

1. Distribution of pressure, p_i, and velocity, u_i, on the x axis. This is the distribution of pressure and velocity in the incident wave.
2. $u=0$ on the t axis since the wall is rigid.
3. Shock equations across the discontinuity, D, expressing the conservation of (a) mass, (b) momentum, (c) energy.

Subject to these boundary conditions three differential equations which again express the conservation of mass, momentum, and energy must be satisfied in M_0 and M. In these differential equations appear three unknown functions, e.g., pressure, velocity, and entropy, and the problem is to find these functions in the regions M_0 and M and in addition to find the equation of the curve D. Employing the usual approximation, however, we assume that the flow in M_0 and M is isentropic. In addition the stronger approximation is made that the entropy change across D can be neglected. The entropy change across D is actually only third order in the volume change; but a stronger argument is that results of the calculation made under this assumption agree very well with the results of the rigorous numerical method in which the change of entropy is properly taken into account. Under these assumptions the entropy becomes a constant of the motion and may be ignored. There are then only two unknown functions, say, p and u, and the exact conservation of energy condition is replaced by the approximate conservation of entropy condition.

The three conservation conditions in M and M_0

may then be written

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial \rho}{\partial t}=0, \tag{3}$$

$$\frac{\partial p}{\partial x}=-\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right), \tag{4}$$

$$p+\pi=k\rho^{\gamma} \text{ (an approximation)}, \tag{5}$$

where ρ is density. There are two unknown functions and two differential equations. On the other hand the three conservation conditions across the shock front are

$$s=u_0+v_0\left[\frac{p-p_0}{v_0-v}\right]^{\frac{1}{2}}, \tag{6}$$

$$u=u_0+[(p-p_0)(v_0-v)]^{\frac{1}{2}}, \tag{7}$$

$$(p_0+\pi)v_0^{\gamma}=(p+\pi)v^{\gamma} \text{ (an approximation)}, \tag{8}$$

where s is the shock velocity (or slope of D) and $v=1/\rho$ is the specific volume. The variables in these equations are of course to be evaluated on D; subscripts refer to the two sides (M and M_0) of D.

By Riemann's method[3] Eqs. (3)–(5) are rewritten

$$\frac{\partial P}{\partial t}=-(c+u)\frac{\partial P}{\partial x}, \tag{9}$$

$$\frac{\partial Q}{\partial t}=(c-u)\frac{\partial Q}{\partial x}, \tag{10}$$

where P and Q are the Riemann functions which are defined in this paper as

$$P=2c/(\gamma-1)+u, \tag{11}$$

$$Q=2c/(\gamma-1)-u, \tag{12}$$

where

$$c=(dp/d\rho)^{\frac{1}{2}}=\text{velocity of sound}. \tag{13}$$

It is convenient to rewrite the shock equations (6) and (7) with the aid of (8) and (13) in terms of these five variables: c, u, c_0, u_0, and s. They become to the first order in c/c_0-1

$$s=u_0+ec+e_0c_0, \tag{14}$$

$$Q=Q_0, \tag{15}$$

[3] See, for example, H. Lamb, *Hydrodynamics* (Cambridge University Press, 1932), sixth edition, page 481.

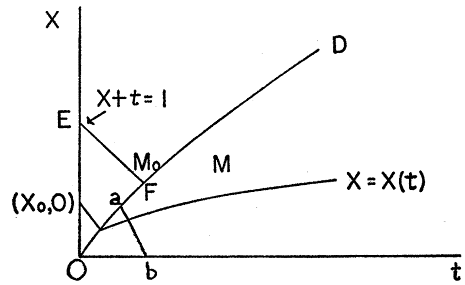

FIG. 4. Reflection in x, t space. OD and EF are world lines of reflected shock and tail of incident shock, respectively, and ab is a characteristic on which Q is constant.

where the following abbreviations have been introduced

$$e=(\gamma+1)/2(\gamma-1),$$
$$e_0=1-e.$$

It has been pointed out that Q is nearly constant[1] across a rather strong shock front in air, even without neglect of the entropy change.

INITIAL CONDITIONS

We come now to the formulation of the boundary conditions on the x axis or, in other words, to a description of the incident shock wave. For simplicity it is assumed that at $t=0$ the incident wave is progressive and that the velocity distribution in it is linear; the pressure distribution is then fixed. Equations (9) and (10) show how the progressive condition may be expressed. According to these equations a general one-dimensional hydrodynamical disturbance may be decomposed (non-linearly) into two waves, described by P and Q which travel in opposite directions with the velocities $c+u$ and $c-u$, respectively. The condition for a progressive wave is that either P or Q be constant; and in this case it must be P since the pulse is traveling toward $-\infty$. The required boundary conditions on the x axis are then

$$\begin{aligned} u_0(x,0)&=-B+Bx \qquad 0\leqslant x\leqslant 1,\\ u_0(x,0)&=0 \qquad x>1,\ B>0, \end{aligned} \tag{16}$$

$$P_0(x,0)=2/(\gamma-1). \tag{17}$$

Here B is the peak velocity. The units of length and time are so chosen that the initial length of the incident pulse is unity and so that $c=1$ when $p=0$. Equation (17) together with (5) and (13) determine the pressure as the following function of the velocity

$$p+\pi=\pi[1-(\gamma-1)u/2]^{2\gamma/(\gamma-1)}. \quad (18)$$

The pressure computed from (18) and (16) is very nearly linear for shocks in which $(p+\pi)/\pi<1.6$. In other words a pulse of the type considered here is essentially linear in both pressure and velocity.

A remark may be made about the progressive assumption at this point. Although a continuous wave of finite amplitude can propagate without increase of length, the same is no longer true as soon as it has evolved into a shock. After it has become a shock, reflection of the continuous portion of the pulse from the discontinuity begins; as a result the disturbance propagates backwards as well as forwards, and hence the progressive assumption cannot be rigorously satisfied by a shock any time after it is formed. Nevertheless it is often a good approximation, and as we have just seen, the pulse considered here is essentially linear in both pressure and velocity.

Although one could specify the pressure distribution in the incident pulse instead of the velocity distribution in it, we have not done so because it is analytically more convenient to regard the velocity as given. Since, however, experimental data generally concern pressure, it is necessary to have a way of determining the maximum velocity B from the measured peak pressure. B can be calculated from the following formula which follows from (7)

$$B=[(\zeta-1)(1-\zeta^{-1/\gamma})/\gamma]^{\frac{1}{2}}, \quad (19)$$

where $\zeta=(p_m+\pi)/\pi$ and p_m is measured peak pressure in the incident wave. This Eq. (19) is not consistent with (18) since (19) is a shock equation while (18) is based on the progressive assumption; and as we have just seen, they are not compatible. The procedure followed here is this: ζ is regarded as computed from the measured peak pressure. From (19) B is found. From (16) u is found. Equations (16) and (17) then completely determine the wave. The numerical discrepancy between (18) and (19) is actually insignificant for shocks of the intensity considered here ($\zeta\leqslant 1.6$).

SOLUTION IN M_0

The differential equations can be solved readily in M_0 if the disturbance in M_0 remains progressive. That it does remain nearly progressive may be verified by comparing the result of the rigorous Riemann numerical integration with the analytic progressive solution. Therefore we assume

$$P_0(x, t)=2/(\gamma-1). \quad (20)$$

Let

$$\Delta=c_0-u_0. \quad (21)$$

Then by (20), (21), (10)–(12) it follows that

$$\frac{\partial\Delta}{\partial t}=\Delta\frac{\partial\Delta}{\partial x}. \quad (22)$$

The solution of (22) is

$$t\Delta+x=w(\Delta), \quad (23)$$

where w is a function to be determined by the boundary conditions. When these conditions are (16) and (17) then

$$\begin{aligned}\Delta&=[(\gamma+1)(-Bx+B)+2]/[B(\gamma+1)t+2] & x&<1-t, \quad (24)\\ &=1 & x&>1-t.\end{aligned}$$

The discontinuity in the derivatives of Δ along the line $x+t=1$ (Fig. 4) has its origin in the discontinuity at the tail of the incident pulse (Eq. (16)). The straight line EF is the world line of the tail of the incident shock and F represents its intersection with the head of the reflected shock. Here we are interested only in the part OEF of the region M: and so only the first representation of Δ in (24) will be carried. The quantities u_0, c_0, and Q_0 are now readily found.

$$u_0=2B(x+t-1)/(2+B(\gamma+1)t) \quad x<1-t, \quad (25)$$

$$c_0=1-(\gamma-1)u_0/2, \quad (26)$$

$$Q_0=2c_0/(\gamma-1)-u_0. \quad (27)$$

With these expressions the problem is solved in M_0.

SOLUTION IN M

To determine the solution in M one has to overcome a difficulty characteristic of shock problems; namely, although the boundary conditions on D are known, the equation of the line D is unknown and has to be determined simultaneously with the solution of the differential equations in M. The shock equations (14) and

(15) contain five variables: u_0, c_0, u, c, and s, of which u_0 and c_0 are known from (25) and (26). Still one more condition is necessary to fix u, c, and s, and this additional information is bound up in the solution of the differential equations. In order to separate the determination of the shock line from the solution of the differential equations we make the following approximation

$$P(g(t), t) = P(0, 0), \tag{28}$$

where

$$x = g(t) \tag{29}$$

is the equation of D. The best arguments for the condition (28) are that the decay of P along D, computed by the rigorous numerical method, is small and that the pressure-time curve, obtained on the basis of (28), agrees very closely with the corresponding curve computed according to the numerical method (as shown in Fig. 1). The function P actually decreases along D as t increases and this decrease causes the shock to decay. Therefore the physical meaning of the approximation (28) is that decay of the reflected shock is negligible during the period of reflection. Aside from its numerical success this assumption has in its favor two facts which make it plausible: first, the time of reflection is short; and second, since the shock travels faster with respect to M_0 than with respect to the wall, the wall behaves as a sustaining piston behind the shock.

The equations (14), (15), (25), (26), and (28) lead to the following differential equation for the shock line D

$$\frac{dx}{dt} = \frac{f + hx + jt}{1/B + (\gamma+1)t/2}, \tag{30}$$

$$f = 1/B + (3\gamma - 5)/4,$$

$$h = (3-\gamma)/2,$$

$$j = 2 + B(\gamma+1)^2/8.$$

The solution of this equation is

$$x = [2/(\gamma-1)B + (5\gamma-3)/4(\gamma-1)] \times [1 - (1 + (\gamma+1)Bt/2)^{(3-\gamma)/(1+\gamma)}] + (2/(\gamma-1))[1 + B(\gamma+1)^2/16]t. \tag{31}$$

It is more convenient and numerical comparison with the complete equation again shows that it is little less accurate to use only the first two terms of the expansion of (31) in powers of t, i.e.,

$$x = \alpha t + \beta t^2, \tag{32}$$

where

$$\alpha = (3\gamma - 5)B/4 + 1, \tag{33}$$

$$\beta = [(5\gamma-3)B^2/4 + 2B](3-\gamma)/4. \tag{34}$$

Equation (32) fixes the boundary line D. The boundary conditions on it are

$$P(x, t) = P(0, 0), \tag{35}$$

$$Q(x, t) = Q_0(x, t). \tag{36}$$

Q_0 is given by (27) and (32). If the value of Q_0 so obtained is put in (36) one gets

$$Q(x, t) = q_0 + q_1 t + q_2 t^2, \tag{37}$$

where

$$q_0 = 2(1/\gamma - 1 + B), \tag{38}$$

$$q_1 = -4B[1 + (5\gamma-3)B/8], \tag{39}$$

$$q_2 = -B(3\gamma-1)q_1/4. \tag{40}$$

In (37) higher order terms were again found small enough to neglect. Finally

$$P(x, t) = P(0, 0) = Q(0, 0) = q_0. \tag{41}$$

The boundary conditions are most convenient in the forms (41) and (37).

The boundary D and the boundary conditions on it have now been found so that the differential equations in M can be solved. Numerical integration shows that in M the characteristics of Q are almost straight. Now the actual slope of a Q characteristic is $c-u$. Let the slope be assumed constant and equal to $-\lambda$ which is defined as follows

$$2\lambda = (c-u)_a + (c-u)_b, \tag{42}$$

where the points a and b are shown in Fig. 4. One then finds

$$(c-u)_b = c_b = (\gamma-1)Q_b/2 = (\gamma-1)Q_a/2,$$

and by (41)

$$(c-u)_a = (\gamma-3)P_a/4 + (\gamma+1)Q_a/4 = (\gamma-3)q_0/4 + (\gamma+1)Q_a/4.$$

Hence

$$\lambda = (\gamma-3)q_0/8 + (3\gamma-1)Q_a/8. \tag{43}$$

Then the equation of the characteristic passing through $(0, t_b)$ is

$$x=\lambda(t_b-t), \qquad (44)$$

where λ is given by the preceding equation.

To find Q_b as a function of t_b it is convenient to regard t_a as the independent variable. The relation between Q_b and t_b is then given by the parametric equations:

$$Q_b=q_0+q_1t_a+q_2t_a^2, \qquad (45)$$

$$t_b=t_a+x(t_a)/\lambda(t_a), \qquad (46)$$

according to (37) and (44), where

$$x(t_a)=\alpha t_a+\beta t_a^2,$$
$$\lambda(t_a)=(\gamma-3)q_0/8+(3\gamma-1)Q_a(t_a)/8,$$

according to (32) and (43). The time t_a is the retarded value of the time t_b in the sense that t_b-t_a is the time required for a given value of Q to propagate itself from the shock front to the wall. If the shock and Q both traveled with the velocity of sound (unity), then one would have exactly $t_a=0.5t_b$. This relation is nearly satisfied in any case.

CALCULATION OF PRESSURE ON REFLECTOR

In order to calculate $p(0, t_b)$ from t_b it is sufficient to find $Q(0, t_b)$; because at the wall, where

$u=0,$

$$c=(\gamma-1)Q/2, \qquad (47)$$

and from c the pressure follows according to the equation

$$p+\pi=\pi c^{2\gamma/(\gamma-1)}. \qquad (48)$$

It is convenient to combine (47) and (48)

$$p+\pi=\pi[(\gamma-1)Q/2]^{2\gamma/(\gamma-1)}. \qquad (49)$$

Since $Q(0, t_b)$ is already known by (45) and (46), this equation completes the determination of $p(0, t_b)$ as a function of t_b.

The complete method for computing the pressure-time curve at the wall then runs as follows. The incident shock is specified by its pressure-ratio, ζ. The maximum material velocity, B, is first found from Eq. (19). The following constants, q_0, q_1, q_2, α, and β, which are functions of B and γ only, are then found from (38)–(40), (33), and (34). These constants then determine the auxiliary functions $x(t_a)$, $Q(t_a)$, and $\lambda(t_a)$ which are defined in Eqs. (32), (37), and (43). In terms of these functions the parametric equations of the pressure-time curve are (46) and (49). The duration of the pressure on the wall is determined by putting $Q=2/(\gamma-1)$ in (37) and then using (46).