The model attributes to the proton one-half the kinetic and potential energies of the relative motion of the two particles within the deuteron. The scalar field gives, including the Pauli part,

$$\Delta(\mu_p+\mu_n) = \langle - W_p - 0.285 T_p \rangle / Mc^2 \qquad (6)$$

where p stands for "proton" and μ is the magnetic moment. For a "square well" the kinetic energy of relative motion is

$$T_p + T_n = 2T_p = \alpha a(D+W)/(1+\alpha a) \qquad (7)$$

with

$$\alpha = (-MW/\hbar^2)^{-\frac{1}{2}}, \quad a = \text{radius of well}. \qquad (7')$$

For $a = e^2/mc^2$ and $D = 21$ Mev one finds, from Eq. (7), $2T_p = 14.4mc^2$ corresponding to $T_p = 0.0039 Mc^2$, and Eq. (6) gives $\Delta(\mu_p+\mu_n) = 0.0000$. The four-vector field gives

$$\Delta(\mu_p+\mu_n) = (-0.67+0.05) T_p/Mc^2$$
$$= -0.62 \times 0.0039 = -0.0024.$$

This is also very small.

The model is not believed in as a reality. It has properties in common, however, with Eq. (18.1) of the first paper[8] and gives[7] the inverted fine structure of nuclear levels in agreement with experiment.

5. For a "square well" interaction and range $a = e^2/mc^2$ the proton and neutron spend 45 percent of the time within $r < a$. The additivity of nuclear magnetic moments indicates, therefore, either the retention of individuality by the proton and neutron within the range of force, or a mistaken idea regarding the range of force or a compensation of changes in the moments.

6. It is not intended to say that the relativistic corrections are negligible, but it is believed that the variations in these corrections due to the sensitivity of the Dirac part of the moments to assumptions regarding the interaction between particles have been shown to be so large as to make estimates of the corrections uncertain to within practically their whole magnitude.

Note on the Calculation of Angular Distributions in Resonance Reactions

G. Breit* and B. T. Darling*
University of Wisconsin, Madison, Wisconsin
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This note is concerned with pointing out some labor saving devices in the calculation of transformation coefficients arising in the composition of angular momenta which are needed for the calculation of angular distributions of disintegration products arising in nuclear resonance reactions.

THE angular distribution of disintegration products in resonance reactions has been treated theoretically by several authors.[1] The present note is concerned with pointing out a few labor saving devices in the technique of making the necessary calculations of transformation coefficients arising in the representation of the coupling of angular momenta by means of wave functions. It is realized that these coefficients are written out for the general case in Wigner's book on the applications of Group Theory to Quantum Mechanics. The formulas applicable to the general case are rather lengthy, and it is often desirable to have some other way of checking the results or of obtaining them.

Resonance reactions involve transitions from the initial state of the colliding particles via one or more states of the compound nucleus to the final state of the disintegration products. The nature of the disintegration products, their yield, and angular distribution depend upon the proper-

* Now at Yale University.
[1] E. Gerjuoy, Phys. Rev. 58, 503 (1940); C. L. Critchfield and E. Teller, Phys. Rev. 60, 10 (1941); E. Eisner, Phys. Rev. 65, 85 (1944).

ties of the intermediate states of the compound nucleus and consequently also on the nature and state of the colliding particles as well as of the disintegration products. The angular distributions depend on the combined effect of products of matrix elements, each product consisting of two factors corresponding respectively to transitions from an initial state to an intermediate one, and to a subsequent transition to a final state. Some of the products have to be added to each other, and the square of the absolute value of the sum enters the formulas for the differential collision cross sections. A knowledge of the relative signs of the transformation coefficients representing the coupling of angular momenta is needed in some cases, and the calculations can be laborious. By applying two general theorems about properties of transformation coefficients of angular momenta, it is possible to shorten the work. The present note is concerned with an explanation of how this can be done.

Usually the beam of incident particles is unpolarized, i.e., their spins i_1 and i_2 are independently and randomly oriented in space. The initial state is described by a modified plane wave in which the spin states are a statistical mixture with equal probabilities and without correlation of the spin states of the individual particles. If $\psi_m{}^l$ are angular wave functions of the orbital motion for angular momentum $l\hbar$ and projection $m\hbar$ along z, and if the z axis is chosen in the direction of the incident plane wave, then only the components with $m=0$ will occur in the initial state.

The angular wave functions $\Psi_M{}^j$ of the compound nucleus having angular momentum $j\hbar$, projection $M\hbar$, are linear combinations of $\psi_m{}^l\psi_{m_1}{}^{i_1}\psi_{m_2}{}^{i_2}$, with $l+i_1+i_2=j$, and $m+m_1+m_2=M$. The transition probabilities from initial to intermediate state require the knowledge of the transformation coefficients $(\psi_0{}^l\psi_{m_1}{}^{i_1}\psi_{m_2}{}^{i_2}, \Psi_M{}^j)$. The evaluation of these quantities is the principal concern of this note.

First of all one can compound the individual spins $i_1+i_2=s$ and use the compound spin states $\psi_\nu{}^s$ instead of $\psi_{m_1}{}^{i_1}\psi_{m_2}{}^{i_2}$. The transformation coefficients that are needed are $(\psi_0{}^l\psi_\nu{}^s, \Psi_M{}^j)$ instead of $(\psi_0{}^l\psi_{m_1}{}^{i_1}\psi_{m_2}{}^{i_2}, \Psi_M{}^j)$. This may be done on account of the following theorem.

(I) A statistical mixture without correlation and with equal probabilities of states having definite projections of two uncoupled spins along an axis is also a statistical mixture in the same sense of states corresponding to compounded spins with definite projections along the same axis.

This theorem is a special case of a theorem proved by J. v. Neumann.[2] It can be proved very simply without direct reference to the more general discussion of v. Neumann. The proof may, for instance, be given as follows:[3] Let u_i, v_j be two complete, orthogonal sets of characteristic functions for a physical system. In the present case the u_i, v_j correspond, respectively, to decoupled and coupled spin states. A statistical mixture of the u_i can be represented by

$$\Sigma_i \epsilon_i u_i,$$

where the ϵ_i are statistical variables subject to the condition,

$$\langle \epsilon_i{}^* \epsilon_l \rangle = \delta_{il}/N.$$

The sign $\langle \ \rangle$ indicates a statistical average over the statistical mixture, and δ is the Kronecker δ, while N is the total number of states. One can express the u_i in terms of the v_j by means of a unitary transformation

$$u_i = \Sigma_j S_{ij} v_j.$$

The statistical mixture in terms of the v_j is, therefore

$$\Sigma_j \epsilon_j' v_j$$

with

$$\epsilon_j' = \Sigma_i \epsilon_i S_{ij}$$

so that

$$\langle \epsilon_j'{}^* \epsilon_k' \rangle = \Sigma_{i,l} S_{ij}{}^* S_{lk} \langle \epsilon_i{}^* \epsilon_l \rangle = \frac{1}{N} \Sigma S_{ij}{}^* S_{ik} = \delta_{jk}/N.$$

The statistical mixture of the states u is thus also a statistical mixture of the states v.

Next, to evaluate $(\psi_0{}^l\psi_\nu{}^s, \Psi_M{}^j)$ it is not necessary to calculate the complete set of normalized wave functions $\Psi_M{}^j$ for the composition of the angular momentum $l+s=j$. To see this, use is made of the following theorem (proof in Appendix).

[2] J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932), p. 183.

[3] It will be understood from now on that the words "statistical mixture" stand for "statistical mixture with equal probabilities and without correlation."

(II) Two angular momenta $\mathbf{l}$, $\mathbf{s}$ are compounded to give a resultant $\mathbf{j}$. Let $P(j, m_j; m_l)$ be the probability of $\mathbf{l}$ having a projection m_l along the z axis for a state in which the absolute value of the resultant is j, and its projection on the z axis is m_j. The theorem states that

$$\Sigma_{m_j} P(j, m_j; m_l) = (2j+1)/(2l+1)$$

independently of the value of m_l.

Thus if we can obtain the relative values of $(\psi_0{}^l\psi_\nu{}^s, \Psi_M{}^j)$, we need only determine a common proportionality factor by normalizing the sum of the squares to $(2j+1)/(2l+1)$.

The relative values of $(\psi_0{}^l\psi_\nu{}^s, \Psi_M{}^j)$ can be obtained simply from the symbolic ξ, η method[4] for combining angular momentum wave functions by calculating only those terms containing $\xi_1{}^l\eta_1{}^l(m=0)$. The ξ, η method for the composition of angular momenta $\mathbf{l}+\mathbf{s}=\mathbf{j}$ works as follows: One expands the expression

$$\psi = (-\xi_1\eta_2 + \xi_2\eta_1)^{l+s-j}$$

$$\times (-b\xi_1 + a\eta_1)^{l-s+j}(-b\xi_2 + a\eta_2)^{s-l+j}$$

in powers of the ξ and η. One then replaces

$$\left[\binom{2l}{l+\mu}\right]^{\frac{1}{2}}\xi_1{}^{l+\mu}\eta_1{}^{l-\mu} \text{ by } \psi_\mu{}^l,$$

and

$$\left[\binom{2s}{s+\nu}\right]^{\frac{1}{2}}\xi_2{}^{s+\nu}\eta_2{}^{s-\nu} \text{ by } \psi_\nu{}^s.$$

Except for a normalizing factor the desired $\Psi_M{}^j$ is then the coefficient of

$$(-)^{j+M}\left[\binom{2j}{j+M}\right]^{\frac{1}{2}}a^{j-M}b^{j+M}$$

in the expansion of ψ.

Example. The procedure will now be illustrated by means of an example. Protons $(i_1 = \frac{1}{2})$ are incident on Li^7 $(i_2 = \frac{3}{2})$ and give rise to a nuclear disintegration with the emission of two alpha-particles. It is probable that the relative orbital

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[4] H. Weyl, *Gruppentheorie und Quantenmechanik* (S. Hirzel, Leipzig, 1928), pp. 154, 159, 193, 261; H. A. Kramers, Proc. Amst. Akad. Sci. **33**, 953 (1930); E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Friedr. Vieweg & Sohn, Braunschweig, 1931), Chaps. XV, XVII; B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Springer, Berlin, 1932), Section 18.

angular momentum before disintegration is $l=1$, and it will be assumed besides that the angular momentum of the compound state is $j=2$ which can be obtained by compounding the spins $i_1 = \frac{1}{2}$, $i_2 = \frac{3}{2}$ to a resultant $s = \frac{1}{2} + \frac{3}{2} = 2$ and compounding $s=2$ with $l=1$ to a resultant $j=2$. One has $l+s-j=1$, $l-s+j=1$, $s-l+j=3$, and the symbolic ψ is

$$\psi = (-\xi_1\eta_2 + \xi_2\eta_1)(-b\xi_1 + a\eta_1)(-b\xi_2 + a\eta_2)^3$$

$$= -\xi_1\eta_1(a\eta_2 + b\xi_2)(-b\xi_2 + a\eta_2)^3 + \cdots.$$

In the symbolic ψ the product $\xi_1\eta_1$ represents within a factor $(\psi_\mu{}^l)_{l=1, \mu=0}$. The product $(a\eta_2 + b\xi_2) \times (-b\xi_2 + a\eta_2)^3$ gives the coefficients with which the functions $\psi_\nu{}^s$ occur in the $\Psi_M{}^j$ by means of the following replacements:

$$\xi_1\eta_1 = u_0 = \text{orbital motion function with } l=1, \mu=0$$

$$d_\nu = (\psi_\nu{}^s)_{s=2}; \quad d_2 = \xi_2{}^4, \quad d_1 = (4)^{\frac{1}{2}}\xi_2{}^3\eta_2,$$

$$d_0 = (6)^{\frac{1}{2}}\xi_2{}^2\eta_2{}^2, \quad d_{-1} = (4)^{\frac{1}{2}}\xi_2\eta_2{}^3, \quad d_{-2} = \eta_2{}^4.$$

To obtain a $(\Psi_M{}^j)_{j=2}$ one needs the coefficients of

$$b^4, \quad -(4)^{\frac{1}{2}}ab^3, \quad (6)^{\frac{1}{2}}a^2b^2, \quad -(4)^{\frac{1}{2}}a^3b, \quad a^4$$

for

$$M = 2, \quad 1, \quad 0, \quad -1, \quad -2,$$

and one rearranges the part of ψ containing $\xi_1\eta_1$ into the form

$$\psi = u_0[b^4 d_2 + \frac{1}{2}(-(4)^{\frac{1}{2}}ab^3)d_1$$

$$-\frac{1}{2}(-(4)^{\frac{1}{2}}a^3b)d_{-1} - a^4 d_2] + \cdots.$$

The $(u_0 d_M, \Psi_M{}^j)$ are within a proportionality factor the coefficients 1, $\frac{1}{2}$, $-\frac{1}{2}$, -1 occurring in the square brackets. One has thus

$M=$	2	1	0	-1	-2
$(u_0 d_M, \Psi_M{}^j)$	$+N$	$+\dfrac{N}{2}$	0	$-\dfrac{N}{2}$	$-N$

To determine N^2 one applies Theorem II and finds

$$N^2 + \frac{N^2}{4} + \frac{N^2}{4} + N^2 = \frac{2j+1}{2l+1} = 5/3; \quad N = (\tfrac{2}{3})^{\frac{1}{2}}.$$

The desired transformation coefficients are with-

in a sign which is immaterial here

M	2	1	0	−1	−2
$(\psi_0{}^1\psi_M{}^{(2)}, \Psi_M{}^i)$	$+(\tfrac{2}{3})^{\frac{1}{2}}$	$+\tfrac{1}{2}(\tfrac{2}{3})^{\frac{1}{2}}$	0	$-\tfrac{1}{2}(\tfrac{2}{3})^{\frac{1}{2}}$	$-(\tfrac{2}{3})^{\frac{1}{2}}$

APPENDIX

Proof of Theorem II

The notation and the general system of Wigner's book[4] are employed here. If

$$\Psi_M{}^j = \Sigma_\mu C_{\mu, M-\mu}{}^j \psi_\mu{}^l \psi_{M-\mu}{}^s$$

is an irreducible state of the angular momentum $j=1+s$, then the matrix

$$X_{\mu'\mu} = \Sigma_{M,\nu}(\psi_{\mu'}{}^l\psi_\nu{}^s, \Psi_M{}^j)(\Psi_M{}^j, \psi_\mu{}^l\psi_\nu{}^s) = X_j{}^l\delta_{\mu'\mu}$$

is diagonal, and

$$X_j{}^l = (2j+1)/(2l+1).$$

Let $D^{(l)}(R)_{\mu'\mu}$ be the irreducible representation induced by the rotation group on $\psi_\mu{}^l$, and similarly for $D^{(s)}(R)_{\nu'\nu}$ on $\psi_\nu{}^s$, and $D^{(j)}(R)_{M'M}$ on $\Psi_M{}^j$. The operator P_R represents the effect of the rotation and is such that

$$P_R\psi_\mu{}^l = \Sigma_{\mu'}D^{(l)}(R)_{\mu'\mu}\psi_{\mu'}{}^l$$

and similarly for $\psi_\nu{}^s$, $\Psi_M{}^j$. Then,

$$X_{\mu'\mu} = P_R X_{\mu'\mu} = \Sigma_{(M,\nu,\mu'',\nu'',M',M'',\mu'',\nu''')}D_{\mu''\mu'}{}^{(l)*}D_{\nu''\nu}{}^{(s)*}(\psi_{\mu''}{}^l\psi_{\nu''}{}^s, \Psi_M{}^j)D_{M'M}{}^{(j)}D_{M''M}{}^{(j)*}$$

$$\cdot (\Psi_{M''}{}^j, \psi_{\mu'''}{}^l\psi_{\nu'''}{}^s)D_{\mu'''\mu}{}^{(l)}D_{\nu'''\nu}{}^{(s)}$$

where R is not explicitly indicated and $*$ means the conjugate complex. Now,

$$\Sigma_\nu D_{\nu''\nu}{}^{(s)*}D_{\nu'''\nu}{}^{(s)} = \delta_{\nu''\nu'''}, \quad \Sigma_M D_{M'M}{}^{(j)}D_{M''M}{}^{(j)*} = \delta_{M'M''},$$

hence

$$X_{\mu'\mu} = P_R X_{\mu'\mu} = \Sigma_{M,\nu,\mu'',\mu'''}D_{\mu''\mu'}{}^{(l)*}(\psi_{\mu''}{}^l\psi_\nu{}^s, \Psi_M{}^j)(\Psi_M{}^j, \psi_{\mu'''}{}^l\psi_\nu{}^s)D_{\mu'''\mu}{}^l$$

$$= \Sigma_{\mu'',\mu'''}D^l(R)_{\mu'\mu''}{}^{-1}X_{\mu''\mu'''}D^l(R)_{\mu'''\mu}$$

for all R. According to Schur's lemma the matrix $(X_{\mu'\mu})$ must be a multiple of the unit matrix,

$$X_{\mu'\mu} = X_j{}^l\delta_{\mu'\mu}.$$

Furthermore,

$$\Sigma_\mu X_{\mu\mu} = 2j+1 = (2l+1)X_j{}^l$$

so that

$$X_j{}^l = (2j+1)/(2l+1).$$

It is only needed to make the further observation that $(\psi_\mu{}^l\psi_{M-\mu}{}^s, \Psi_M{}^j) = C_{\mu, M-\mu}{}^j$, and that

$$\Sigma_M |C_{\mu, M-\mu}{}^j|^2 = X_{\mu\mu} = X_j{}^l = (2j+1)/(2l+1)$$

to see that Theorem II holds.