four $\langle 111\rangle$ directions. Pairs of solute atoms lying along these four directions will be affected equally by a shear stress $\sigma_{xx}-\sigma_{yy}$ across the (110) planes, but unequally by a shear stress σ_{xy} across a (001) plane. The first type of shear stress will therefore not cause a preferred orientation of pair axes, i.e., δ' is identically zero, and therefore the Δ_E due to interaction of nearest neighbors is zero when a tensile stress is applied along one of the principal axes. On the other hand, the shear stress σ_{xy} will tend to cause a preferred orientation, and hence δ is not zero. These conclusions may be arrived at in a more elegant manner. Let n_{111}, $n_{\bar{1}11}\cdots$ be the number of pairs of solute atoms with axes lying along the directions [111], [$\bar{1}$11], $\cdots$. The potential energy of the lattice can then contain three interaction terms which are linear in the n's and in the strains. One interaction term will contain the product of the sum of the n's and of the sum $\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}$. This term will cause no relaxation, and hence will not be further considered. The remaining two terms represent interaction of the orientation variables with shear strains. These terms must have the same symmetry as the lattice. One term is

$$\alpha\{n_{111}(\epsilon_{yz}+\epsilon_{zx}+\epsilon_{xy})+n_{\bar{1}11}(\epsilon_{yz}-\epsilon_{zx}-\epsilon_{xy}) +n_{1\bar{1}1}(-\epsilon_{yz}+\epsilon_{zx}-\epsilon_{xy})+n_{11\bar{1}}(-\epsilon_{yz}-\epsilon_{zx}+\epsilon_{xy})\}.$$

No interaction term can be formed of the ϵ_{xx}, ϵ_{yy}, ϵ_{zz} strains, other than that representing a uniform dilation, which has the cubic symmetry of a cubic lattice.

In f.c.c. lattices the axes passing through nearest neighbors lie along one of the six $\langle 110\rangle$ directions. These pairs are affected unequally by both types of shearing stress, and hence both δ and δ' are different from zero. This conclusion is vindicated by the existence of two shear interaction terms which satisfy the symmetry relations. These are

$$\beta\{(n_{011}+n_{0\bar{1}1})(2\epsilon_{xx}-\epsilon_{yy}-\epsilon_{zz}) +(n_{101}+n_{\bar{1}01})(2\epsilon_{yy}-\epsilon_{zz}-\epsilon_{xx}) +(n_{110}+n_{\bar{1}10})(2\epsilon_{zz}-\epsilon_{xx}-\epsilon_{yy})\}$$

and

$$\gamma\{(n_{011}-n_{0\bar{1}1})\epsilon_{yz}+(n_{101}-n_{\bar{1}01})\epsilon_{zx}+(n_{110}-n_{\bar{1}10})\epsilon_{xy}\}.$$

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Quantized Space-Time

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It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time.

THE problem of the interaction of matter and fields has not been satisfactorily solved to this date. The root of the trouble in present field theories seems to lie in the assumption of point interactions between matter and fields. On the other hand, no relativistically invariant Hamiltonian theory is known for any form of interaction other than point interactions.

Even for the case of point interactions the relativistic invariance is of a formal nature only, as the equations for quantized interacting fields have no solutions. The uses of source functions, or of a cut-off in momentum space to replace infinity by a finite number are distasteful arbitrary procedures, and neither process has yet been formulated in a relativistically invariant manner. It may not be possible to do this.

It is possible that the usual four-dimensional continuous space-time does not provide a suitable framework within which interacting matter and fields can be described. I have chosen the idea that a modification of the ordinary concept of space-time may be necessary because the "elementary" particles have fixed masses and associated Compton wave-lengths.

The special theory of relativity may be based on the invariance of the indefinite quadratic form

$$S^2=c^2t^2-x^2-y^2-z^2, \quad (1)$$

for transformations from one inertial frame to another. It is usually assumed that the variables x, y, z, and t take on a continuum of values and that they may take on these values simultaneously. This last assumption we change to the following:

x, y, z, and t are Hermitian operators for the space-time coordinates of a particular Lorentz frame; the spectrum of each of the operators x, y, z, and t is composed of the possible results of a measurement of the corresponding quantity; the operators x, y, z, and t shall be such that the spectra of the operators x', y', z', t' formed by taking linear combinations of x, y, z, and t, which leave the quadratic form (1) invariant, shall be the same as the spectra of x, y, z, and t.

In other words, we assume that the spectra of the space-time coordinate operators are invariant under Lorentz transformations. It is evident that the usual space-time continuum satisfies the above definition; it is, however, not the only solution. The principal result in this paper is that there exists a Lorentz invariant space-time in which there is a natural unit of length. We hope that the introduction of such a unit of length will remove many of the divergence troubles of present field theory.

It can be shown easily that the introduction of a smallest unit of length in space-time will force one to drop the usual assumptions of commutativity of x, y, z, and t, otherwise the assumption of Lorentz invariance of the spectra of the operators x, y, z, and t, if they commute, implies continuous spectra.

To find operators x, y, z, and t possessing Lorentz invariant spectra, we consider the homogeneous quadratic form[1]

$$-\eta^2=\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2-\eta_4^2, \tag{2}$$

in which the η's are assumed to be real variables. The variables η_0 to η_4 may be regarded as the homogeneous (projective) coordinates of a real four-dimensional space of constant curvature[2] (a De Sitter space). We now define x, y, z, and t by means of the infinitesimal elements of the group under which quadratic form (2) is invariant. We take

$$\begin{aligned} x&=ia(\eta_4\partial/\partial\eta_1-\eta_1\partial/\partial\eta_4),\\ y&=ia(\eta_4\partial/\partial\eta_2-\eta_2\partial/\partial\eta_4),\\ z&=ia(\eta_4\partial/\partial\eta_3-\eta_3\partial/\partial\eta_4),\\ t&=(ia/c)(\eta_4\partial/\partial\eta_0+\eta_0\partial/\partial\eta_4),\end{aligned} \tag{3}$$

in which a is the natural unit of length, c is the velocity of light. The operators x, y, z, and t are assumed to be Hermitian and may be regarded as operating on single valued functions of η_0, $\eta_1\cdots$. From the assumption that x, y, and z are Hermitian operators of the form (3), one can show that each of them has a spectrum consisting of the characteristic values, ma where m is an integer, positive, negative, or zero. The operator, t, has a continuous spectrum extending from minus infinity to plus infinity. The spectrum of each of the operators x, y, z, and t is infinitely degenerate.

Transformations which leave quadratic form (2) and η_4 invariant are covariant Lorentz transformations on the variables η_1, η_2, η_3, and η_0. When the transformed variables η_0', η_1', η_2', η_3', η_4 are substituted into Eqs. (3), it is found that x, y, z, and t undergo contravariant Lorentz transformation. It is evident that the new operators x', y', z', and t' which are formed by replacing η_0, η_1, η_2, η_3, η_4 in Eqs. (3) by η_0', η_1', η_2', η_3', η_4 are linear expressions with real constant coefficients in x, y, z, and t, and are consequently Hermitian operators if x, y, z, and t are Hermitian. The functions on which these operators operate are left invariant except for change in argument. Thus, we see that x', y', z', and t' which are Hermitian operators of the same form in the variables η_0', η_1', η_2', η_3', η_4 as x, y, z, and t were in the variables η_0, η_1, η_2, η_3, η_4 have the same spectra as do x, y, z, t.

We define six additional operators by the equations

$$\begin{aligned} L_x&=i\hbar(\eta_3\partial/\partial\eta_2-\eta_2\partial/\partial\eta_3),\\ L_y&=i\hbar(\eta_1\partial/\partial\eta_3-\eta_3\partial/\partial\eta_1),\\ L_z&=i\hbar(\eta_2\partial/\partial\eta_1-\eta_1\partial/\partial\eta_3),\\ M_x&=i\hbar(\eta_0\partial/\partial\eta_1+\eta_1\partial/\partial\eta_0),\\ M_y&=i\hbar(\eta_0\partial/\partial\eta_2+\eta_2\partial/\partial\eta_0),\\ M_z&=i\hbar(\eta_0\partial/\partial\eta_3+\eta_3\partial/\partial\eta_0),\end{aligned} \tag{4}$$

[1] We could also have taken the quadratic form

$$\eta^2=\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2+\eta_4^2.$$

This leads to another Lorentz invariant discrete space-time.

[2] I am indebted to W. Pauli in a private communication for this interpretation of the variables η_0, η_1 $\cdots$.

in which $\hbar$ is Plank's constant divided by 2π. Now, L_x, L_y, L_z, M_x, M_y, M_z are, with the exception of the factor $i\hbar$, the infinitesimal elements of the four-dimensional Lorentz group. It may be shown that each of the above operators commutes with quadratic form (1) when the values of x, y, z, t given by (3) are used. This is another way of stating the Lorentz invariance of quadratic form (1). It will be observed that L_x, L_y, L_z, M_x, M_y, M_z do not involve η_4, and that as a consequence the Lorentz transformation which leaves form (2) and η_4 invariant leaves quadratic form (1) invariant, and as has already been shown, leaves the spectra of the operators x, y, z, and t invariant. We see from these facts that the usual assumptions concerning the continuous nature of space-time are not necessary for Lorentz invariance. This result is the minimum objective of this work.

The ten operators defined in (3) and (4) have a total of forty-five commutators. Only six of these commutators differ from the ordinary ones and these six are

$$\begin{aligned} [x, y] &= (ia^2/\hbar)L_z, & [t, x] &= (ia^2/\hbar c)M_x, \\ [y, z] &= (ia^2/\hbar)L_x, & [t, y] &= (ia^2/\hbar c)M_y, \\ [z, x] &= (ia^2/\hbar)L_y, & [t, z] &= (ia^2/\hbar c)M_z. \end{aligned} \tag{5}$$

We see from these commutators that if we take the limit $a \rightarrow 0$ keeping $\hbar$ and c fixed, our quantized space-time changes to the ordinary continuous space-time.

The commutators for the quantities L_x, L_y, L_z, M_x, M_y, M_z are those of the infinitesimal elements of the Lorentz group, and for this reason we introduced the factor of $i\hbar$ in their definition. Thus, L_x, L_y, L_z have the usual properties of quantum angular momentum. The commutators which involve one of the operators x, y, z, or t, with one of the operators L_x, L_y, L_z, M_x, M_y, M_z are independent of a, the natural unit of length. Thus all of the commutators pass in the limit $a \rightarrow 0$ to their usual expressions.

In addition to the ten quantities defined in (3) and (4) we define four additional operators, having the transformation properties of space or time displacement operators, or the equivalent, of energy and momentum operators.[3]

$$\begin{aligned} p_x &= (\hbar/a)(\eta_1/\eta_4), & p_z &= (\hbar/a)(\eta_3/\eta_4), \\ p_y &= (\hbar/a)(\eta_2/\eta_4), & p_t &= (\hbar c/a)(\eta_0/\eta_4). \end{aligned} \tag{6}$$

By means of algebraic manipulations one can show that

$$L_x = yp_z - zp_y; \quad M_x = \frac{1}{c}xp_t + ctp_x; \text{ etc.} \tag{7}$$

The angular momenta have their usual expression in terms of coordinate and momentum. One must exercise care as to the order in which the factors in (7) are written since they do not commute. The four quantities p_x, p_y, p_z, p_t commute with one another, and have the same commutators with L_x, L_y, L_z, M_x, M_y, M_z as do the usual expressions for the space or time displacement operators. In addition, each has a continuous spectrum running from minus infinity to plus infinity. Their commutators with the coordinates and time are not the same as usual and are given by

$$\begin{aligned} &[x, p_x] = i\hbar[1 + (a/\hbar)^2 p x^2]; \\ &[t, p_t] = i\hbar[1 - (a/\hbar c)^2 p_t{}^2]; \\ &[x, p_y] = [y, p_x] = i\hbar(a/\hbar)^2 p_x p_y; \\ &[x, p_t] = c^2[p_x, t] = i\hbar(a/\hbar)^2 p_x p_t; \text{ etc.} \end{aligned} \tag{8}$$

Here we note that if all the components of the momentum are small compared to $\hbar/a$ and the energy is small compared to $\hbar c/a$ then these commutators approach those which are given in orinary quantum mechanics. Further, as we take the limit $a \rightarrow 0$ these commutators change to their standard values.

The fact that these new commutators between coordinate and momentum differ from the old ones appreciably only for large values of the momentum, and that they differ by large amounts when $|p| > \hbar/a$ implies that a field theory based on quantized space-time will give substantially the same results as the usual field

[3] The most general form for the energy momentum operators with the correct transformation properties is

$$p_x = \frac{\hbar}{a}\frac{\eta_1}{\eta_4}f\left(\frac{\eta_4}{\eta}\right), \cdots, p_t = \frac{\hbar c}{a}\frac{\eta_0}{\eta_4}f\left(\frac{\eta_4}{\eta}\right),$$

in which $f(\eta_4/\eta)$ is a dimensionless function of its argument. A choice of $f(\eta_4/\eta)$ other than the choice which gives Eqs. (6) may be more useful for physical problems.

theory for all processes which do not involve large components of the momenta, but will produce large effects for processes which do involve large components of the momenta. We might expect that the usual atomic and molecular formulas will suffer no appreciable change, while expressions for self-energy, polarization of the vacuum, and possibly nuclear forces will be considerably altered. Alterations in the last mentioned quantities are certainly necessary.

We note here that the commutator relations (7) have a solution for x, y, z, t.

$$x=i\hbar\left[\frac{\partial}{\partial p_x}+(a/\hbar)^2p_x\left(p_x\frac{\partial}{\partial p_x}+p_y\frac{\partial}{\partial p_y}+p_z\frac{\partial}{\partial p_z}+p_t\frac{\partial}{\partial p_t}\right)\right];$$
$$t=i\hbar\left[\frac{\partial}{\partial p_t}-(a/\hbar c)^2p_t\left(p_x\frac{\partial}{\partial p_x}+p_y\frac{\partial}{\partial p_y}+p_z\frac{\partial}{\partial p_z}+p_t\frac{\partial}{\partial p_t}\right)\right];\ \text{etc.} \tag{9}$$

This solution of the commutator relations (8) is completely analogous to the solution of the conventional commutator relations between coordinate and momentum for the coordinates in terms of derivatives with respect to the momentum. It would not be surprising to find in our case that no corresponding solution for the momenta in terms of derivatives with respect to the coordinates exists since our coordinates and time do not have continuous spectra in general and do not commute.

It is not difficult to verify that if the expressions for x, y, z, and t given by (9) are substituted in the commutators (5) and the corresponding L_x, L_y, L_z, M_x, M_y, M_z computed, then these last mentioned quantities satisfy their usual commutation relations.

It might appear that x, y, z, and t as given by Eqs. (9) are not Hermitian operators. It may be shown, however, that they are Hermitian operators when the correct volume elements of group space are used. The group which concerns this is the transformation group which leaves quadratic form (2) invariant. An infinitesimal volume element, $d\tau$, of group space is given in terms of p_x, p_y, p_z and p_t by the formula

$$d\tau=\frac{\hbar dp_xdp_ydp_zdp_t}{ac(p_x^2+p_y^2+p_z^2+(\hbar/a)^2-(p_t/c)^2)^{5/2}}. \tag{10}$$

To this point the invariance properties of quantized space-time have been considered only with respect to rotations and Lorentz transformations which leave the origin fixed. A continuum of translations is not admissable in this space, indeed one can prove that if x, y, z, and t are such that quadratic form (1) is invariant under infinitesimal displacements, the space x, y, z, and t must be a continuum. Translations of the origin may be introduced as follows. Let $S(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4)$ be an arbitrary unimodular single-valued homogeneous function of the degree zero in the variables, η_0, η_1, η_2, η_3, η_4. Let A be any one of the operators defined by Eqs. (3), (4), or (6). Form the new operators $A'=\tilde{S}AS$ in which $\tilde{S}$ means the complex conjugate of S. It may then be shown that the primed operators satisfy exactly the same commutation relations as did the original operators and that they are Hermitian. The primed operators thus have exactly the same transformation properties with respect to Lorentz transformations as did the original operators. In particular, the displacement operators p_x, p_y, p_z, p_t are not changed by this process. This is a result which we must demand for the space and time displacement operators. If we choose $S=\exp[im\arctan(\eta_1/\eta_4)]$, we find that this results in a translation of ma in the x direction. It should not be expected, nor is it possible in this space, to find a function S which corresponds to sharply defined translational values of x, y, z, and t simultaneously. The relation between two different frames of reference cannot be set up more precisely than the commutation relations between the coordinates permit.

We will not discuss in any detail in this paper the limitations placed upon the simultaneous measurability of x, y, z, and t due to the noncommutativity of these quantities. Some preliminary calculations which I have made indicate that these limitations are not serious enough to interfere with the ordinary description of atomic phenomena in terms of a continuous space-time nor with our usual macroscopic theory.