

Two Notes on Phase-Integral Methods

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I. The connection formulas are derived by a new method. The general approach is that of Zwaan's discussion of Stokes' phenomenon, but no use is made of any assumptions about the reality of the coefficients of the differential equation. Instead of this, the proof is based only on the fact that actual solutions of the differential equation must be single-valued. Both this manner of proof and the form in which the results are obtained are suited to the discussion of certain problems in which a boundary condition consists in the requirement that the field at a great distance contain an out-going wave only. The results serve to establish the validity of Eckersley's phase-integral method for the treatment of problems of wave propagation. II. General arguments used to establish phase-integral methods are asymptotic in nature, and lead to the expectation that the methods will be valid only in a certain limit; in the language

of quantum mechanics this is the limit of large quantum numbers. Much of the methods' usefulness, however, comes from the fact that they give in practice surprisingly accurate results even for small quantum numbers. In the case of the energy levels of the anharmonic oscillator, special arguments have been devised by Kemble and by Birkhoff to establish the usual phase-integral formula without assuming large quantum numbers. In this note a special argument is given for calculating normalization factors for the approximate wave functions of the oscillator. It is shown that the usual asymptotic formula for the normalization factor holds for the lowest quantum states, with about the accuracy with which the phase-integral solution approximates the shape of the exact wave function at the point at which the potential energy function has its minimum.

I. A NEW DERIVATION OF THE CONNECTION FORMULAS

Introduction

THE phase-integral method for solving ordinary differential equations was first clearly formulated by Jeffreys,¹ and was applied to quantum-mechanical problems by Brillouin,² Wentzel,³ and Kramers.⁴ It is usually referred to among physicists as the B.W.K. method or the W.K.B. method.

The differential equation

$$d^2u/dz^2 + k^2(y(z) + \lambda)u = 0 \quad (1)$$

is solved approximately, on the assumption that $k|\Delta z|$ is large if Δz is a distance in which $y(z)$ changes significantly. Any point z_0 for which

$$y(z_0) + \lambda = 0 \quad (2)$$

is called a turning point. In the neighborhood of such a point the coefficient $y(z) + \lambda$ changes significantly for arbitrarily small Δz , so that the

approximate solutions to be obtained cannot be valid near such points. They must have the character of asymptotic solutions, providing good approximations to actual solutions in regions sufficiently far from all turning points. In the usual applications of the method it is assumed that the turning points are all simple zeros of $y + \lambda$, and that any two of them are so far apart that the asymptotic solutions provide good approximations to actual solutions throughout the greater part of the region between them. It is also assumed that the function $y(z)$ has no singularities in the finite region of the z plane.

The general approximate solution of Eq. (1) on these assumptions is

$$u \cong (y + \lambda)^{-1/2} \{Ae^{ikw} + Be^{-ikw}\}, \quad (3)$$

where

$$w = \int_{z_0}^z (y + \lambda)^{1/2} dz. \quad (4)$$

The derivation has been published a great many times. The two arbitrary constants in the solution are A and B ; the constant lower limit of the integral in Eq. (4) is of course not an independent arbitrary constant, and has been chosen for convenience to be a turning point, z_0 .

A given solution u can be approximated closely in a certain region of the z plane by choosing

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¹H. Jeffreys, Proc. London Math. Soc. [2] 23, 428 (1923).

²L. Brillouin, Comptes rendus 183, 24 (1926).

³G. Wentzel, Zeits. f. Physik 38, 518 (1926).

⁴H. A. Kramers, Zeits. f. Physik 39, 828 (1926).

suitable values of the constants A and B . Since the differential equation (1) has no finite singularities, the solution $u(z)$ is a single-valued function;⁵ on the other hand, the right-hand member of Eq. (3) is multiple-valued. Accordingly, it is not possible for the same choice of values of A and B to give a good approximation to a given solution u throughout the whole of the part of the plane, remote from turning points, where an approximation in the form of Eq. (3) is possible. Different values of A and B must be used in different regions in order to approximate the same solution u . This phenomenon, which is of general occurrence in the study of common types of asymptotic approximations,⁶ was first noted and explained by Stokes.⁷

Connection formulas relating the different expressions of the type of Eq. (3) which approximate the same solution u were given by Jeffreys.⁸ The use of such formulas in understanding clearly the approximate solution of quantum-mechanical problems was stressed by Kramers.⁴ Kramers first obtained the formulas by approximating $y(z)$ by a linear function in the neighborhood of z_0 , and studying the exact solutions of the resulting differential equation. A simpler derivation was given by Zwaan,⁹ who used the basic idea by means of which Stokes had explained the phenomenon: In certain regions of the z plane one of the exponential factors in Eq. (3) is very large and the other is very small. Because the relation between the right-hand member of Eq. (3) and the actual solution u is only approximate, there is no meaning to the assignment of any particular value to the coefficient of the small exponential factor, for points in such a region. Thus this coefficient can have different values

for two regions adjacent to such a "Stokes region" on opposite sides, but the other coefficient must keep the same value.

More elaborate discussions have been given by Langer¹⁰ and by Kemble.¹¹ Langer's treatment is a generalization of that of Kramers; Kemble's is a critical discussion of Zwaan's method, with applications to certain further situations not covered by the usual connection formulas.¹²

The present treatment uses Stokes' fundamental idea in the same simple way as did Zwaan. In his treatment Zwaan assumed that the coefficient $k^2(y+\lambda)$ is real for real values of z . This assumption is unnecessarily restrictive, and is untrue in some cases of interest; for example, in problems of wave propagation it often happens that the imaginary part of λ is not small. The derivation given here involves no assumption about reality; instead we use the fact that the true solution $u(z)$ does not have a branch-point at z_0 .

Derivation of the Formulas

The only assumption we shall make about the analytic function $y(z)$, is that $y(z)+\lambda$ has a simple zero at z_0 , and that there are no singularities or other zeros near enough to z_0 to prevent our going out from this turning point in an arbitrary direction far enough to make the asymptotic expression (3) a good approximation.

Very near to z_0 we can represent $y(z)+\lambda$ by the first term of Taylor's expansion, and get

$$w = \int_{z_0}^z (y+\lambda)^{\frac{1}{2}} dz \cong (y'(z_0))^{\frac{1}{2}} \int_0^{z-z_0} \zeta^{\frac{1}{2}} d\zeta = \frac{2}{3} (y'(z_0))^{\frac{1}{2}} (z-z_0)^{\frac{3}{2}}. \quad (5)$$

There are three directions, making angles of 120° with each other, along which we can move z away from z_0 in such a way that the expression (5) is purely real. Once we have started out in any one of these three directions we can choose

⁵ The assumption that $y(z)$ is an integral function, made here for convenience, is unnecessarily restrictive. It suffices to assume that any singularities of $y(z)$ are well removed from z_0 . The true solution u , unlike the functions in the right-hand member of Eq. (3), does not have z_0 as a branch point.

⁶ For example, in many of the asymptotic formulas for Bessel functions: cf. G. N. Watson, *Treatise on the Theory of Bessel Functions* (The Macmillan Company, New York, 1922 and 1944), pp. 201-202, and further mention on pp. 203, 238, 336.

⁷ G. G. Stokes, *Mathematical and Physical Papers* (Cambridge University Press, Cambridge), Vol. IV, pp. 77-109 and pp. 283-298; *ibid.* Vol. V, pp. 221-225.

⁸ Reference 1. There is a misprint in one of Jeffreys' formulas.

⁹ A. Zwaan, Thesis, "Intensitäten im Ca-Funkenspektrum," Utrecht, 1929.

¹⁰ R. E. Langer, *Trans. Am. Math. Soc.* **33**, 23 (1931); *ibid.* **34**, 447 (1932); *Phys. Rev.* **51**, 669 (1937).

¹¹ E. C. Kemble, *Phys. Rev.* **48**, 549 (1935); *Fundamental Principles of Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1937), pp. 90-112, 572-578.

¹² Kemble discusses the radial equation of the Kepler problem by his method, and also the low quantum states of the oscillator, as mentioned in the second of these notes.

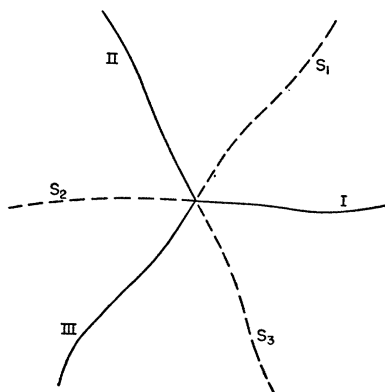


FIG. 1. Loci extending from a turning point.

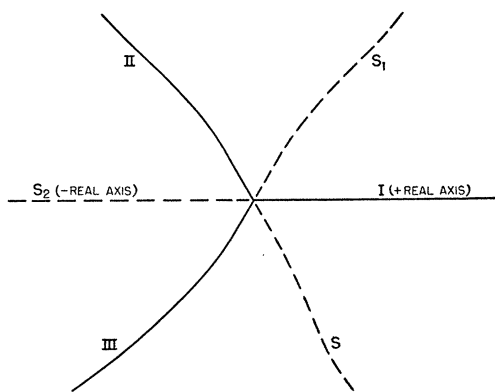


FIG. 2. Loci for case in which $y(z)$ is real for real z , and λ and z_0 are real.

each successive increment dz such that $(y+\lambda)^{\frac{1}{2}}dz$ is real, and thus define a locus, which in general is a curved line. We accordingly get three curves, shown as I, II, III in Fig. 1, such that

$$w(z) = \int_{z_0}^z (y+\lambda)^{\frac{1}{2}} dz \text{ is pure real on I, II, III.} \quad (6)$$

We define a branch of $(y+\lambda)^{\frac{1}{2}}$ by requiring this expression to be positive on I and then continuing counterclockwise from I. Then

$$w(z) = \int_{z_0}^z (y+\lambda)^{\frac{1}{2}} dz \text{ is } \begin{array}{l} + \text{ on I,} \\ - \text{ on II,} \\ + \text{ on III,} \end{array} \quad (7)$$

and - on return to I.

Along each of these three loci the two functions which appear in Eq. (3) are equal in magnitude, and if we are to take Eq. (3) as an approximate expression for u we must know both A and B . We set ourselves the problem of determining the coefficients A_{II} and B_{II} which must be used on II to approximate a function which is approximated on I by Eq. (3) with given coefficients A_I and B_I . Since the problem is linear and admits of superposition of solutions, there must be a relation in the form

$$\begin{array}{l} A_{II} = aA_I + bB_I \\ B_{II} = cA_I + dB_I, \end{array} \quad (8)$$

with coefficients a, b, c, d independent of the A 's and B 's.

We can also construct three loci running out from z_0 , on each of which $w(z)$ is purely imaginary.

These are shown in Fig. 1 as S_1, S_2, S_3 . With the choice of branch, and of direction of continuation used in writing (7) we then have

$$\begin{array}{l} w(z) \text{ is } + \text{ imaginary on } S_1; \text{ far out on } S_1, \\ \qquad \qquad \qquad \qquad \qquad \qquad e^{-ikw} \gg e^{ikw}, \\ w(z) \text{ is } - \text{ imaginary on } S_2; \text{ far out on } S_2, \\ \qquad \qquad \qquad \qquad \qquad \qquad e^{ikw} \gg e^{-ikw}, \\ w(z) \text{ is } + \text{ imaginary on } S_3; \text{ far out on } S_3, \\ \qquad \qquad \qquad \qquad \qquad \qquad e^{-ikw} \gg e^{ikw}. \end{array} \quad (9)$$

These loci are commonly called "Stokes lines." On or near such a locus only one coefficient in (3) needs, in general, to be given in order to specify the expression which approximates a given function u asymptotically. This is, of course, the coefficient of the exponential which is large on the locus in question. The coefficient of the other exponential has no meaning on or near the locus, except in the special case where the coefficient of the large exponential vanishes. The Stokes phenomenon consists in the fact that if a coefficient becomes meaningless on a Stokes line, its value in the region on one side of the line can differ from that in the region on the other side.

Applying this conception of the Stokes phenomenon to the continuation from I to II across S_1 , we see that B_{II} must equal B_I , but A_{II} need not equal A_I unless $B_I = 0$. Thus (8) becomes

$$\begin{array}{l} A_{II} = A_I + \alpha B_I, \\ B_{II} = B_I. \end{array} \quad (10)$$

All now that remains of our problem is to evaluate the constant α .

Similarly, we get for the continuation from II

to III across S_2 :

$$\begin{aligned} A_{III} &= A_{II}, \\ B_{III} &= \beta A_{II} + B_{II}; \end{aligned} \quad (11)$$

and for the continuation from III back to I across S_3 :

$$\begin{aligned} A_I' &= A_{III} + \gamma B_{III}, \\ B_I' &= B_{III}. \end{aligned} \quad (12)$$

Then from (10)-(12) we get

$$\begin{aligned} A_I' &= (1 + \gamma\beta)A_I + (\alpha + \gamma + \alpha\beta\gamma)B_I, \\ B_I' &= \beta A_I + (1 + \alpha\beta)B_I. \end{aligned} \quad (13)$$

On the other hand we can obtain directly a relation between A_I' and B_I' , the coefficients to be used in (3) after continuation through one revolution counterclockwise, and the original coefficients A_I and B_I . We simply use the fact that, since u is single valued, both expressions (3) must give the same value. Since, by Eq. (7), the two functions e^{ikw} and e^{-ikw} interchange values by this continuation, and since the factor $(y + \lambda)^{-1/2}$

acquires a factor $e^{-1/2 \cdot 2\pi i} = -i$, we must accordingly require

$$A_I' = iB_I; \quad B_I' = iA_I. \quad (14)$$

The necessary and sufficient condition for Eqs. (13) and (14) to be consistent is $\alpha = \beta = \gamma = i$. Then (10) becomes

$$A_{II} = A_I + iB_I; \quad B_{II} = B_I. \quad (15)$$

This is the fundamental connection formula, from which the various formulas required for specific applications can readily be obtained.

The traditional statement of the connection formulas is based on the assumptions that λ is real; that $y(z)$ is real for real z ; and that the turning-point under consideration, z_0 , is real. In this case the lines I and S_2 are straight and coincide with portions of the real axis. We write the connection from I to S_2 by using that from I to II and neglecting the term e^{-ikw} on S_2 unless A_{II} is zero. By using $A_I = e^{-i\pi/4}$, $B_I = e^{i\pi/4}$, we get from Eqs. (15) and (3)

$$\begin{aligned} |y + \lambda|^{-1/2} \exp\left(-k \int_z^{z_0} |y + \lambda|^{1/2} dz\right) &\rightarrow 2(y + \lambda)^{-1/2} \cos\left\{k \int_{z_0}^z (y + \lambda)^{1/2} dz - \frac{\pi}{4}\right\}. \end{aligned} \quad (16)$$

(on S_2) (on I)

Since we are interested only in values *on* the axis, we have used absolute values in the left member so as to express it in terms of real positive quantities only.

The meaning of Eq. (16) is that if a function u is approximated on S_2 by the left member, then it is approximated on I by the right member. The arrow points to the right only, because a small error in phase in fitting the right member to u would correspond to the appearance of the positive real exponential with a non-vanishing coefficient on the left, and this would completely ruin the agreement on S_2 .

If we use $A_I = e^{i(\pi/4 - \theta)}$, $B_I = e^{-i(\pi/4 - \theta)}$, we get

$$\begin{aligned} \cos\theta \cdot |y + \lambda|^{-1/2} \exp\left(k \int_z^{z_0} |y + \lambda|^{1/2} dz\right) &\leftarrow (y + \lambda)^{-1/2} \cos\left\{k \int_{z_0}^z (y + \lambda)^{1/2} dz + \frac{\pi}{4} - \theta\right\}. \end{aligned} \quad (17)$$

(on S_2) (on I)

This result is reliable when $|\theta|$ is not too near $\pi/2$. Since any constant multiple of u is a solution as well as u , the value of θ cannot be found by examining u on S_2 , so that the arrow must point to the left only. This rule is often written with $\theta = 0$, which gives

$$\begin{aligned} |y + \lambda|^{-1/2} \exp\left(k \int_z^{z_0} |y + \lambda|^{1/2} dz\right) &\leftarrow (y + \lambda)^{-1/2} \cos\left\{k \int_{z_0}^z (y + \lambda)^{1/2} dz + \frac{\pi}{4}\right\}. \end{aligned} \quad (18)$$

(on S_2) (on I)

These formulas (16)-(18) are written in such a way that they can be applied to the situation shown in Fig. 2 if we simply take all quantities as positive unless the minus sign is written explicitly.

To get formulas which hold when I extends to the left and S_2 to the right, we have only to interchange the limits on the integrals so that dz will continue to point from left to right.

Equations (16) and (18) are usually called the first and second connection formulas. From the fundamental result (15) we can readily derive a formula of somewhat similar appearance, which is valid independently of any assumptions about the reality of λ , $y(z)$, or z_0 . For $A_I=0$, $B_I=1$, Eqs. (15) and (3) give the result

$$2e^{i\pi/4}(y+\lambda)^{-1/2} \cos\left\{k \int_{z_0}^z (y+\lambda)^{1/2} dz + \frac{\pi}{4}\right\} \leftarrow (y+\lambda)^{-1/2} \exp\left\{-ik \int_{z_0}^z (y+\lambda)^{1/2} dz\right\}. \quad (19)$$

(on and near II)

(on and near I)

In application of this formula it must be remembered that the choice of branch is as indicated in (7), so that the term kw in the argument of the cosine is almost or quite purely negative.

The meaning of (19) is that if a given function u is closely approximated by the right member on and near I, then it is closely approximated by the left member on and near II. The converse is not true: if we fit the asymptotic expression to approximate agreement with u on and near II, we cannot be sure that the phase is determined precisely, and a small error in phase would mean the presence of a term in e^{ikw} on and near I. Although such a term, with small coefficient, would not hurt the agreement on and above I, it would spoil the agreement completely below I, where e^{ikw} becomes large. Accordingly, we draw the arrow in (19) pointing from right to left only.

Since, as shown below, Eq. (19) has direct applications of some interest, it might be convenient to call it the third connection formula. This is not justified in the sense of mathematical independence, since the fundamental relation (15) contains just two equations. The conventional presentation of the two formulas, however, as shown in Eqs. (16) and (18), does not lend itself readily to the treatment of cases in which the reality conditions assumed in this presentation do not hold. Thus from the point of view of applications the formula of Eq. (19) is a useful addition to the two standard connection formulas.

An Application

Phase-integral methods have commonly been applied to three types of boundary-value problems, all with $y(z)$ real for real z :

(a) Outgoing waves only for $z \rightarrow +\infty$, no requirement for $z \rightarrow -\infty$. The problem is that of

scattering in one dimension, and can be solved for any real value of λ for which $y+\lambda > 0$ for $z \rightarrow +\infty$ and for $z \rightarrow -\infty$. The phase-integral method gives useful results only when $y+\lambda < 0$ in some part (or parts) of the range $-\infty < z < \infty$; this is the problem of "penetration of a potential barrier."

(b) $u \rightarrow 0$ for $z \rightarrow +\infty$, and either $u \rightarrow 0$ for $z \rightarrow -\infty$ or else a condition of the form $u(0) + \mu u'(0) = 0$, with real μ . The problem can be solved only for certain real characteristic values of λ . Examples are the problems of energy levels of the oscillator and of the radial equation of the central-field problem.

(c) Outgoing wave only for $z \rightarrow +\infty$, and a condition of the form $u(0) + \mu u'(0) = 0$. The problem can be solved only for certain characteristic values of λ , and these are in general complex, even for real μ .

Applications of phase-integral methods to problems of types (a) and (b) are well known in the literature of quantum mechanics. We discuss here the applications to certain simple problems of type (c).

The case in which the curve of $-y(z)$, which plays the role of potential energy in quantum mechanics, has a maximum and then approaches zero for $z \rightarrow +\infty$, and $I(\lambda) \ll 1$, has been discussed by Gamow¹³ in his treatment of radio-active disintegration. The equation

$$y(z) + R(\lambda) = 0 \quad (20)$$

has two real zeros, z_1 and z_2 ; the relations are shown graphically in Fig. 3. Since $\lambda \cong R(\lambda)$, the turning points are very close to z_1 and z_2 . For

¹³ G. Gamow, *Zeits. f. Physik* **51**, 204 (1928); *Atomic Nuclei and Radioactivity* (Oxford University Press, New York, 1931 and 1937), first edition, pp. 38-50; pp. 87-101, second edition.

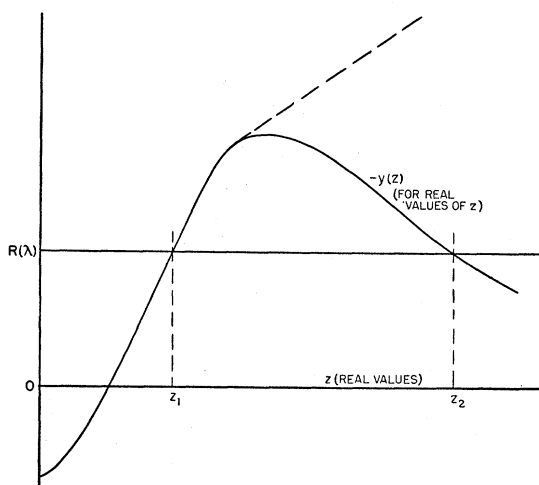


FIG. 3. Curve of $-y(z)$ plotted against real values of z .

the usual case, $\mu = 0$, the phase-integral method requires that $R(\lambda)$ be calculated from the condition

$$k \int_0^{z_1} (y + R(\lambda))^{\frac{1}{2}} dz = (m - \frac{1}{4})\pi, \quad (21)$$

where z_1 is the smaller root of Eq. (20) and m is an integer. $I(\lambda)$ is then given by

$$I(\lambda) = \left\{ 2k \int_0^{z_1} (y + R(\lambda))^{-\frac{1}{2}} dz \right\}^{-1} \times \exp \left\{ -2k \int_{z_1}^{z_2} |y + R(\lambda)|^{\frac{1}{2}} dz \right\}. \quad (22)$$

Approximate values can be obtained by these formulas for all integer values of m which are small enough so that the exponential factor in Eq. (22) is extremely small. These results can be derived by straightforward application of the boundary conditions and the connection formulas (16) and (18). Because of the fact that the arrows in Eqs. (16) and (18) point only one way, however, such a derivation can be made cogent only at the expense of a certain amount of complication. The more usual procedure is to establish Eq. (21) by considering the problem of type (b) obtained by replacing the outer part of the potential curve by something like the dashed curve in Fig. 3—this is approximately justified, since the “leakage through the barrier” is very small—and then establish Eq. (22) by using the approximate expressions for $u(z)$ in

the formula

$$\int_0^z \frac{d}{dz} \left(u^* \frac{du}{dz} - u \frac{du^*}{dz} \right) dz = 2ik^2 I(\lambda) \int_0^z u^* u dz, \quad (23)$$

which follows from Eq. (1).

The use of a phase-integral method for the solution of problems of type (c) in which $I(\lambda)$ is not small has been developed by Eckersley¹⁴ in papers on the propagation of radio waves. According to Eckersley, the equation from which the complex characteristic values of λ are to be obtained is

$$k \int_0^{z_0} (y + \lambda)^{\frac{1}{2}} dz = (m - \frac{1}{4})\pi, \quad (24)$$

for the case $u = 0$. Here m is a positive integer, and z_0 is a function of λ , being a suitably chosen solution of Eq. (2).

This method has been applied successfully in a number of cases, but no readily understandable and convincing argument for its validity has been published; Eckersley's own approach appears to be based essentially on mere analogy with the phase-integral treatment of problems of type (b). The “third connection formula,” Eq. (19), provides a means of deriving Eq. (24) directly from the stated boundary conditions. If the time-dependence is taken to be given by a factor $e^{i\omega t}$, the right-hand member of Eq. (19) represents the required outgoing wave. To make the left-hand member vanish for $z = 0$, the argument of the cosine, being a negative number, must be set equal to $-(m - \frac{1}{4})\pi$, with m a positive integer. This gives at once Eq. (24).

It is clear that this application of the connection formula can be permissible only if z_0 lies in the lower half-plane. It falls in general in the fourth quadrant of the z plane. It is necessary also that z_0 be a well-isolated turning point. A safe rule for the applicability of Eckersley's method is probably that z_0 be far from all other turning points and also be the only turning point in the fourth quadrant. In the case of a function $-y(z)$, like that of Fig. 3, there is an infinite number of solutions obtainable by Eck-

¹⁴ T. L. Eckersley, Proc. Roy. Soc. **A132**, 83 (1931); **A136**, 499 (1932); **A137**, 158 (1932); T. L. Eckersley and G. Millington, Phil. Trans. Roy. Soc. **A237**, 273 (1938).

ersley's method, with the turning-points well separated, and one lying well below the real axis and in the fourth quadrant; and there may be a finite number of solutions obtainable by Gamow's method, with two turning points, well separated, very near the positive part of the real axis.

There may also be one or more solutions which are not obtainable by either phase-integral method, because the turning-points are not well separated.

For the generalized case with $\mu \neq 0$, Eqs. (21), (22), and (24) must be replaced by

$$k \int_0^{z_1} (y + R(\lambda))^{\frac{1}{2}} dz = (m - \frac{1}{4})\pi + \tan^{-1} \mu k (y(0) + R(\lambda))^{\frac{1}{2}}, \quad (21')$$

$$I(\lambda) = \left\{ 2k \int_0^{z_1} (y + R(\lambda))^{-\frac{1}{2}} dz - 2k\mu (y(0) + R(\lambda))^{-\frac{1}{2}} \right. \\ \left. \times [1 + k^2 \mu^2 (y(0) + R(\lambda))]^{-1} \right\}^{-1} \exp \left\{ -2k \int_{z_1}^{z_2} |y + R(\lambda)|^{\frac{1}{2}} dz \right\}, \quad (22')$$

and

$$k \int_0^{z_0} (y + \lambda)^{\frac{1}{2}} dz = (m - \frac{1}{4})\pi + \tan^{-1} \mu k (y(0) + \lambda)^{\frac{1}{2}}. \quad (24')$$

The derivations of Eqs. (21') and (24') are obvious; Eq. (22') can be derived by applying to Eq. (23) a modification of the technique illustrated in the following note.

For convenience this discussion has been based throughout on the idea that the time-dependence is given by a factor of the form $e^{i\omega t}$. This seems to be the preferred convention in studies of radio wave propagation. In quantum mechanics, however, the practically universal custom is to write $e^{-i\omega t}$ for the time factor. When this is done, $I(\lambda)$ is, of course, negative instead of positive.

II. NORMALIZATION OF APPROXIMATE WAVE FUNCTIONS OF THE ANHARMONIC OSCILLATOR

Introduction

The phase-integral methods for solving boundary-value problems are based on the approximate solution of differential equations of the type of Eq. (1) in terms of expressions such as that shown in Eq. (3). The solutions so obtained are asymptotic in character, and provide good approximations to actual solutions only in regions remote from turning points. This means that one can expect to obtain good answers for boundary-value problems when the turning points are well separated from each other and from any finite points at which boundary conditions are applied. The meaning of the words "well separated" is that the phase integral between two well sepa-

rated points is a large number. Accordingly, phase-integral methods should work well in quantum mechanical problems when the quantum number is large, and in problems of classical mathematical physics when one considers high modes of vibration.

The really interesting cases, of course, are almost always those of small quantum numbers or low modes of vibration. The great usefulness of phase-integral methods comes mostly from the fact that in practice they usually give remarkably good results for these cases also. This circumstance suggests the possibility that arguments might be found to justify mathematically the use of the methods in cases of small quantum numbers or low modes of vibration. Such arguments could not be based solely on the general results of Eqs. (3) and (16)–(19), but would have to be of a more special nature.

For the case of the lower energy levels of the anharmonic oscillator two different special arguments for the approximate validity of the phase-integral formula for the energies have been given, one by Birkhoff¹⁵ and the other by Kemble.¹¹ This case is, accordingly, well understood. In the present note a special argument is used to establish for low quantum numbers the formula for the normalization of the approximate wave functions of the oscillator.

¹⁵ G. D. Birkhoff, Bull. Am. Math. Soc. **39**, 696 (1933).

Derivation of the Formula for the Normalization Constant

The potential-energy¹⁶ function $-y(z)$ is assumed to have a single minimum and no maximum. For convenience we suppose the origin of the z -coordinate and the zero of the energy scale so chosen that

$$y' \geq 0, \quad z < 0; \quad y'(0) = 0; \quad y' \leq 0, \quad z > 0, \quad (25)$$

$$y(0) = 0. \quad (26)$$

Equation (2) has two real roots, one positive and one negative; we denote the negative turning point by z_1 , the positive one by z_2 .

The characteristic values of λ are given approximately by the roots of the equation

$$k \int_{z_1}^{z_2} (y + \lambda)^{\frac{1}{2}} dz = (n + \frac{1}{2})\pi, \quad n = 0, 1, 2, \dots, \quad (27)$$

which expresses the condition of quantization in the older quantum theory. The arguments of Birkhoff and Kemble show that the approximation is good even for the smaller values of n . Approximate formulas for u are, for values of z not too near to z_1 or z_2 :

$$u \cong (-)^n \cdot C |y + \lambda|^{-\frac{1}{2}} \times \exp \left\{ -k \int_z^{z_1} |y + \lambda|^{\frac{1}{2}} dz \right\}, \quad z < z_1, \quad (28)$$

$$u \cong (-)^n \cdot 2C (y + \lambda)^{-\frac{1}{2}} \times \cos \left\{ k \int_{z_1}^z (y + \lambda)^{\frac{1}{2}} dz - \frac{\pi}{4} \right\}, \quad z_1 < z < z_2, \quad (29)$$

$$u \cong 2C (y + \lambda)^{-\frac{1}{2}} \times \cos \left\{ k \int_z^{z_2} (y + \lambda)^{\frac{1}{2}} dz - \frac{\pi}{4} \right\}, \quad z_1 < z < z_2, \quad (30)$$

$$u \cong C |y + \lambda|^{-\frac{1}{2}} \times \exp \left\{ -k \int_z^{z_2} |y + \lambda|^{\frac{1}{2}} dz \right\}, \quad z > z_2. \quad (31)$$

Here all fractional powers are to have real positive values. The combinations (28)–(29) and (30)–(31) represent applications of the first con-

nection formula, Eq. (16). The requirement that the alternative formulas (29) and (30) give the same values for u provides the usual derivation of Eq. (27).

The normalization constant C is to be chosen such that the values given by Eqs. (28)–(31) agree as well as possible with the values of the exact wave function u , which is normalized so that

$$\int_{-\infty}^{\infty} (u(z))^2 dz = 1. \quad (32)$$

The obvious way to do this is to consider the result of substituting the approximate formulas (28)–(31) into Eq. (32). Since these formulas fail completely near the turning points, such an argument can have meaning only when the contributions to the integral from the regions near the turning points form only a small part of the total. Thus the argument will be valid only for large n , and can be carried out with this in mind. Accordingly, the contributions from the exponential “tails” described by Eqs. (28) and (31) are neglected. Also, since the trigonometric function has numerous nodes, the square of the cosine can be replaced approximately by the mean value $\frac{1}{2}$. This is the usual way¹⁷ of deriving the formula

$$2C^2 \int_{z_1}^{z_2} (y + \lambda)^{-\frac{1}{2}} dz \cong 1. \quad (33)$$

The approximate value of C will now be obtained from Eq. (32) by a new method, which makes no use of the formulas (28)–(31) except in the neighborhood of the point $z = 0$. Equations (29) and (30) are used to give approximate values of $u(0)$ and $u'(0)$. Taking account of Eqs. (25) and (26) we get from Eq. (29)

$$u(0) \cong (-)^n 2C \lambda^{-\frac{1}{2}} \cos \left\{ k \int_{z_1}^0 (y + \lambda)^{\frac{1}{2}} dz - \frac{\pi}{4} \right\}, \quad (34)$$

$$u'(0) \cong (-)^{n+1} 2C k \lambda^{\frac{1}{2}} \times \sin \left\{ k \int_{z_1}^0 (y + \lambda)^{\frac{1}{2}} dz - \frac{\pi}{4} \right\}; \quad (35)$$

¹⁶ The notation of wave mechanics is obtained from that used here by replacing k^2 by $2m/\hbar^2$, $y(z)$ by $-V(z)$, and λ by E .

¹⁷ Cf., e.g., W. Pauli, *Handbuch der Physik* (Verlagbuchhandlung, Julius Springer, Berlin, or Edwards Bros., Ann Arbor), Vol. 24/1, p. 173.

and from Eq. (30)

$$u(0) \cong 2C\lambda^{-\frac{1}{2}} \cos \left\{ k \int_0^{z_2} (y+\lambda)^{\frac{1}{2}} dz - \frac{\pi}{4} \right\}, \quad (36)$$

$$u'(0) \cong 2Ck\lambda^{\frac{1}{2}} \sin \left\{ k \int_0^{z_2} (y+\lambda)^{\frac{1}{2}} dz - \frac{\pi}{4} \right\}. \quad (37)$$

Since the phase-integral formulas are used only to give an approximation to the shape of the function u at the point $z=0$, which is ordinarily about as remote from z_1 and z_2 as any part of the classical range of vibration, the result will be established to a fair degree of approximation even for the small values of n .

We proceed with the derivation. The characteristic function $u(z; \lambda)$ satisfies the equation

$$(d^2/dz^2)u(z; \lambda) + k^2(y+\lambda)u(z; \lambda) = 0, \quad (38)$$

where λ is a characteristic value, and vanishes at $\pm \infty$. Let $u_1(z; \lambda')$ be the solution of

$$(d^2/dz^2)u_1(z; \lambda') + k^2(y+\lambda')u_1(z; \lambda') = 0 \quad (39)$$

which vanishes at $-\infty$ and reduces to $u(z; \lambda)$ for $\lambda' = \lambda$; when λ' is not a characteristic value $u_1(z; \lambda')$ does not vanish at $+\infty$. Similarly, let $u_2(z; \lambda')$ satisfy

$$(d^2/dz^2)u_2(z; \lambda') + k^2(y+\lambda')u_2(z; \lambda') = 0, \quad (40)$$

vanish at $+\infty$, and reduce to $u(z; \lambda)$ for $\lambda' = \lambda$. Now multiply Eq. (38) by $u_1(z; \lambda')$ and Eq. (39) by $-u(z; \lambda)$ and add:

$$(d/dz) \{ u_1(z; \lambda')u'(z; \lambda) - u_1'(z; \lambda')u(z; \lambda) \} + k^2(\lambda - \lambda')u_1(z; \lambda')u(z; \lambda) = 0. \quad (41)$$

We now integrate from $-\infty$ to 0, and insert two terms whose sum is zero:

$$\begin{aligned} & [u_1(0; \lambda') - u_1(0; \lambda)]u'(0; \lambda) \\ & - [u_1'(0; \lambda') - u_1'(0; \lambda)]u(0; \lambda) \\ & + k^2(\lambda - \lambda') \int_{-\infty}^0 u_1(z; \lambda')u(z; \lambda) dz = 0. \end{aligned} \quad (42)$$

In writing the added terms use was made of the fact that for $\lambda' = \lambda$, $u_1(z)$ reduces to $u(z)$. Keeping this same fact in mind, we see that on transposing Eq. (42), dividing by $\lambda' - \lambda$, and taking the limit $\lambda' \rightarrow \lambda$ we obtain

$$\int_{-\infty}^0 [u(z; \lambda)]^2 dz = k^{-2} \left\{ u_1' \frac{\partial u_1}{\partial \lambda'} - u_1 \frac{\partial u_1'}{\partial \lambda'} \right\}_{\lambda'=\lambda, z=0}. \quad (43)$$

By applying a similar procedure to Eqs. (38) and (40), we find that

$$\int_0^{\infty} [u(z; \lambda)]^2 dz = k^{-2} \left\{ u_2 \frac{\partial u_2'}{\partial \lambda'} - u_2' \frac{\partial u_2}{\partial \lambda'} \right\}_{\lambda'=\lambda, z=0}. \quad (44)$$

All that remains to be done is to evaluate the right-hand members of Eqs. (43) and (44) by means of suitable phase-integral approximations, add, and equate to unity. Since $u_1(z; \lambda')$ vanishes at $-\infty$, it is represented by Eqs. (28) and (29), with λ replaced by λ' , and for $z=0$ by Eqs. (34) and (35), with λ replaced by λ' . For u_2 , on the other hand, Eqs. (36) and (37) must be used. The calculations give

$$\begin{aligned} \int_{-\infty}^0 [u(z; \lambda)]^2 dz & \cong 2C^2 \int_{z_1}^0 (y+\lambda)^{-\frac{1}{2}} dz \\ & + (C^2/k\lambda) \sin 2 \left\{ k \int_{z_1}^0 (y+\lambda)^{\frac{1}{2}} dz - \frac{\pi}{4} \right\}, \end{aligned} \quad (45)$$

$$\begin{aligned} \int_0^{\infty} [u(z; \lambda)]^2 dz & \cong 2C^2 \int_0^{z_2} (y+\lambda)^{-\frac{1}{2}} dz \\ & + (C^2/k\lambda) \sin 2 \left\{ k \int_0^{z_2} (y+\lambda)^{\frac{1}{2}} dz - \frac{\pi}{4} \right\}. \end{aligned} \quad (46)$$

The last terms in these two equations come from applying the operator $\partial/\partial\lambda$ to the factors $\lambda^{\pm\frac{1}{2}}$ in Eqs. (34)–(37). By Eq. (27), these last terms are equal in magnitude and opposite in sign. Accordingly, Eqs. (32), (45), and (46) yield the result

$$2C^2 \int_{z_1}^{z_2} (y+\lambda)^{-\frac{1}{2}} dz \cong 1. \quad (47)$$

This is identical in appearance with the usual formula, Eq. (33). From the manner of its derivation, however, it follows that (47) holds not merely asymptotically, but to the same degree of accuracy to which the phase-integral formulas approximate the shape of the function u in the neighborhood of the point $z=0$.

Comparison of Exact and Approximate Normalized Functions for the Harmonic Oscillator

A numerical comparison of exact and approximate normalized wave functions can most readily

be made in the case of the harmonic oscillator. If the formulas by which the approximate values are computed had, in the general anharmonic case, asymptotic validity only, good agreement for the lower quantum states of the harmonic oscillator would have to be regarded as in all probability fortuitous, and without much significance for anharmonic cases. Since, however, special arguments have now been found for the better-than-asymptotic validity of both the approximate formula for the characteristic values and that for the normalization factors, the results can claim to be roughly representative of the accuracy to be expected in more general cases. There are still, of course, special cases in which the accuracy could be much worse. One is that in which the point at which $y'(z)=0$ is much nearer one turning point than the other, and hence falls in a region in which the phase-integral approximation to u is very poor for the lower quantum states. Another is that in which the slope of the potential-energy curve is very large in the neighborhood of one or both turning points: the complete failure of phase-integral methods for rectangular potential curves is notorious. Apart from such abnormal cases, however, accuracy of the order of that found below for the harmonic case is to be expected.

For convenience in writing formulas we may suppose the units of length and of energy so chosen that for the harmonic oscillator we have

$$k=1, \quad y(z) = -z^2. \quad (48)$$

Both the exact and phase-integral solutions (cf. Eq. (27)) now give for the characteristic values

$$\lambda = 2n + 1, \quad n = 0, 1, 2, \dots \quad (49)$$

The exact solutions are

$$u_n(z) = (2^n n!)^{-\frac{1}{2}} \pi^{-\frac{1}{2}} H_n(z) e^{-z^2/2} \quad (50)$$

From the properties¹⁸ of the Hermite polynomials we get

$$u_n(0) = \begin{cases} (-1)^{n/2} (n!/2^n)^{\frac{1}{2}} [(n/2)!]^{-1} \pi^{-\frac{1}{2}}, & n \text{ even,} \\ 0, & n \text{ odd;} \end{cases} \quad (51)$$

¹⁸ Courant-Hilbert, *Mathematische Physik* (Verlagsbuchhandlung, Julius Springer, Berlin, or Interscience Publishers, New York), second edition, Vol. I, pp. 78, 79.

$$u'_n(0) = \begin{cases} 0, & n \text{ even,} \\ (-1)^{(n-1)/2} 2(n!/2^n)^{\frac{1}{2}} \\ \quad \times [((n-1)/2)!]^{-1} \pi^{-\frac{1}{2}}, & n \text{ odd;} \end{cases} \quad (52)$$

$$u_n(z) = (2^n/n!)^{\frac{1}{2}} \pi^{-\frac{1}{2}} z^n e^{-z^2/2} \\ \times \left\{ 1 - \frac{n(n-1)}{4} z^{-2} + \dots \right\}, \quad |z| \gg 1. \quad (53)$$

Here the expression in curly brackets is a polynomial.

For the normalization factor of the approximate solutions, Eqs. (47) and (48) give

$$C = (2\pi)^{-\frac{1}{2}}, \quad (54)$$

independent of n . From Eqs. (27), (34)–(37), (48), and (49) we then have

$$u_n(0) \cong u_{n \text{ approx}}(0) \\ = \begin{cases} (-1)^{n/2} (2/\pi)^{\frac{1}{2}} (2n+1)^{-\frac{1}{2}}, & n \text{ even,} \\ 0, & n \text{ odd;} \end{cases} \quad (55)$$

$$u'_n(0) \cong u'_{n \text{ approx}}(0) \\ = \begin{cases} 0, & n \text{ even,} \\ (-1)^{(n-1)/2} (2/\pi)^{\frac{1}{2}} (2n+1)^{\frac{1}{2}}, & n \text{ odd.} \end{cases} \quad (56)$$

From Eqs. (31), (48), (49) we have

$$u_n(z) \cong u_{n \text{ approx}}(z) = (2\pi)^{-\frac{1}{2}} \\ \times (z^2 - 2n - 1)^{-\frac{1}{2}} e^{-W}, \quad z > z_1 = (2n+1)^{\frac{1}{2}}, \quad (57)$$

where

$$W = \int_{(2n+1)^{\frac{1}{2}}}^z (z^2 - 2n - 1)^{\frac{1}{2}} dz. \quad (58)$$

With the notations

$$y = (2n+1)^{-\frac{1}{2}} z = \cosh \xi, \quad (59)$$

we can write

$$W = (n + \frac{1}{2}) \{ y(y^2 - 1)^{\frac{1}{2}} - \log(y + (y^2 - 1)^{\frac{1}{2}}) \} \\ = (n + \frac{1}{2}) \{ y(y^2 - 1)^{\frac{1}{2}} - \cosh^{-1} y \}, \quad (60)$$

or

$$W = \frac{1}{4} (2n+1) (\sinh 2\xi - 2\xi). \quad (61)$$

By means of expansions in descending powers of z , we can obtain from Eqs. (57)–(60) the result

$$u_{n \text{ approx}}(z) = (2\pi)^{-\frac{1}{2}} [4e/(2n+1)]^{n/2 + \frac{1}{2}} z^n e^{-z^2/2} \\ \left\{ 1 - \frac{(2n+1)(2n-3)}{16} z^{-2} + \dots \right\}, \\ z > (2n+1)^{\frac{1}{2}}. \quad (62)$$

The expression in brackets is a power series which converges for $z > (2n+1)^{1/2}$; the convergence is rapid only for $z > 2n+1$.

The comparison of exact and approximate values of $u(0)$ and $u'(0)$, as computed from

$$\frac{u_n \text{ approx}(0)}{u_n(0)} = \left(\frac{2}{\pi}\right)^{1/2} \left\{ \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (n-2) \cdot (n-2) \cdot n \cdot n}{1 \cdot 3 \cdot 5 \cdot 5 \cdots (n-3) \cdot (n-1) \cdot (n-1) \cdot (n+\frac{1}{2})} \right\}^{1/2} \quad (n \text{ even}) \quad (63)$$

and

$$\frac{u'_n \text{ approx}(0)}{u'_n(0)} = \left(\frac{2}{\pi}\right)^{1/2} \left\{ \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (n-1)(n-1)(n+\frac{1}{2})}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (n-2) \cdot n \cdot n} \right\}^{1/2} \quad (n \text{ odd}) \quad (64)$$

both converge to unity in virtue of Wallis' infinite product formula,

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \quad (65)$$

The convergence is much more rapid than that of Wallis' product, not only because of the fourth root, but especially because the modifications of final factors in Eqs. (63) and (64) are such as to remove the errors of order $1/n$ and leave only those of order $1/n^2$.

For sufficiently large values of z we can compare the exact and approximate solutions by considering the factors outside the curly brackets in Eqs. (53) and (62). This comparison is shown in Table II. Here again the agreement is surprisingly good and improves rapidly with increasing n . The ratio

$$\lim_{z \rightarrow \infty} \left(\frac{u_n \text{ approx}(z)}{u_n(z)} \right) = (n!)^{1/2} \{ (2\pi)^{1/2} [(n+\frac{1}{2})/e]^{n+1/2} \}^{-1/2} \quad (66)$$

approaches unity as $n \rightarrow \infty$, by Stirling's formula.

TABLE I. Comparison of values of $u_n(0)$ and $u_n \text{ approx}(0)$ and their derivatives.

n	$u_n(0)$	$u_n \text{ approx}(0)$	$\frac{u_{\text{approx}}}{u}$	$u'_n(0)$	$u'_n \text{ approx}(0)$	$\frac{u'_{\text{approx}}}{u'}$
0	0.7511	0.7979	1.0623	0	0	—
1	0	0	—	1.0622	1.0501	0.9885
2	0.5311	0.5336	1.0046	0	0	—
3	0	0	—	1.3010	1.2978	0.9976
4	0.4600	0.4607	1.0015	0	0	—

Eqs. (51), (52), (55), and (56) is shown in Table I. The agreement is seen to be surprisingly good even for the smallest values of n , and improves quickly as n increases. The ratios

The values of z for which the comparison shown in Table II applies are very large, since the higher terms of the series in curly brackets can be neglected only for $z \gg n^2$. A comparison for smaller values of z is best made by purely numerical calculation from Eqs. (50) and (57). It is interesting to make the comparison at fixed values of the integral W . Equation (61) is easily solved numerically for ξ , once n and W are given, and z is then given by Eq. (59). Values of z for various values of W are given in Table III,¹⁹ and the values of u and u_{approx} are shown in Table IV. The percentage difference between u_{approx} and u is tabulated in Table V.

Inspection of Table V shows that the accuracy of the approximation improves with increasing n only for large values of W . For smaller values of W the amount of error depends principally on the value of W , not that of n . The approximate formulas for u are asymptotic formulas which give the shape of u , as distinct from normalization, for large W . It is interesting, however, to

TABLE II. Comparison of $u_n(z)$ and $u_n \text{ approx}(z)$ for very large values of z .

n	$u_n(z)/(z^n e^{-z^2/2})$	$u_n \text{ approx}(z)/(z^n e^{-z^2/2})$	$\frac{u_{\text{approx}}}{u}$
0	0.7511	0.7244	0.9645
1	1.0622	1.0479	0.9866
2	1.0622	1.0535	0.9918
3	0.8673	0.8622	0.9941
4	0.6133	0.6103	0.9952

¹⁹ Values given are rounded off to three decimal places. In calculating u to the accuracy desired, more accurate values of z are required, particularly for large values of W .

TABLE III. Values of z for various values of W .

$n \backslash W$	1	1.5	2	2.5	3	4	6	8	10	15
0	1.957	2.229	2.464	2.676	2.869	3.217	3.811	4.318	4.770	5.738
1	2.560	2.804	3.018	3.212	3.390	3.712	4.269	4.749	5.179	6.110
2	3.005	3.235	3.437	3.621	3.790	4.098	4.632	5.095	5.511	6.415
3	3.379	3.599	3.792	3.968	4.132	4.429	4.946	5.396	5.801	6.686
4	3.705	3.918	4.106	4.276	4.435	4.724	5.228	5.668	6.065	6.932

TABLE IV. Values of u and u_{approx} for various values of W .

$n \backslash W$	1		1.5		2		2.5		3	
	u	u_{approx}	u	u_{approx}	u	u_{approx}	u	u_{approx}	u	u_{approx}
0	0.1106	0.1132	0.0627	0.0631	0.0361	0.0360	0.02094	0.02079	0.01225	0.01211
1	0.1026	0.1068	0.0585	0.0599	0.0337	0.0343	0.01965	0.01992	0.01152	0.01164
2	0.0991	0.1043	0.0564	0.0582	0.0327	0.0334	0.01905	0.01940	0.01120	0.01135
3	0.0964	0.1012	0.0550	0.0570	0.0319	0.0328	0.01867	0.01904	0.01096	0.01115
4	0.0949	0.0995	0.0542	0.0561	0.0314	0.0323	0.01837	0.01876	0.01080	0.01099

$n \backslash W$	4		6		8		10		15	
	u	u_{approx}	u	u_{approx}	u	u_{approx}	u	u_{approx}	u	u_{approx}
0	0.00426	0.00418	0.0 ⁸ 528	0.0 ⁸ 516	0.0 ⁶ 670	0.0 ⁶ 653	0.0 ⁸ 862	0.0 ⁸ 839	0.0 ⁷ 531	0.0 ⁷ 513
1	0.00401	0.00403	0.0 ⁸ 501	0.0 ⁸ 501	0.0 ⁶ 639	0.0 ⁶ 636	0.0 ⁸ 824	0.0 ⁸ 820	0.0 ⁷ 508	0.0 ⁷ 504
2	0.00391	0.00394	0.0 ⁸ 489	0.0 ⁸ 491	0.0 ⁶ 625	0.0 ⁶ 626	0.0 ⁸ 807	0.0 ⁸ 807	0.0 ⁷ 499	0.0 ⁷ 498
3	0.00383	0.00389	0.0 ⁸ 481	0.0 ⁸ 484	0.0 ⁶ 615	0.0 ⁶ 617	0.0 ⁸ 796	0.0 ⁸ 797	0.0 ⁷ 493	0.0 ⁷ 493
4	0.00377	0.00382	0.0 ⁸ 475	0.0 ⁸ 478	0.0 ⁶ 608	0.0 ⁶ 610	0.0 ⁸ 787	0.0 ⁸ 789	0.0 ⁷ 488	0.0 ⁷ 488

TABLE V. Percent error, $100(u_{\text{approx}} - u)/u$.

$n \backslash W$	1	1.5	2	2.5	3	4	6	8	10	15	∞
0	2.4	0.7	-0.1	-0.7	-1.1	-1.8	-2.3	-2.57	-2.74	-3.33	-3.55
1	4.1	2.4	1.8	1.5	1.0	0.4	-0.1	-0.36	-0.55	-0.79	-1.34
2	5.2	3.2	2.3	1.8	1.3	0.9	0.5	0.14	-0.05	-0.28	-0.82
3	5.0	3.6	2.6	2.0	1.7	1.1	0.7	0.37	0.19	-0.04	-0.59
4	4.8	3.5	2.6	2.1	1.8	1.5	0.6	0.46	0.29	0.00	-0.48

note that even for $W=1$ the errors are only a few percent. This illustrates another aspect of the remarkable accuracy of phase-integral methods, namely that they give the shape of the function fairly well, even rather near a turning point. This is a fact of experience, and no clear-cut special argument seems to be available to justify it. In some applications, of course, such as the derivation of Eqs. (21) and (22) by Gamow's method, the calculation depends on the assumption that W is large in other ways besides the approximation of the function u . Very accurate approximation of values of u in the neighborhood of a turning point can be obtained by a

modification of the phase-integral treatment given by Langer.²⁰

As remarked earlier, the sort of accuracy found here for the case of the harmonic oscillator is also to be expected in more general cases, provided certain exceptional circumstances—extreme asymmetry of the potential curve and/or extreme steepness of potential gradient—are not encountered.

²⁰ See reference 10. Numerical application of Langer's method is most readily accomplished by means of tables of Airy integrals. Tables for real argument are being published by the British Association Mathematical Tables project. For complex argument, use may be made of *Tables of Modified Hankel Functions and their Derivatives* (Harvard University Press, Cambridge, 1945).