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On the Scattering and Absorption of Particles by Atomic Nuclei

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Expressions very similar to the "Breit-Wigner" formulas are derived for the scattering and absorption cross section of nuclear particles, e.g. neutrons, protons, etc. The resonance energies, particle widths and other characteristic features can be expressed in terms of the logarithmic derivative of the particle wave function at the nuclear surface. Simple assumptions about the behavior of the incident particle inside the nucleus are used to find approximate expressions for (1) the particle widths in terms of the level spacing; (2) the neutron scattering cross section near and between resonances; (3) the total cross section and the absorption cross section averaged over neighboring levels for neutron energies up to 1 Mev. These results are then compared with the experimental data and are found to be in qualitative agreement.

1. INTRODUCTION

THE scattering and absorption of neutrons, protons, or other particles by nuclei is currently described by a picture that makes use of the existence of certain resonance states of the compound nucleus. The cross section of these processes, and its dependence on the energy of the incident particles is given by the so-called "Breit-Wigner" formula[1,2] which is built in close analogy to the corresponding expressions in optical dispersion. It has the following form for an incident particle of an angular momentum $l\hbar$, if the spins of the particle and of the nucleus are disregarded:[3]

$$\sigma_{sc}=\frac{4\pi}{k^2}(2l+1)\left|\sum_r\frac{\frac{1}{2}\Gamma_n^{(r)}}{E-E_r+i\Gamma^{(r)}/2}+p_{sc}\right|^2, \quad (1)$$

$$\sigma_{abs}=\frac{\pi}{k^2}(2l+1)\left|\sum_r\frac{(\Gamma_n^{(r)}\Gamma_a^{(r)})^{\frac{1}{2}}e^{i\varphi_r}}{E-E_r+i\Gamma^{(r)}/2}\right|^2. \quad (2)$$

σ_{sc} is the cross section for elastic scattering, σ_{abs} is the cross section for absorption or any other process that changes the initial state of the bombarded nucleus. It includes radiative capture, fission, and inelastic scattering. Here k is the wave number of the incident particle, E is its energy, E_r are the resonance energy values for the compound nucleus, $\Gamma^{(r)}$ is the total width of the resonance, $\Gamma_n^{(r)}$ is the neutron width, corresponding to the reemission of the neutron with its original energy, $\Gamma_a^{(r)}$ is the absorption width and φ_r is a phase. $\Gamma_a^{(r)}$ contains the radiation width and also the width for any other process that does not lead to the elastic reemission of the incident particles, such as fission or inelastic scattering. The sum is taken over all resonance states of the compound nucleus. The

[1] G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936).
[2] H. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).
[3] Consideration of the spins of the nucleus and the particle introduces a factor $(2J+1)/[(2i+1)(2s+1)(2l+1)]=\xi(J,i,s,l)$ to the expressions (1) and (2), where i is the spin of the bombarded nucleus, s of the particle and J of the compound nucleus; J can take on the values $J=|i+j|$, $|i+j-1|$, $\cdots|i-j|$, and j the values $|l+s|$, $|l+s-1|$, $\cdots|l-s|$. It can be shown that $\Sigma_J\,\xi(J,i,s,l)=1$, if the sum taken over all possible values of J. All expressions, in which averages are taken over many resonance levels, are therefore identical for the case $s=0$, $i=0$ and for the general case.

amplitude p_{sc}, contained in the formula for σ_{sc}, describes the so-called "potential scattering" which is not caused by resonance effects.

The derivation of these expressions and the theoretical determination of the constants appearing in them have been given by several authors.[2,4–6] The difficulty lies in the fact that the strong interaction between the impinging particles and the nucleus prevents a perturbation calculation in analogy with the dispersion theory of light. Especially the definition and the properties of the levels far off resonance have led to great complications and the theories were not able to give an unambiguous result. These levels contribute, mainly, to the background between resonances and are of special importance for the scattering cross section, since they are responsible, partly or fully, for the potential scattering.

We propose in this paper a different way of approach to a derivation of expressions for the absorption and scattering cross sections. The result is, of course, very similar to (1) and (2), but it does not contain a summation over far away levels. The cross sections are determined by a function $f(E)$, which is equal to the ratio of the derivative to the value of the eigenfunction of the incoming particle at the surface of the nucleus. It is possible to express some characteristic quantities, as the neutron width $\Gamma_n^{(r)}$ and the resonance energies E_r, in terms of some general properties of this function f. In order to obtain numerical values for the cross sections, certain simple assumptions will be made as to the general behavior of the function $f(E)$. These assumptions cannot be exact and the results derived from it must be interpreted as qualitative indications only.

The fundamental idea of this derivation can be expressed in qualitative terms in the following way. Let us consider an incident beam of neutrons of low energy. It is represented outside the nucleus by a wave of a wave number k. Near the nuclear surface this wave is joined smoothly with equal value and derivative to the wave function inside the nucleus, which is the solution of a complicated many-body problem. We expect that the neutron will have, in the average, a very high kinetic energy inside the nucleus. The dependence of the wave function inside on the coordinate of the incoming neutron will therefore be qualitatively represented by a wave with a high wave number $K \gg k$. We thus have to join, at the nuclear surface, a wave of low wave number k with a wave of high wave number K. In general, this can only be done if the amplitude A inside is very much smaller than the amplitude outside which we conveniently normalize to unity. A is of the order k/K. In the exceptional cases, however, when the derivative of the inside wave function is near zero at the surface, the two waves can be joined with about equal amplitude: $A \sim 1$. (See Fig. 1.) There are certain narrow intervals in the energy for which the wave function inside will have this exceptional property. These are the excitation energies of the compound nucleus for which the neutron can get into the nucleus, since, for these values, the amplitude is large inside. These energies are the observed resonances, and the widths of the intervals are the resonance widths. If we assume that the phase of the inside wave function at the nuclear surface changes smoothly with the energy, we will expect that the energy intervals, in which the derivative is small enough to give rise to large A, are proportional to the distance D between resonances: $\Gamma = \alpha D$ where α is a small number of the order k/K. Thus the width of the resonance is expected to be proportional to D. (This consideration does not consider any absorption inside the nucleus so that this conclusion applies to the neutron width only.) This simple picture also gives some indication of the scattering cross section. The outside wave function assumes a very small value for all energies

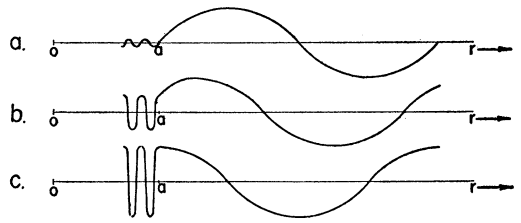

FIG. 1. Schematic representation of neutron wave functions at the nuclear surface. The wave functions are indicated as functions of the distance r from the center of the nucleus. $r=a$ is the nuclear radius. Case a corresponds to a neutron energy between resonances, Case b is near resonance, Case c is in resonance.

[4] P. L. Kapur and R. Peierls, Proc. Roy. Soc. **A166**, 277 (1938).
[5] A. J. Siegert, Phys. Rev. **56**, 750 (1939).
[6] G. Breit, Phys. Rev. **69**, 472 (1946).

at the nuclear surface except the ones near the resonances. Therefore, the situation is similar to the scattering of an impenetrable sphere of the same size as the nucleus, since in the latter case the wave function vanishes at the surface. We thus expect, in addition to a resonance scattering, a potential scattering between resonances whose cross section is similar to the one of an impenetrable sphere.

2. THE CASE OF NEUTRONS WITH $l=0$

(a) Derivation of the General Expressions

The wave function ψ of the incident particle outside the nucleus is a linear combination of an incoming and an outgoing wave:

$$\varphi = r\psi = e^{-ikr} + \eta e^{ikr}, \tag{3}$$

where k is the wave number of the incoming neutron. If there is no absorption, the absolute square of η is unity; in case of absorption, $|\eta|^2<1$. The absorption cross section is given by

$$\sigma_{\text{abs}} = (\pi/k^2)(1-|\eta|^2), \tag{4}$$

which is easily understood, if one remembers that π/k^2 is the maximum possible absorption cross section which is reached when there is no outgoing wave e^{+ikr} at all.

The scattering cross section is given by

$$\sigma_{\text{sc}} = (\pi/k^2)|1+\eta|^2. \tag{5}$$

This can be seen by observing that, in a plane wave the $l=0$ part of the wave function has the form $r\psi = e^{-ikr} - e^{+ikr}$; this differs from (3) by $(1+\eta)e^{+ikr}$ which, therefore is the scattered wave.

We now express the two cross sections in terms of the logarithmic derivative of φ at the nuclear radius a. We define a function $f(E)$ by

$$f(E) = a[(d\varphi/dr)/\varphi]_{r=a}. \tag{6}$$

The value of η is connected with f by

$$\eta = e^{-2ix}\left[\frac{x-if}{x+if}\right], \tag{7}$$

where $x=ka$. The nuclear radius a is the smallest distance from the center of the nucleus at which the incoming particle is no longer under the influence of the nuclear forces.

We obtain an expression for the cross sections by inserting (7) into (4) and (5) and by putting $f=f_0-ih$, where f_0 and h are real functions; we note that h is necessarily non-negative in order to insure $|\eta|^2\leqslant 1$.

$$\sigma_{\text{abs}} = \frac{4\pi x h}{k^2}\left|\frac{1}{x+if}\right|^2 = \frac{4\pi}{k^2}\frac{xh}{(x+h)^2+f_0^2}, \tag{8}$$

$$\sigma_{\text{sc}} = \frac{4\pi}{k^2}\left|\frac{x\cos x - f\sin x}{x+if}\right|^2 = \frac{4\pi}{k^2}\left|\frac{x}{i(x+h)-f_0}+e^{ix}\sin x\right|^2. \tag{9}$$

It is seen that resonances occur always when $f_0=0$. We therefore call the values E for which $f_0(E)=0$, the resonance energies E_r of the nucleus. In the neighborhood of the resonances we put

$$f_0(E) = (E-E_r)(df_0/dE)_{E=E_r} \tag{10}$$

and introduce the following abbreviations (the negative sign in (11) and (12) will be explained later and is caused by the fact that df_0/dE can be shown to be negative):

$$\Gamma_n^{(r)} = -2x/(df_0/dE)_{E=E_r}, \tag{11}$$

$$\Gamma_a^{(r)} = -2h/(df_0/dE)_{E=E_r}. \tag{12}$$

Then

$$\sigma_{\text{abs}} = \frac{\pi}{k^2}\frac{\Gamma_a^{(r)}\Gamma_n^{(r)}}{(E-E_r)^2+(\Gamma_n^{(r)}+\Gamma_a^{(r)})^2/4}, \quad E \text{ near } E_r, \tag{13}$$

$$\sigma_{\text{sc}} = \frac{4\pi}{k^2}\left|\frac{\frac{1}{2}\Gamma_n^{(r)}}{E-E_r+(i/2)(\Gamma_n^{(r)}+\Gamma_a^{(r)})}+e^{ix}\sin x\right|^2, \quad E \text{ near } E_r. \tag{14}$$

These expressions show the familiar forms (1) and (2) although they contain only contributions from one level. They are valid only as long as (10) is a good approximation for $f_0(E)$. Formulae (9) and (14) contain the characteristic "potential scattering" term which, if it were present without the other term, would give rise to a scattering cross section:

$$\sigma_{\text{sc}} = (4\pi/k^2)\sin^2 ka.$$

This is equivalent to the scattering of an impene-

trable sphere of radius a. It must be said, however, that the split of the scattering amplitude into a "potential scattering" term: $e^{ix}\sin x$ and a "resonance term":

$$\tfrac{1}{2}\Gamma_n^{(r)}/(E-E_r+i(\Gamma_n^{(r)}+\Gamma_a^{(r)})/2)$$

is so far purely formal since no use was made of the fact that a is the nuclear radius. Actually a could have been any radius r for which the wave function of the incoming particle is given by (3); this means a could assume any value larger than or equal to the nuclear radius. It is evident that any change in value of a entails a corresponding change in $f(E)$ which makes the values (8) and (9) of the cross sections invariant to the choice of a. It will be shown in the next section, however, that $f(E)$ is expected to have certain simple properties which depend only on the nuclear structure inside the nucleus, in case a is chosen to be the nuclear radius. Specifically, relation (10) is then supposed to be valid over an energy region much larger than the line width $\Gamma_n^{(r)}+\Gamma_a^{(r)}$. In this case expression (14) is valid also for energies at which the resonance term becomes smaller than the potential scattering term, so that the potential scattering assumes real significance.

(b) Definition and Determination of the Function $f(E)$

It is possible to make qualitative statements as to the behavior of the function $f(E)$, if the wave-length $\lambda\!\!\!^{-}$ of the wave function outside of the nucleus is much longer than the average distance d between particles inside the nucleus:

$$\lambda\!\!\!^{-}=1/k\gg d. \tag{15}$$

The actual value of $\lambda\!\!\!^{-}$ is then of no importance for the shape of the wave functions inside since $\lambda\!\!\!^{-}$ is very much longer than the wave-lengths inside which are of the order d. The value of $f(E)$ will depend on the energy E of the incoming particle, only by being an explicit function $f(W)$ of the excitation energy W of the compound nucleus: $W=E+B$, where B is the binding energy of the neutron to the nucleus. Expression (11) shows therefore the characteristic proportionality of $\Gamma_n^{(r)}$ with k, since f_0 or its derivative should not depend explicitly on k.

The actual determination of $f(W)$ itself can be done only by finding the solution of the Schrödinger equation for the compound nucleus corresponding to the energy W, and by calculating the logarithmic derivative of this solution with respect to r at the nuclear surface. Because of condition (15), the value of this derivative does not change appreciably outside the nucleus in distances of the order d. The function $f(W)$ is therefore well defined, in spite of the fact that the actual position of the nuclear surface is not sharply determined. The Schrödinger equation for the compound nucleus can be written in the form:

$$H_0\Psi(r_1\cdots r_A)=W\Psi(r_1\cdots r_A), \tag{16}$$

where $r_1\cdots r_A$ are the coordinates of all constituents. Since W is the energy of the compound nucleus after the neutron has entered from outside, it belongs, strictly speaking, to the continuous spectrum of the Hamiltonian H_0. We assume, for the sake of simplicity, that no other particle but the incoming neutron can be emitted at the energy W. The wave function Ψ is then completely defined by the boundary conditions

$$\Psi(r_1\cdots r_A)=0 \quad \text{for} \quad r_i=\infty, \quad i=2\cdots A,$$

if r_1 are the coordinates of the incoming neutron. We then obtain a definite value for

$$f_0(W)=a\left(\frac{\partial r_1\Psi}{\partial r_1}\Big/(r_1\Psi)\right)_{r_1=a}$$
$$=a\left(\frac{dr_1\psi}{dr_1}\Big/(r_1\psi)\right)_{r_1=a} \tag{17}$$

since, for $r_1=a$, Ψ has assumed the form of a product $\psi(r_1)\chi(r_2\cdots r_A)$, where χ is the eigenfunction of the initial (or residual) nucleus.

This function $f_0(W)$ is not yet the function $f(W)$ in which we are interested, since the damping effect of the radiative absorption has not been considered yet. It is possible to express the effect of the radiative absorption by an imaginary addition to the Hamiltonian H_0:

$$H\Psi=(H_0+H^1)\Psi, \quad H^1=-i\Gamma_a/2.$$

The term H^1 would add to the eigenvalues of H_0 an imaginary term $-i\Gamma_a/2$, which describes the

characteristic exponential time dependence:

$$|\Psi|^2 \sim |e^{-i(W-i\Gamma_a/2)t}|^2 = e^{-\Gamma_a t}.$$

The solution of H which is needed to determine $f(E)$ is given by

$$H\Psi = W\Psi$$

which also can be written in the form

$$H_0\Psi = (W+i\Gamma_a/2)\Psi.$$

Hence, if $f_0(W)$ was determined according to (17) from Eq. (16), the actual $f(W)$ will be given by

$$f(W) = f_0(W+i\Gamma_a/2) \simeq f_0(W)+i(\Gamma_a/2)(df_0/dW), \quad (17a)$$

if $\Gamma_a/2$ is very small compared to W. The imaginary part $-h$ of $f(W)$ is thus

$$h = -(\Gamma_a/2)(df_0/dW),$$

which agrees with our definition (12) of Γ_a.

The resonance value $W_r = B+E_r$ and the neutron widths $\Gamma_n^{(r)}$ are defined by:

$$f_0(W_r) = 0, \quad \Gamma_n^{(r)} = -2x/(df_0/dW)_{W=W_r}. \quad (18)$$

Since the imaginary part of $f(W)$ is necessarily non-positive in order to fulfill in (7) the condition $|\eta|^2 \leqslant 1$, we conclude from (17a)

$$df_0/dW \leqslant 0.$$

In view of the fact that the derivative of f_0 is never positive and that f_0 has zero points at $W = W_r$, it is convenient to write it in the form

$$f_0(W) = -Ka \tan z(W), \quad (19)$$

where K is a constant of the dimension of a wave number and $z(W)$ is a monotonically increasing function, which is equal to multiples of π for the resonance values W_r:

$$z(W_r) = n\pi, \quad n \text{ an integer}. \quad (20)$$

The value of K can be estimated in the following way: the wave function Ψ inside the nucleus depends on the coordinates of all constituents. We make the qualitative assumption that Ψ, in its dependence on r, has the character of a periodic function with a high wave number:

$$r\Psi \sim \cos(Kr+\delta) \quad (21)$$

from which one obtains (see (6))

$$f \doteq -Ka \tan(Ka+\delta).$$

If this is compared to (19), it seems plausible to assume that K is a wave number corresponding to the kinetic energy of a particle in the nucleus.

We would like to emphasize that this procedure is not equivalent to the use of a one-particle model as the form of (21) may perhaps suggest. Expression (19) uses only the fact that Ψ has a periodic dependence on r with a period given by K. The function z is left, so far, completely undetermined. It would have the form $z = (2mW/\hbar^2)^{\frac{1}{2}}a$ in the one-particle model.

The order of magnitude of K can be obtained in the following way: the maximum momentum P of N equal free Fermi-Dirac particles in their lowest states, if kept within a volume V, is given by

$$P = 2\pi\hbar(3N/4\pi V)^{\frac{1}{3}}.$$

If A is the number of constituents in a nucleus, we put $N = A/4$, because of the 4 types of particles in the nucleus and get:

$$P/\hbar = (2\pi/r_0)(9/64\pi^2)^{\frac{1}{3}},$$

where r_0 is the elementary radius which determines nuclear radii by $a = r_0A^{\frac{1}{3}}$. If $r_0 = 1.5\times10^{-13}$, we obtain $P/\hbar \simeq 1.0\times10^{13}$ cm^{-1} which corresponds to an energy of 20 Mev. The average kinetic energy of a neutron entering the nucleus will be of the order of this energy plus the binding energy B. The wave number K should therefore correspond to an energy of about 28 Mev which would be:

$$K \approx 1.2\times10^{13} \text{ cm}^{-1}. \quad (22)$$

The function $f(W)$ would be qualitatively determined if some assumption could be made as to the behavior of $f(W)$ between the values $n\pi$ which it reaches at W_r. It seems natural to assume that $z(W)$ increases smoothly from $n\pi$ to $(n+1)\pi$ if the energy W goes from one resonance to the next. This assumption can be expressed by

$$(dz/dW)_{W_r} = \pi/D^* \quad (23)$$

where D^* is an energy of the order of the average level distance D. D itself is, of course, a function of W. The function f_0 can then be written as function of E by means of (19) in the form:

$$f_0(E) = -Ka \tan[(\pi/D^*)(E-E_r)]. \quad (24)$$

It is evident that this expression has only very approximate validity.

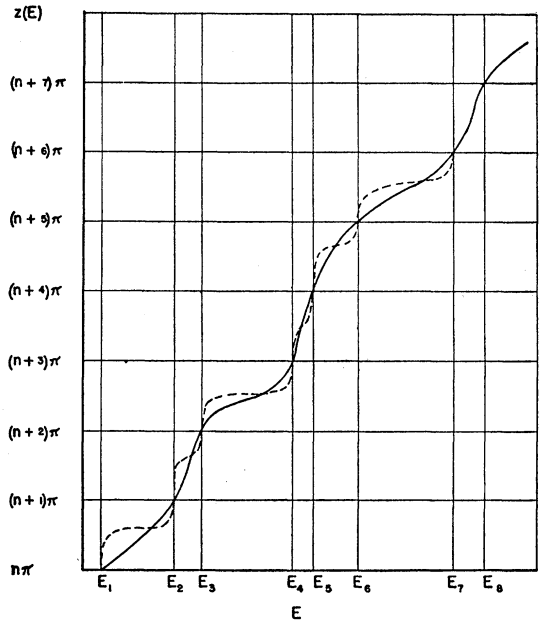

FIG. 2. Function $z(E)$ (Eq. (19)) plotted *vs.* the energy E, E_1 to E_8 are resonance energies. The solid curve gives a smooth dependence as assumed in this paper. The broken line gives an example of an anomalous dependence which would lead to low absorption cross sections.

The relation between D^* and the average level distance D depends on the detailed properties of the function $z(W)$ which, so far, is only defined by (20). If $z(W)$ is laid as smoothly as possible through its values given by (20), we obtain $D^*=D$ in case of equidistant levels. It is seen in Fig. 2 that fluctuations in the level distance tend to decrease D^*/D, since the slope of z at resonances will be closer to the slope of the straight connection with the neighbor resonance which lies closest. It seems to be a good definition of a "smooth curve" to assume that D^*, as defined in (23), is the smaller of the two distances between the resonance W_r and the two adjacent resonance values. If this definition is adopted, one can show that a random level distribution gives

$$D^*/D=\tfrac{1}{2}. \tag{25}$$

The assumption of a "smooth" function $z(W)$ is somewhat arbitrary, and may not be justified in every case. One may argue, however, that the many-body problem of the nucleus is in some way equivalent to a one-body problem with a coordinate r which has a much larger interval at its disposal than $0<r<a$, because of internal reflections and exchanges. The eigenfunction inside the nucleus has then the form $C\cos[K(r+s)]$, where s is a length much larger than a, and K the average wave number for the excitation energy W. One then expects a smooth dependence on W of the function $z=K(W)\cdot(a+s)$. One can, of course, explain any value for D^*/D by a suitable function $z(W)$. The dotted curve in Fig. 2 gives an example, which would lead to an abnormally small value.

3. THE ABSORPTION AND SCATTERING CROSS SECTION OF *s*-NEUTRONS

The assumptions made in the preceding section give rise to a simple expression for the neutron width Γ_n, which becomes according to (18), (19), and (23):[7]

$$\Gamma_n^{(r)}=\frac{2k}{K}\frac{D^*}{\pi}. \tag{26}$$

The neutron width for s-neutrons can be determined experimentally from an analysis of the neutron absorption resonances. The material available indicates that the values of Γ_n are all of the order of $10^{-3}\sqrt{E_r}$ ev where E_r is the resonance energy in electron volts. This value would give rise to $D^*\sim 10$ ev which seems to be a reasonable order of magnitude for the resonance level distance.

Let us consider the average of σ_{abs} over an energy interval ΔE which contains many levels:

$$\langle\sigma_{\text{abs}}\rangle_{\text{Av}}=\frac{1}{D}\int\frac{\pi}{k^2}\frac{\Gamma_n\Gamma_a dE}{(E-E_r)^2+(\Gamma_n+\Gamma_a)^2/4}$$
$$=\frac{2\pi}{k^2}\frac{\Gamma_n}{D}\frac{\Gamma_a}{\Gamma_n+\Gamma_a}=\frac{4\pi}{kK}\frac{D^*}{D}\frac{\Gamma_a}{\Gamma_n+\Gamma_a}. \tag{28}$$

Here the integral is taken over the neighborhood of a typical resonance E_r within the interval. It can be shown from the more general expression (8) that only the immediate neighborhood of the resonances contributes to the integral. Γ_n and Γ_a are the average values of $\Gamma_n^{(r)}$ and $\Gamma_a^{(r)}$ over the resonances included within ΔE. Expression (26) was used to eliminate Γ_n in the final expression.

For very low energies the following inequality

[7] N. Bohr and J. A. Wheeler, Phys. Rev. 56, 426 (1939).

holds: $\Gamma_a \gg \Gamma_n$. Since, for heavy nuclei ($A>100$) $\Gamma_a \sim 0.1$ ev and D^* is of the order of 10 ev, we find $\Gamma_a = \Gamma_n$ according to (26) for $E \sim 7000$ ev. We thus expect the following expression to hold:

$$\langle \sigma_{\text{abs}} \rangle_{\text{Av}} = \frac{4\pi}{kK}\frac{D^*}{D} \simeq \frac{250}{(E_{\text{ev}})^{\frac{1}{2}}} 10^{-24}\ \text{cm}^2, \quad \begin{matrix} E \ll 7000\ \text{ev} \\ A>100 \end{matrix} \quad (29)$$

if E_{ev} is the incident energy in ev and (25) is used. No measurement of $\langle \sigma_{\text{abs}} \rangle_{\text{Av}}$ in this energy region is available at present for heavy nuclei ($A>100$). It is not possible to derive a similar expression for lighter nuclei because of the larger D which would restrict the limit of validity to even lower energies. The larger D will then make it impossible to take an average over many levels within the validity of the expression. It should be noticed that the cross section (29) is an average over an energy region containing many levels and it should therefore not be applied to the thermal absorption cross section.

We now turn to the scattering cross section as a function of energy. The value of σ_{sc} for any energy E can be calculated from (9) and (24). The evaluation gives rise to a line shape of the characteristic form given in Fig. 3. This figure is calculated for the indium resonance, and a value $D^*=25$ ev was used which would give the observed neutron width[8] of 3×10^{-3} ev in expression (26). The depression at the low energy side and the high shoulder at the high energy side can also be understood as destructive and constructive interference of the "potential scattering" amplitude with the resonance term as shown in (9). This phenomenon has been also described by Bethe.[9] The extensions of these features in the energy scale are much larger than the width $(\Gamma_a+\Gamma_n)$ of the line. It is of the order $2D/(Ka)$, which is about $\frac{1}{5}$ of the level distance in case of heavy nuclei.

No measurements of σ_{sc} as a function of energy have been made so far, but the value of σ_{sc} for thermal energies or for energies near the thermal value are known for several elements. We expect these values to be near $4\pi a^2$. However, they may be above or below this value if there is a level nearby below or above the energy at which the scattering was measured. The distance of this level must be at most of the order of about a fifth of the average level distance, in order to influence the scattering cross section. The nuclear radius a which determines the potential scattering is roughly defined as the distance at which the wave number of the incoming neutron assumes values of the order K. It is thus defined with an inherent inaccuracy of $\pm 1/K$. We therefore expect the scattering cross section away from resonance to be near to $4\pi R^2$, where R is the nuclear radius determined from other sources. The deviation from $4\pi R^2$ should be of the order $8\pi R/K$ which is about 37 percent of the total for elements near Fe and 25 percent of the total at the upper end of the periodic table.

Table I shows a number of scattering cross sections measured, together with the values of $4\pi R^2$ based upon the assumption $R=1.5\times10^{-3}A^{\frac{1}{3}}$ cm. Of course, there will be fluctuations away from this average value. This table contains scattering cross sections mostly of elements which show small absorption at low energies.

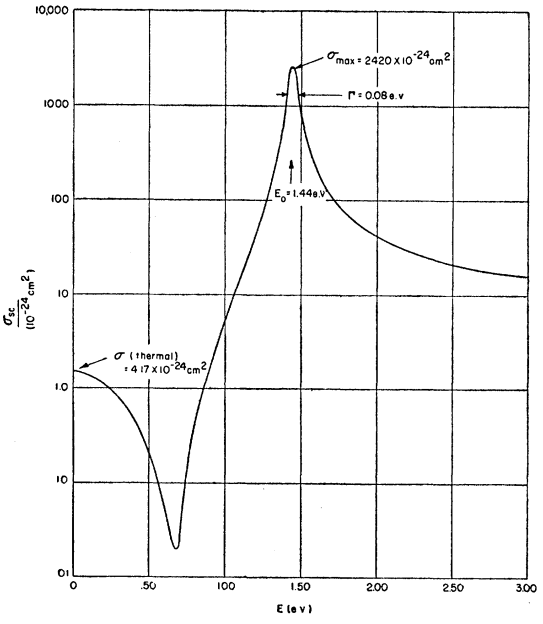

FIG. 3. The scattering cross section for neutrons as a function of neutron energy in the neighborhood of a resonance. The numerical values used for Γ and E_r are those for the indium resonance. Because of a typographical error the value of σ_{sc} for thermal energy is given wrongly in this figure. It should read: $\sigma(\text{thermal}) = 1.47\times10^{-24}$ cm^2.

[8] J. Hornbostel, H. H. Goldsmith, J. H. Manley, Phys. Rev. **58**, 18 (1940); J. Rainwater and W. Havens, unpublished; we are grateful to these authors for communicating their results before publication.

[9] H. A. Bethe, Rev. Mod. Phys. 9, 95 (1937).

This selection is attributed to experimental reasons. It follows therefore that in the case of heavier elements ($A>100$) there is no resonance nearby to interfere with the potential scattering. The cross section should therefore be $4\pi a^2$, which should not differ from $4\pi R^2$ by more than about 30 percent. The values of Table I bear out this expectation with the possible exception of Sb and Hg. In these cases, the actual value of R may be considerably different from the assumed value $R=1.5\times10^{-13}A^{1/3}$. In the case of the lighter elements ($A<100$), however, the neutron width is probably much larger than the absorption width because of the larger D, so that resonance scattering may occur in spite of weak absorption. We expect then, in some cases, to find an interference effect of resonance and potential scattering which may account for the abnormally high cross sections in Fe, Ni, Cu, Ge, Se, and Sr, and for the low value in Mn.

Finally, measurements[10] of the change in phase of the scattered wave with respect to the incident wave has been measured for several heavy nuclei for energies in or near to the thermal region. In most cases this phase change was 180°. This result is in agreement with the assumption that the potential scattering p_{sc} is the scattering by an impenetrable sphere for, in this case, the outgoing wave must be exactly out of phase with the incident wave. This phase relation will hold as long as the potential scattering term is the most important term in σ_{sc}. In the resonance region, this phase relation holds for neutron energies greater than the resonance energy. It does not hold for energies smaller than the resonance energy. This is, of course, just a very small portion of the neutron spectrum so that for most energies the phase change will be near 180°.

TABLE I. Elastic scattering cross sections σ_{sc} for thermal and epithermal neutrons.

Element	$4\pi R^2$	σ_{sc}	Energy (ev)	Reference
Ti	3.6	6	C	*A*
Mn	3.8	2.4	1	*B*
Cr	3.8	4	1	*B*
Fe	4.0	11.5	1	*B*
		11.1	1.44	*E*
Co	4.2	5	*C*	*C*
Ni	4.4	18	1	*B*
Cu	4.5	8	1	*B*
		8.3	1.44	*E*
Zn	4.6	4.2	1	*B*
		4.2	1.44	*E*
Ge	5.0	22	1	*B*
Se	5.3	10	*C*	*A*
Sr	5.5	10	*C*	*A*
Br	5.4	7	*C*	*A*
Cb	5.9	5	*C*	*A*
Mo	6.0	6.5	*C*	*A*
Ru	6.1	6	*C*	*A*
Pd	6.3	4.5	*C*	*A*
Ag	6.4	6.6	0–0.5	*B*
Cd	6.5	5.3	1	*B*
Sn	6.6	5.0	1	*B*
Sb	6.9	4.2	1	*B*
Te	7.0	5	*C*	*A*
Ba	7.5	8	*C*	*A*
Ta	9.0	7.2	1	*B*
Os	9.2	10	1	*B*
W	9.3	5.7	0.5	*B*
Pt	9.4	12	1	*B*
Tl	9.8	9.7	1	*B*
Hg	9.8	15	0–1	*B*
Pb	9.9	12.5	1	*B*
Bi	10.0	9.2	1	*B*

A M. Goldhaber and G. H. Briggs, Proc. Roy. Soc. **A162**, 127 (1937).
B J. Rainwater and W. Havens, unpublished. We thank Rainwater and Havens for communicating these results.
C N. E. Bradbury, F. Bloch, H. Tatel, and P. A. Ross, Phys. Rev. **52**, 1023 (1937).
E H. B. Hanstein, Phys. Rev. **59**, 489 (1941).
Remarks:
The fourth column gives the energy at which σ_{sc} was determined. In all references but reference *A*, the value measured is always the total cross section $\sigma_{sc}+\sigma_a$. In these cases, σ_a was either known to be much smaller than σ_{sc} or it was possible to subtract unambigously a term proportional to $1/v$ which was considered to be σ_a. *C* means neutron energies absorbable in Cd. The velocity selector measurements were taken mostly in the neighborhood of 1 ev (~1) except where other energies are quoted. In the latter cases, the nearest level was already influencing the value at 1 ev, so that it was advisable to choose a lower value.
The scattering cross section of Cd was determined as an additive term to the best fitting Breit-Wigner expression covering the resonance.

4. GENERALIZATION TO NEUTRONS $l>0$ AND TO OTHER KINDS OF INCIDENT PARTICLES (E.G., CHARGED PARTICLES)

So far, we have considered only neutrons in the s state, i.e., $l=0$. When neutrons with $l>0$, or charged particles, protons, deuterons, α-particles, etc., are considered, the radial equation for the particle for $r>a$ now will contain potential energy terms. In the case of charged particles, $l=0$, the potential energy term is Zze^2/r where Z is the nuclear charge and z is the particle charge. When $l\neq0$, then for both cases, neutron or charged particles, one must introduce an additional effective potential energy equal to $\hbar^2l(l+1)/2mr^2$. In any of these cases, the particle may be considered to be moving in the region $r>a$, in a potential $V(r)$. It is then possible to generalize the discussion of Sections 2 and 3 as follows.

Let $u(r)$ equal the product of r and the wave function of the incident particle moving in the potential $V(r)$, which asymptotically for large r

[10] E. Fermi and L. Marshall, Phys. Rev. **70**, 103 (1946).

behaves as e^{-ikr}:

$$u(r)\underset{r\to\infty}{\longrightarrow}e^{-i(kr-l\pi/2)} \quad \text{and similarly} \quad v(r)\underset{r\to\infty}{\longrightarrow}e^{i(kr-l\pi/2)}.$$

The wave function φ for the particle can then be written as in (3) as:

$$\varphi=r\psi=u+\eta v, \quad |\eta|\leqslant 1. \tag{31}$$

The absorption cross section is given by

$$\sigma_{\text{abs}}=(2l+1)(\pi/k^2)[1-|\eta|^2] \tag{32}$$

and the scattering cross section by

$$\sigma_{\text{sc}}=(2l+1)(\pi/k^2)|1+\eta|^2. \tag{33}$$

A function $f(E)$ may be defined in analogy to (7) as:

$$f(E)=a(\varphi'/\varphi)=x\frac{u'+\eta v'}{u+\eta v}, \tag{34}$$

where u' and v' denote derivatives with respect to kr at $r=a$. Solving (34) for η we obtain

$$\eta=-\frac{xu'-fu}{xv'-fv}. \tag{35}$$

It is convenient to express u and v and their derivatives in terms of their absolute value and phase:

$$\begin{aligned} u&=|v|e^{-i\delta}, \quad & v&=|v|e^{i\delta}, \\ u'&=|v'|e^{-i\delta'}, \quad & v'&=|v'|e^{i\delta'}. \end{aligned} \tag{36}$$

Note δ' and $|v'|$ are not the derivatives of δ and $|v|$. If one uses the relation

$$|vv'|\sin(\delta'-\delta)=1 \text{ for all } r \tag{37}$$

one may transform (32) and (33) into

$$\sigma_{\text{abs}}=\frac{4\pi}{k^2}(2l+1)\frac{xh/|v|^2}{(x/|v|^2+h)^2+[f_0-x|v'/v|\cos(\delta'-\delta)]^2}, \tag{38}$$

$$\sigma_{\text{sc}}=\frac{\pi}{k^2}(2l+1)\left|\frac{x/|v|^2}{i(x/|v|^2+h)-[f_0-x|v'/v|\cos(\delta'-\delta)]}+\sin\delta e^{i\delta}\right|^2. \tag{39}$$

Here $f=f_0-ih$, $|v|$, $|v'|$, δ and δ' are evaluated at $x=ka$. The resonances occur when

$$f_0=x|v'/v|\cos(\delta'-\delta). \tag{40}$$

In the neighborhood of the resonances, we expand $f_0(E)$ as in (10) and introduce the following abbreviations

$$\Gamma_n^{(r)}=-\frac{2x}{|v|^2}\left(\frac{df_0}{dE}\right)^{-1}_{E=E_r}, \tag{41}$$

$$\Gamma_a^{(r)}=-2h\left(\frac{df_0}{dE}\right)^{-1}_{E=E_r}. \tag{12}$$

We obtain

$$\sigma_{\text{abs}}=\frac{\pi}{k^2}(2l+1)\frac{\Gamma_n^{(r)}\Gamma_a^{(r)}}{(E-E_r)^2+[(\Gamma_n^{(r)}+\Gamma_a^{(r)})/2]^2}, \tag{42}$$

$$\sigma_{\text{sc}}=\frac{4\pi}{k^2}(2l+1)\times\left|\frac{\frac{1}{2}\Gamma_n^{(r)}}{(E-E_r)+i/2(\Gamma_n^{(r)}+\Gamma_a^{(r)})}+p_{\text{sc}}\right|^2, \tag{43}$$

with

$$p_{\text{sc}}=e^{i\delta}\sin\delta.$$

These expressions are almost identical with the ones obtained for neutrons with $l=0$. The term in σ_{sc}, which represents the potential scattering is changed, and the expression (41) for the neutron width contains the characteristic factor $|v|^{-2}$ which expresses the effect of the potential field outside the nucleus. This factor is the ratio $|v(\infty)/v(x)|^2$ of the intensities of an outgoing wave at infinity and at the nuclear surface. It is a small number for repulsive fields. Its appearance in the expression of the neutron width is expected, since the nuclear eigenfunction adjusts its form to the wave function of the particle at the nuclear surface, whereas the width is connected with the current of outgoing particles through a sphere around the nucleus with a very large radius. The factor $|v|^{-2}$ has also been used in earlier papers by Kapur and Peierls,[4] Weisskopf and Ewing,[11] and Bethe.[9]

There is a limitation to the validity of these considerations, which corresponds to the condition (15) of the case without barrier. Here the

[11] V. F. Weisskopf and D. H. Ewing, Phys. Rev. **57**, 472 (1940).

condition is:

$$1/\kappa \gg d, \quad \kappa = k|v'/v|_{r=a},$$

where κ is the instantaneous wave number at the nuclear surface. This condition is necessary, in analogy to (15), to insure small variations of the wave function outside, in intervals of the order d. It puts a limit of about several Mev's to the height of the barrier to which our expressions are applicable.

5. APPLICATION TO NEUTRON CROSS SECTIONS FOR HIGHER ENERGIES

When different l values are involved, the function v will depend upon l. This dependence will be indicated by a subscript l. In the case of neutrons v_l is given by:

$$v_l = i(\pi x/2)^{\frac{1}{2}} H_{l+\frac{1}{2}}^{(1)}(x). \tag{44}$$

The value[12] $|v_l|^2$ is then

$$\begin{aligned} |v_l|^2 &= \tfrac{1}{2}\pi x |H_{l+\frac{1}{2}}^{(1)}(x)|^2, \\ |v_0|^2 &= 1, \\ |v_1|^2 &= (1+x^2)/x^2, \\ |v_2|^2 &= (9+3x^2+x^4)/x^4, \\ |v_3|^2 &= (225+45x^3+6x^4+x^6)/x^6. \end{aligned} \tag{45a}$$

The function $\delta_l(x)$ is:

$$\delta_l(x) = -\tan^{-1}\frac{J_{l+\frac{1}{2}}(x)}{N_{l+\frac{1}{2}}(x)}, \tag{45b}$$

$$\begin{aligned} \delta_0 &= x, \\ \delta_1 &= x - \pi/2 + \cot^{-1} x, \\ \delta_2 &= x - \pi + \cot^{-1} \tfrac{1}{3}(x^2-3), \\ \delta_3 &= x - 3\pi/2 + \cot^{-1}[x(x^2-15)/(6x^2-15)]. \end{aligned}$$

The width Γ_{nl} for the emission of a neutron of angular momentum l is therefore given by

$$\Gamma_{nl} = \frac{2k}{K}\frac{1}{|v_l|^2}\frac{D_l^*}{\pi} \tag{46}$$

if the same assumptions about $f_0(E)$ are made as in (24). This is the generalization of (26).

These expressions can be applied to the absorption cross section of neutrons. Let us consider the average cross section over a small energy integral ΔE containing many resonances. This is the cross section observed, if the neutron beam is not sufficiently monochromatic to distinguish resonances. The average over (42) for a given l gives:

$$\langle\sigma_{\text{abs}}\rangle_{\text{Av}}{}^{(l)} = \frac{\pi}{k^2}(2l+1)\frac{2\pi\langle\Gamma_{nl}\Gamma_{al}\rangle_{\text{Av}}}{D_l(\langle\Gamma_{nl}\rangle_{\text{Av}}+\langle\Gamma_{al}\rangle_{\text{Av}})}.$$

Here D_l is the level distance of resonance levels excited by particles with an angular momentum $l\hbar$; $\langle\Gamma_{nl}\rangle_{\text{Av}}$ and $\langle\Gamma_{al}\rangle_{\text{Av}}$ are the average neutron width and absorption width, respectively, for neutrons of angular momentum $l\hbar$ in the energy region E. This can be transformed by means of (41) and (23) and (46) into:

$$\langle\sigma_{\text{abs}}\rangle_{\text{Av}}{}^{(l)} = \frac{4\pi}{kK}(2l+1)\frac{D_l^*}{D_l}\frac{1}{(1+\langle\Gamma_{nl}\rangle_{\text{Av}}/\langle\Gamma_{al}\rangle_{\text{Av}})|v_l|^2} \tag{47}$$

where $|v_l|^2$ is given in (45). We now assume that the ratio $\langle\Gamma_{al}\rangle_{\text{Av}}/D_l^*$ and the wave number K is independent of l and of the energy. One then can write:

$$\frac{\langle\Gamma_{nl}\rangle_{\text{Av}}}{\langle\Gamma_{al}\rangle_{\text{Av}}} = \frac{\langle\Gamma_{n0}\rangle_{\text{Av}}}{\langle\Gamma_{a0}\rangle_{\text{Av}}|v_l|^2} = \frac{x}{x_0|v_l|^2},$$

where x_0 is the value of $x=ka$ for which

$$\langle\Gamma_{n0}\rangle_{\text{Av}} = \langle\Gamma_{a0}\rangle_{\text{Av}}, \quad x_0 = \frac{\pi}{2}\frac{Ka}{D}\langle\Gamma_{a0}\rangle_{\text{Av}}.$$

The total absorption cross section can be obtained by summing (47) over all l's:

$$\langle\sigma_{\text{abs}}\rangle_{\text{Av}} = \frac{4\pi}{kK}\sum_{l=0}^{\infty}\frac{D_l^*}{D_l}(2l+1)\frac{1}{|v_l|^2+x/x_0}. \tag{48}$$

The assumption that $\langle\Gamma_{al}\rangle_{\text{Av}}/D_l^*$ is independent of the energy will break down when inelastic scattering or fission sets in which, in this treatment, are included in $\langle\Gamma_{al}\rangle_{\text{Av}}$. This increase in $\langle\Gamma_{al}\rangle_{\text{Av}}$ will make x_0 increase sharply at higher energies. We may assume that the inelastic scattering is unimportant for energies less than 0.4 Mev.

The value of $x_0 = \frac{1}{2}\pi Ka\Gamma_a/D$ will vary from element to element but it is expected to be much

[12] The functions $|v_l(x)/x|^2$, $|v_l'(x)/x|^2$, $\delta(x)$, $\delta'(x)$ have been tabulated for $l=0(1)20$ and $x=0(.1)10$ in "Scattering and Radiation from Circular Cylinders and Spheres," Morse, Lowan, Feshbach, and Lax, a report issued by the N.D.R.C. Division (6).

smaller for lighter elements than for heavier ones. This comes mainly from its dependence on the level distance D. We expect that D is substantially larger for light elements. The ratio Γ_a/D for any width Γ_a corresponding to an individual transition is usually considered to be independent of D. However, Γ_{al} is the sum of all individual transition widths to lower states of the compound nucleus. The number of possible end states is roughly inversely proportional to D, so that Γ_{al}/D is expected to decrease considerably with increasing D.

The variation of D seems to occur, however, only at values of $A \leq 100$. The fact that the neutron widths in known nuclear resonance levels show approximately the same proportionality factor of $\sqrt{E}$ in elements different as indium ($A=115$) rhodium ($A=103$) and Au ($A=187$) indicates, according to (26), that the level distance D at the neutron resonances does not change appreciably for these values of A. This is probably the effect of two opposing trends: (a) The level distance D decreases with increasing A for a given excitation energy of the compound nucleus. (b) The excitation energy at which D is measured is the binding energy of the neutron, which decreases with increasing A for $A>100$ and which is roughly constant for $A<100$. For $A>100$ the two effects a and b operate in opposite directions, giving rise to D and consequently to an x_0 which is comparatively constant. In the region $A<100$, only the trend given in (a) remains, which will give rise to an increase in D and thus to a decrease in x_0 for decreasing A. This probably explains the marked decrease of the absorption cross section, σ_{abs}, for elements for which $A<100$ as observed by Hughes.[13]

Expression (48) is, strictly speaking, valid only for nuclei with spin zero. If the spin is not zero, reemission of the neutron is possible with an l different from the incoming one, without exciting the nucleus. The partial widths corresponding to these processes should be included in $\langle\Gamma_{al}\rangle_{\text{Av}}$ since they represent an "inelastic" scattering inasmuch as the state of the nucleus has changed during the process. Thus $\langle\Gamma_{al}\rangle_{\text{Av}}$ would be made larger than the pure radiation width, by an amount which depends on the neutron energy. Closer investigation and application of selection rules reveals, however, that this effect is not very important. It decreases σ_{abs} by less than 10 percent if $x \lesssim 1$.

Expression (48) for the absorption cross section of neutrons can be compared with experiments performed by Greisen, Linenberger, Miskel, and Segrè,[14] on the elements Ag, In, I, and Au, with neutrons of energies between 5 and 400 kev. Figure 4 shows these results plotted as a function of $x^2=k^2a^2$, which is proportional to the energy. An energy scale is given for In and Au. The solid curves are plots of expression (48) with different values of x_0. It is hard to predict the actual value of x_0, even in elements like In and Au, where one resonance level is analyzed, since the properties of this level are probably not typical. The best values[8] for the In resonance are:

$$\Gamma_n \simeq 3\times 10^{-3}(E_r)^{\frac{1}{2}} \text{ ev}, \quad \Gamma_a \sim 0.08 \text{ ev}$$

and for Au:

$$\Gamma_n \simeq 6\times 10^{-3}(E_r)^{\frac{1}{2}} \text{ ev}, \quad \Gamma_a \sim 0.15 \text{ ev}.$$

This gives $x_0 \sim 0.05$ for both elements. The actual value of x_0 is probably somewhat higher since these two resonances are unusually strong which points to an abnormally large Γ_n. Moreover, there is an additional uncertainty due to the fluctuations in the nuclear radius from the assumed average value. Figure 4 shows that the theoretical curves give a fair representation of the experimental material.

The rather sharp decrease of $\sigma_{\text{abs}}/\sigma_0$ for Ag and Au for energies greater than a few hundred kev is caused by the onset of inelastic scattering of the neutrons. The cross section for this competing process is not included in the measured absorption cross section. Bismuth and lead,[14] however, seem to represent notable exceptions since they have shown no observable absorption in this energy region. These five nuclei (lead has four isotopes) fall completely out of the rules observed with so many other heavy nuclei. Within the framework of this theory, this would only be explained by an unusually large level distance D for these elements, which would point to an unusually low binding energy of the last neutron.

[13] Hughes, Phys. Rev. **70**, 106 (1946).

[14] We are grateful to these authors for communicating their results before publication.

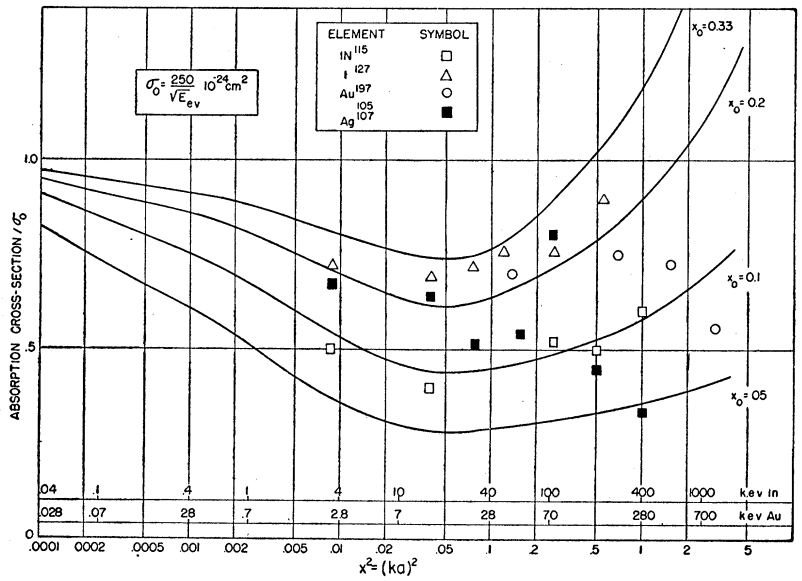

FIG. 4. The absorption cross section for neutrons as a function of $(ka)^2$. The experimental points were obtained by Greisen, Linenberger, Miskel, and Segrè. We are grateful to these authors for communicating their results prior to publication.

The elastic cross section of neutrons can be calculated with the help of (43). We consider again the average $\langle\sigma_{sc}\rangle_{Av}{}^{(l)}$ over many resonances, since this is the magnitude which can be measured most readily. We obtain

$$\langle\sigma_{sc}\rangle_{Av}{}^{(l)}=\frac{1}{D}\int_{E_r-D/2}^{E_r+D/2}\frac{4\pi}{k^2}(2l+1)\times\left|\frac{\frac{1}{2}\langle\Gamma_{nl}\rangle_{Av}}{(E-E_r)+\frac{1}{2}(\langle\Gamma_{nl}\rangle_{Av}+\langle\Gamma_{al}\rangle_{Av})}+p_{sc}\right|^2 dE$$

$$=\frac{4\pi}{k^2}(2l+1)\left[\left(1-\frac{\pi\langle\Gamma_{nl}\rangle_{Av}}{D}\right)\sin^2\delta_l+\frac{\pi}{2}\frac{\Gamma_{nl}{}^2}{D(\Gamma_{nl}+\Gamma_{al})}\right].$$

With the aid of approximations (46) and (47), we may now write

$$\langle\sigma_{sc}\rangle_{Av}=\sum_l\langle\sigma_{sc}\rangle_{Av}{}^{(l)}=\sigma_0+\frac{4\pi}{k^2}\frac{D^*}{D}\frac{x}{X}\times\sum_l\frac{2l+1}{|v_l|^2}\left[\frac{1}{1+(x_0|v_l|^2/x)}-2\sin^2\delta_l\right],\quad(49)$$

where $X=Ka$, δ_l is defined by Eqs. (36), (45b) and σ_0 is:

$$\sigma_0=\frac{4\pi}{k^2}\sum_l(2l+1)\sin^2\delta_l.$$

This is the scattering cross section of an impenetrable sphere of radius a, σ_0 is $4\pi a^2$ for small energies ($ka\ll1$) and approaches $2\pi a^2$ asymptotically for large energies ($ka\gg1$).

The quantity usually measured is not $\langle\sigma_{sc}\rangle_{Av}$ but rather the total scattering cross section

$$\langle\sigma_t\rangle_{Av}=\langle\sigma_{sc}\rangle_{Av}+\langle\sigma_{abs}\rangle_{Av}.$$

The theoretical formula for $\langle\sigma_t\rangle_{Av}$ is obtained by adding (49) and (48)

$$\langle\sigma_t\rangle_{Av}=\sigma_0+\frac{4\pi}{k^2}\frac{D^*}{D}\frac{x}{X}\sum_l\frac{2l+1}{|v_l|^2}\cos2\delta_l.\quad(50)$$

Note that $\langle\sigma_t\rangle_{Av}$ is independent of x_0.

The total nuclear cross section is, according to (50) always greater than σ_0. For low energies, one need only use the $l=0$ term in (50).

$$\langle\sigma_t\rangle_{Av}=4\pi a^2[1+D^*/DXx].$$

The excess of $\langle\sigma_t\rangle_{Av}$ over $4\pi a^2$ for small x will be large since the $1/x$ term ($1/v$ law of absorption) becomes very large for small x. For larger energies, several values of l contribute to the cross section. In the limit the positive and negative terms cancel so that $\langle\sigma_t\rangle_{Av}\to\sigma_0$.

Measurements of the total cross section of several elements of neutrons in the energy region between 20 kev and 800 kev are made by

Leipunsky[15] and Wattenberg.[16] Figure 5 shows these experimental results in units of πa^2, together with curves for σ_0 and $\langle\sigma_t\rangle_{Av}$ for indium, and $D^*/D=\frac{1}{2}$. Curves for other nuclei may be calculated quite easily from this one by making use of the fact that $\langle\sigma_t\rangle_{Av}-\sigma_0$, from (50), depends inversely on the nuclear radius. In comparing the theoretical and experimental values, one must keep in mind that a was assumed to be $1.5\times10^{-13}A^{\frac{1}{3}}$ cm. The actual value of the nuclear radius will probably deviate from this average value. In Fig. 5, representative elements from among the middle and heavy elements have been included. It is seen that the qualitative features of $\langle\sigma_t\rangle_{Av}$ are in substantial agreement with the theory for the elements Fe, Ni, W, Pb. Although the cadmium points are of the right order of magnitude, they do not show the characteristic decrease with increasing ka. This behavior is exhibited by other elements with $A\sim100$ such as Sn, I, Sb, all of which have about the same $\langle\sigma_t\rangle_{Av}$ as Cd.

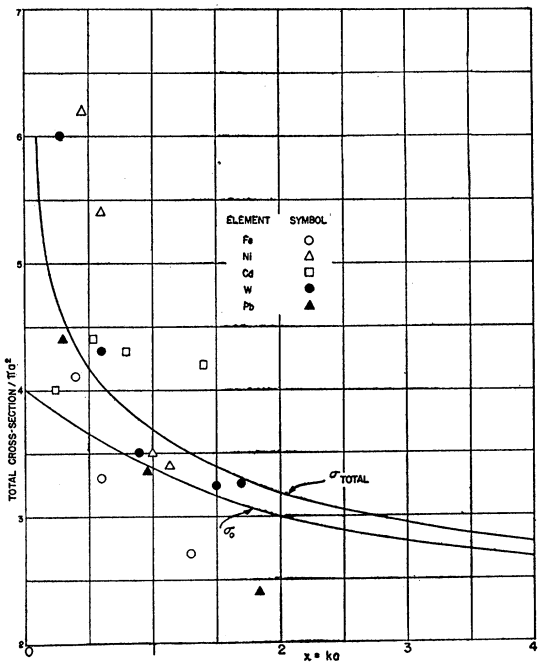

FIG. 5. The total cross section for neutrons as a function of ka. σ_{TOTAL} is the total cross section curve for $A=115$. The σ_0 curve is the cross section for scattering by an "impenetrable sphere." The experimental points are typical examples taken from Wattenberg (reference 17). Similar results have also been obtained by Leipunski (reference 16).

6. RELATION TO OTHER METHODS

The derivation of the resonance formulas presented here is in many respects similar to the method employed by Kapur and Peierls.[4] It may be useful for the understanding of the situation to point out the similarities.

It is possible to write the expressions (8) and (9) in a form in which the familiar sum over resonance levels appears, if the following relation is used:

$$g(E)=\frac{1}{f(E)-ix}=\sum_r\frac{C_r}{S_r-E}, \qquad (50a)$$

where S_r are the poles of the function $g(E)$, which are defined by

$$f(S_r)=ix. \qquad (51)$$

The C_r are the residues: $C_r=(1/(df/dE))_{E=S_r}$. Here x is considered as a constant parameter independent of E. It follows from (17a) and (18) that

$$S_r=E_r-i\Gamma_a^{(r)}/2-i\Gamma_n^{(r)}/2 \qquad (52)$$

where E_r is the solution of $f_0(E)=0$. If the imaginary parts of S_r are small compared to the real ones, the relation $C_r=-\Gamma_n^{(r)}/2x$ holds because of (18), and we can write (8) in the form

$$\sigma_{\text{abs}}=\frac{\pi}{k^2}\left|\sum_r\frac{(\Gamma_n^{(r)}\Gamma_a^{(r)})^{\frac{1}{2}}}{E-E_r+i(\Gamma_n^{(r)}+\Gamma_a^{(r)})/2}\right|^2 \qquad (53)$$

after using the relation

$$h=-\tfrac{1}{2}\Gamma_a^{(r)}(df/dE)_{E=S_r}.$$

Expression (9) can similarly be written in the form:

$$\sigma_{\text{sc}}=\frac{4\pi}{k^2}\left|\tfrac{1}{2}\sum_r\frac{\Gamma_n^{(r)}}{E-E_r+i(\Gamma_n^{(r)}+\Gamma_a^{(r)})/2}+e^{ix}\sin x\right|^2. \qquad (54)$$

[15] A. I. Leipunsky, J. Phys. Academy of Science, U.S.S.R. **3**, 231 (1940).
[16] J. Wattenberg, unpublished. We are grateful to the author for communicating his results before publication.

These expressions are the same as the ones derived by Kapur and Peierls. The eigenvalues S_r are defined in their paper by the condition, that the wave ψ of the emitted particles satisfies the boundary condition

$$\partial\psi/\partial r - ik\psi = 0,$$

which is identical with our condition (51).

The cross sections for a given energy E between resonances are represented in this form as the effect of contributions of a large number of levels E_r, whereas in our case the same cross sections are determined by the function f at the energy E in question. The two approaches are mathematically equivalent because of the identity (50a). The method presented here seems to offer the advantage of an easier interpretation of the magnitudes involved. Qualitative conclusions as to the behavior of the logarithmic derivatives $f(E)$ of the wave function at the surface of the nucleus can be drawn more readily than conclusions concerning a whole series of eigensolutions of a complicated eigenvalue problem at energies far off the value E. Especially the levels far off resonance and their contribution to scattering offer some difficulties of interpretation. It was pointed out by Siegert,[5] that they do not correspond to any physical state since the boundary condition contains the k of the incident particle and not the k which would correspond to the emission of the particle by the resonance level itself. Siegert's method avoids this difficulty by using a different boundary condition for the definition of the compound states. However, the physical meaning of compound states of high energy is questionable, when their width becomes larger than their distances, which is bound to happen at higher energies.

Bethe[17] has derived expressions similar to ours for the average cross sections and particle widths by ascribing to the nucleus a strong absorption coefficient in form of an imaginary potential energy. This absorption was supposed to describe the "amalgamation" of the incident particle with the nucleus. It therefore was assumed to be so strong that the wave function of the incoming particle decreases by a factor e^{-1} at a distance b of the order of the reciprocal of our K. Bethe's expression for the average neutron capture cross section for $l=0$ $\langle\sigma_{\text{abs}}\rangle_{\text{Av}} = 2\pi^2\lambda b$ [his formula (27a)] is identical with ours (28) if

$$b = (1/K)(D^*/D)(2/\pi).$$

Bethe's model does not exhibit any resonances because of the strong absorption assumed. This absorption is very much larger than the actual one due to radiative capture, and the corresponding breadth is so large that it would broaden any levels until they merge. The similarity of Bethe's results with ours comes from the fact that his fictitious absorption causes the wave function of the incoming particle to assume a high logarithmic derivative after entering the nucleus of the order $K\sim 1/b$. This fact alone determines the average cross sections, whereas the special form (24) of the logarithmic derivative is important only for the finer details relating to resonance phenomena. Bethe's assumption of an absorption coefficient of the order b^{-1} appears to be equivalent to our assumption (21) about the average wave numbers inside of the nucleus. Our way of introducing high wave numbers inside the nucleus seems to be more general since it also permits the description of resonance phenomena.

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[17] H. A. Bethe, Phys. Rev. **57**, 1125 (1940).

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