# Multiple Scattering and the Mass of the Meson

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It is shown that multiple scattering by the atoms of the gas in a cloud chamber can cause large apparent curvature of tracks. This effect makes the measurement of curvature in a magnetic field meaningless if the energy of the particle is small and the magnetic field low. An analysis shows that *all* published meson tracks are compatible with a unique mass of about 200 electron masses.

## 1

IT has been attempted repeatedly to determine the mass of the meson in a cloud chamber,[1–4] by simultaneous measurement of the range of the particle in the gas, and of the curvature of the track in a magnetic field. It is the purpose of this note to point out that such measurements are subject to very large errors due to multiple scattering of the mesons in the coulomb fields of the gas atoms. When properly analyzed, all existing data are compatible with a unique mass of the meson, of about 200 electron masses.

The reason for the large effect of multiple scattering is the extremely low energy of any particle which can be stopped in the cloud chamber gas. The mean angle of deflection due to multiple scattering in a given thickness of material, is inversely proportional to the kinetic energy of the particle. The deflection due to a magnetic field is only inversely proportional to the momentum. Therefore, at sufficiently low velocities, the scattering effect will always be greater than the curvature in the magnetic field. This is especially true if a small magnetic field (1000 gauss) or a gas of high atomic number (argon) is used.

Multiple scattering was also responsible for the erroneous interpretation of some cloud-chamber pictures recently obtained with the betatron of the General Electric Research Laboratory. These pictures were believed[5, 6] to show tracks of mesons which were therefore supposed to be produced by $\gamma$-rays of 100 Mev or less. More recent cloud-chamber studies[7] to investigate the influence of multiple scattering in air indicate that the tracks can be interpreted as due to protons.

Even the most "direct" method for measuring the meson mass, that of Leprince-Ringuet,[8] is affected by scattering. In this method, the momentum of a recoil electron is measured in a magnetic field. This momentum, combined with the angle between recoil electron and incident meson, gives the mass of the latter. Both the momentum and especially the angle measurement may be falsified by scattering.

## 2

The mean square average angle of scattering in a thickness $x$ of a substance of atomic number $Z$ is[9–11]

$$\theta^2_{\mathrm{Av}\ pl}=\frac{4\pi e^4Z^2Nx}{p^2v^2}\ln\frac{\theta_{\max}}{\theta_{\min}}, \tag{1}$$

where $N$ is the number of atoms per $\mathrm{cm}^3$ in the substance, $p$ the momentum, and $v$ the velocity of the particle. The subscript $pl$ indicates that the projection of the scattering on a *plane* is meant.

[1] D. J. Hughes, Phys. Rev. **69**, 371 (1946).
[2] Y. N. Nishina, M. Takeuchi, and T. Ichimiya, Phys. Rev. **55**, 585 (1939).
[3] A. J. Ruhlig and H. R. Crane, Phys. Rev. **53**, 266 (1938). In this paper, the change of curvature along the path was actually used, but the results can also be interpreted on the basis of curvature and range.
[4] Further literature in J. A. Wheeler and R. Ladenburg, Phys. Rev. **60**, 756 (1941).
[5] M. Schein and A. J. Hartzler, Phys. Rev. **69**, 248T (1946).
[6] M. Schein, A. J. Hartzler, and G. S. Klaiber, Phys. Rev. **70**, 435 (1946).
[7] G. S. Klaiber, A. E. Luebke, and G. C. Baldwin, Bull. Am. Phys. Soc. (September, 1946), Abstract C11.
[8] L. Leprince-Ringuet, S. Gorodetzky, E. Nageotte, and R. Richard-Foy, Phys. Rev. **59**, 460 (1941).
[9] E. J. Williams, Proc. Roy. Soc. **169**, 531 (1939).
[10] E. J. Williams, Phys. Rev. **58**, 292 (1940).
[11] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).

The minimum scattering angle $\theta_{\min}$ in (1) is given by screening and is

$$\theta_{\min} = (mc/p)(Z^{1/3}/181) \qquad (2)$$

with $m$ the electron mass. The maximum angle, $\theta_{\max}$, should be taken for cloud-chamber tracks as that angle which can easily be recognized as single scattering (normally about 0.1 radian). (The finite size of the nucleus does not provide a limitation on $\theta_{\max}$ for the small momenta with which we are concerned.)

To obtain the apparent radius of curvature, let us consider the projection of a track on a plane (Fig. 1). Let $M$ be the midpoint of the track, and **t** the tangent to the track at $M$. Let $y_1$ and $y_2$, respectively, be the distance of the end points, $P_1$ and $P_2$, of the track from the tangent **t**, and $x$ the distance from $M$ to either end point (assumed to be large compared with $y_1$ and $y_2$). Then it is easily seen (e.g., reference 9, Eq. (1.68)) that

$$(y_1^2)_{\text{Av}} = (y_2^2)_{\text{Av}} = (\tfrac{1}{3})x^2\theta^2_{\text{Av}\,pl}. \qquad (3)$$

The distance $s$ of $M$ from the straight line $P_1P_2$, i.e., the sagitta of the curve, is given by

$$s = (y_1 + y_2)/2. \qquad (4)$$

Since $y_1$ and $y_2$ are statistically independent,

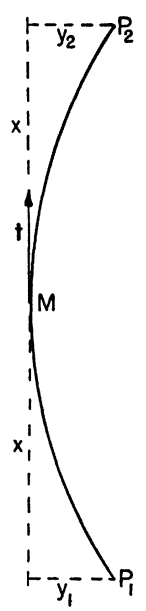


FIG. 1. Curvature of track due to scattering.

we have

$$s_{\text{Av}}^2 = (y_1^2 + y_2^2)_{\text{Av}}/4 = x^2\theta^2_{\text{Av}\,pl}/6. \qquad (5)$$

From $s$, we get the radius of the closest fitting circle,

$$\rho_s = x^2/2s, \qquad (6)$$

or inserting (5),

$$\rho_s = (\tfrac{3}{2})^{\frac{1}{2}} \frac{x}{(\theta^2_{\text{Av}\,pl})^{\frac{1}{2}}}. \qquad (7)$$

If the gas contains $PL$ nuclei per cm$^3$ where $L$ is Loschmidt's number ($L = 2.7 \cdot 10^{19}$, $P = 2$ for air at NTP, $P = 1$ for argon at NTP), the apparent radius of curvature becomes

$$\rho_s = 103\frac{M\beta^2}{Z}\left(\frac{x}{BP}\right)^{\frac{1}{2}}. \qquad (8)$$

Here $M$ is the mass of the particle (including relativistic correction) in units of the electron rest mass, $\beta$ is $v/c$, and $B$ is a correction factor close to unity, *viz.*,

$$B = 1 + 0.444 \log_{10} \frac{p}{mc}\theta_{\max} Z^{-\frac{1}{3}}. \qquad (9)$$

The apparent radius $\rho_s$ is given in cm. If the velocity is non-relativistic and if $E$ is the kinetic energy of the particle in Mev, (8) can be rewritten:

$$\rho_s = 404\frac{E}{Z}\left(\frac{x}{BP}\right)^{\frac{1}{2}}. \qquad (10)$$

The "scattering radius" (8) may be compared with the radius of curvature in a magnetic field,

$$\rho_H = 1700 M\beta/H, \qquad (11)$$

and we obtain

$$\rho_H/\rho_s = \beta_0/\beta, \qquad (12)$$

with

$$\beta_0 = 16.5\frac{Z}{H}\left(\frac{BP}{x}\right)^{\frac{1}{2}}. \qquad (13)$$

Below the "critical velocity" $\beta_0$, the major contribution to the curvature arises from scattering which makes any calculation of the mass from the curvature meaningless.

Table I gives some results for $\beta_0$ for gases at NTP. (For the calculation of $B$, $\theta_{\max}$ was assumed equal to the deflection of the particle in

TABLE I. Values for the critical velocity, $\beta_0$.

| Gas | $Hx^{\frac{1}{2}}\beta_0$ | $\beta_0$ for $H=1000$, $x=10$ | Corresponding proton energy (Mev) |
|---|---|---|---|
| $H_2$ | 25 | 0.008 | 0.03 |
| He | 35 | 0.011 | 0.06 |
| Air | 176 | 0.056 | $1.4_6$ |
| Ne | 173 | 0.055 | 1.4 |
| A | 310 | 0.098 | 4.5 |

TABLE II. Energies and ranges below which curvature measurements in a field of 1000 gauss become meaningless.

| Gas (NTP) | Proton | | | Meson ($\mu=200$) | | |
|---|---|---|---|---|---|---|
| | $\beta=v/c$ | Energy Mev | Range cm | $\beta=v/c$ | Energy Mev | Range cm |
| Hydrogen | 0.032 | 0.47 | 2.2 | 0.047 | 0.112 | 0.96 |
| Air | 0.082 | 3.14 | 15.2 | 0.123 | 0.765 | 6.5 |
| Argon | 0.101 | 4.75 | 31 | 0.151 | 1.15 | 13.8 |

a magnetic field of 1000 gauss over a distance of 10 cm; the result is not sensitive to this assumption.) It is seen that for argon, with a field of 1000 gauss and a track length of $2x=20$ cm, the measurements become completely meaningless when $\beta\leq 0.1$, and even if the velocity is equal to that of light, the scattering introduces an error of 10 percent in $H\rho$.

The situation is even worse for particles close to the end of their range because then the available track length is often much shorter than the 20 cm assumed in Table I. Table II gives, for protons and for mesons of mass 200, the values of the kinetic energy at which the "scattering radius of curvature" $\rho_s$ becomes equal to the radius in a magnetic field of 1000 gauss, assuming that the entire range of the particle is used for the curvature measurement.[12] For simplicity, the energy at the midpoint of the track was assumed to determine both $\rho_H$ and $\rho_s$. The table shows clearly that in air, and even more in argon, magnetic curvature measurements on any track stopping in a cloud chamber are meaningless if the magnetic field is as low as 1000 gauss. Even at much higher fields, the errors are very large. Increase of pressure is only of little help because, although particles of higher energy can then be stopped in the chamber, the curvature due to scattering for a given energy increases.

## 3

We have seen that the curvature of a short range track does not permit one to infer the mass of the particle. The best one can do is to investigate whether the observed "curvature" can reasonably be ascribed to scattering (plus possible magnetic deflection) for an *assumed* mass of the particle. Since scattering is a statistical phenomenon, a curvature $1/\rho_s$ of twice the expected value must be considered as quite likely and the assumed particle mass must in such a case not be ruled out.

This argument is strengthened by the fact that at the larger angles single scattering becomes important. The probability distribution for multiple scattering is, of course.

$$P(\theta)d\theta=\frac{d\theta}{(2\pi\theta_{Av}^2)^{\frac{1}{2}}}\exp\left[-\theta^2/2\theta_{Av}^2\right]. \quad (14)$$

The distribution for single scattering is

$$P(\theta)d\theta=\frac{\theta_{Av}^2 d\theta}{\theta^3\ln(\theta_{\max}/\theta_{\min})}, \quad (15)$$

where $\theta_{\max}$ and $\theta_{\min}$ have the same meaning as in Eq. (1). This gives quite appreciable probabilities for deflections of several times $\theta_{Av}$.

One should further keep in mind that the observer is attracted to the unusual tracks showing large curvature while the normal, nearly straight tracks pass unnoticed. It should, therefore, not be surprising to find tracks whose statistical probability is of the order of one percent.

It is often objected that the scattering should cause an erratic path, rather than a smooth circular arc. This argument is fallacious because of the limited accuracy with which the track can be observed, especially if it is a thick and short track of a heavily ionizing particle near the end of its range. Consider again Fig. 1: It is equally probable that the points $P_1$ and $P_2$ lie on the *same* side of the tangent **t**, as that they lie on opposite sides. In the former case, we observe in first approximation a circular ($C$-type curve) arc, although closer examination might possibly reveal that most of the curvature is in one-half of the track. In the latter case, we get an S-

[12] Frequently, only the first half of the range is used. While this makes for better definition of the energy, it gives a smaller $x$ and therefore no improvement in the ratio $\rho_s/\rho_H$.

shaped curve, but the expectation value of the sagitta of each half of this curve is smaller by a factor $2^{\frac{3}{2}}$ than the expectation value of the sagitta of the entire curve. This means that the deviation of the $S$-curve from a straight line can be observed only if the accuracy of measurement is quite high, whereas the curvature of a $C$-type track can be clearly recognized. In other words, "erratic" deviations of the track are much smaller and harder to recognize than its over-all curvature. However, photographs of proton tracks, taken by Klaiber, Luebke, and Baldwin[7] without a magnetic field, show $S$-type as well as $C$-type curves.

A further point which makes for apparently smooth curvature is the combination of magnetic and scattering curvature. Even if the scattering predominates, the magnetic field still gives some contribution and helps to make the curvature more uniform. However, cases may also occur in which the scattering "curvature" is opposite to the magnetic one, and predominates sufficiently to give seemingly the wrong sign of the electric charge. One such example was observed in the General Electric Research Laboratory, in which a proton showed a "negative" curvature.

The smoothness of a track is thus no argument against a major contribution of scattering. Nor is the variation of curvature along the track a valid criterion: If an (already short) track is broken into sections, the curvature in each section is relatively even more influenced by scattering than that of the entire track. The "right" law of increase of curvature (as expected in a magnetic field) is qualitatively also produced by scattering which also increases towards the end of a track.

## 4

The formulae developed in Section 2 may be compared with those of Williams.[9,10] In particular, Williams has given a very simple rough formula for the apparent radius of curvature due to scattering,* namely,

$$\rho_s/R = 1.3(M/mZ)^{\frac{1}{2}}. \tag{16}$$

Here $R$ is the range of the particle and $\rho_s$ is the mean curvature over the first half of the range. Therefore, if we wish to compare Williams' result with our Eq. (10), we must put in the latter $x=\frac{1}{4}R$ and $E$ equal to the energy of a particle of range $\frac{3}{4}R$. The comparison of Eqs. (10) and (16) shows then considerable discrepancies; e.g., for protons of range between 2 and 20 cm in air, Eq. (10) gives a radius of curvature which is about 0.6 of that obtained from Williams' formula (16). Since Williams' formula has been widely used, it is necessary to explain this discrepancy.

The main part of the discrepancy is caused by a different definition of $\rho_s$. Williams uses the straight mean deflection, whereas we have used the mean square. This causes a factor $(2/\pi)^{\frac{1}{2}}$ which is about 0.8. There is, of course, no real difference between using the two different kinds of mean, but it must be remembered that when the straight mean is used, scattering may exceed this mean by a large factor. Curvatures of 3–4 times the straight mean are statistically quite probably.

The second difference is in the definition of the radius of curvature. Williams defines this by means of the change of direction of the track from the beginning to the end. It is experimentally not easy to determine the direction of a track at a given point with any accuracy. We, therefore, believe that our definition in terms of the sagitta of the curve is closer to actual experimental procedure such as fitting the best circle to a given curve. This difference in definition causes another factor $(3/4)^{\frac{1}{2}}$ which is 0.87, again in the direction of making our $\rho_s$ smaller than that of Williams.

Finally there is some inaccuracy in the analytical range-energy relation used by Williams in deriving Eq. (16). This inaccuracy causes an error of about 10 percent, again in the same direction. The error is greater in the case of heavier elements; e.g., for argon the correction is about 20 percent. The results from Williams formula should, therefore, be multiplied by about 0.6 for air and about 0.52 for argon.

A further difference between Williams' paper and ours is the much more accurate way in which he treated the maximum scattering angle $\theta_{\max}$. Williams[9] wanted to give an *accurate* value for the average scattering angle and included a

* Reference 10, Eq. (21a).

detailed discussion of the large deflections. In this paper, we are concerned only with approximate values permitting some judgment as to the statistical probability of certain curvatures of cloud-chamber tracks. We were, therefore, satisfied with a much rougher treatment of $\theta_{max}$, and we wished at the same time to determine $\theta_{max}$ in conformity with experimental procedures; i.e., as the angle which can be observed without difficulty as single scattering. The difference between the two determinations of $\theta_{max}$ corresponds to only a few percent in $\rho_s$ is all particle cases.

In much of the literature great liberty is taken with the range-energy relation and particularly with the stopping power of argon and similar substances. Very frequently the stopping power is assumed to be proportional to the number of electrons in an atom which is reasonably accurate for very high energy particles but fails even in this case by 10–20 percent.[11] In the case of slow particles, with which we are here concerned, this assumption is entirely wrong; for instance, argon has a stopping power[13] of about 0.97 of that of air instead of 1.25 as would correspond to the number of electrons. The energy, and therefore the expected radius of curvature, of tracks in argon has, therefore, been grossly over-estimated in much of the literature.

## 5

We shall now analyze some of the tracks which have been reported.

*1.* The track of D. J. Hughes.† This track was interpreted by Hughes as caused by a particle of mass between 10 and 50 electron masses. We shall show, however, that the data are compatible with the "normal" meson mass of 200.

The track length (measured on photograph) is 2.5 cm in argon of NTP. At the low velocity of the particle concerned, the stopping power of argon is slightly less than that of air,[13] rather than 1.4 times as great (proportional to the density) as apparently assumed by Hughes. At the midpoint, the residual range is then 1.2 cm air (Hughes gives 1.8). The measured radius of curvature is 3.5 cm.

[13] M. S. Livingston and H. A. Bethe, Rev. Mod. Phys. 9, 261 ff. (1937) (Table 48, p. 272). The number given there (1.94) is for one argon *atom* as compared to one air "atom"; per molecule, the stopping power is then 0.97.

† Reference 1, Fig. 5.

A meson of mass 200 m and of range 1.2 cm air has an energy[13] of 280 kev. This gives the following results:

$H\rho = 2.5\cdot 10^4$.
$\rho_H = 22$ cm (Since $H = 1165$ gauss).
Curvature attributable to scattering: $1/\rho = 1/3.5 - 1/22$, $\rho = 4.1_5$ cm.
$\rho_s = 7.5$ cm (calculated from Eq. (10), with $Z = 18$, $P = 1$, $B = 0.89$, $x = 1.25$).

The "observed" curvature is 1.8 ($= 7.5/4.1_5$) times the standard scattering. This is quite probable: the statistical probability that the curvature is more than 1.8 times its average value is 7 percent. The track observed by Hughes, therefore, is no evidence for a meson mass different from the "normal" value of 200.

For all the remaining tracks reported by Hughes, the mass of 200 is included in the interval of probable error given by Hughes himself.

*2.* The track of Nishina and collaborators.[2] This track gave, in the analysis of the authors, a mass of $177 \pm 8$ electron masses (error due to measurement alone). This experiment was made under much more favorable conditions since the magnetic field was high (12,600 gauss) and the gas was air rather than argon. The data:

Range 6.15 cm at 1.2–1.3 atmosphere pressure, corresponding to 7.3–8.1 cm air at NTP.
Measured radius of curvature $\rho_m = 3.08$ cm.
Critical velocity from Eq. (13),

$$\beta_0 = 16.5\frac{7.2\cdot 1.02}{12,600}\left(\frac{2.5}{3.08}\right)^{\frac{1}{2}} = 0.0087.$$

Residual range at midpoint of track $= 3.8_5$ cm air at NTP, corresponding to an energy of 0.56 Mev for a meson of mass 200.
Therefore velocity $\beta = 0.105$.
Probable error of curvature due to scattering $\beta_0/\beta = 0.083$.
For $\mu = 200$ m we have $H\rho = 35,700$, $\rho_H = 2.83$ cm which differs from the observed radius of curvature, $\rho_m = 3.08$ cm, by 8 percent which is within the probable error due to scattering. A mass of 200 is easily compatible with this measurement.

*3.* The track of Ruhlig and Crane.[3] According to the analysis of the authors, this track gave a mass of $120 \pm 30$. Data:

Track length (measurable) 4.7 cm. (Actual range may be longer because track probably passes out of the field of illumination.)
Measured radius of curvature 9 cm.

At midpoint, residual range $2.3_5$ cm (if 4.7 cm is the total range). If mass is assumed to be 200 m, the energy is 420 kev.

The calculated scattering radius is $\rho_s = 25.6$ cm ($Z = 7.2$, $B = 1.00$).

Calculated

$$H\rho = 30{,}500, \quad \rho_H = 10.7 \text{ cm} \quad (H = 2850).$$

A combination of magnetic and scattering curvature can easily explain the measured radius of curvature of 9 cm.

This track was the only one considered by Corson and Brode[14] to show a definite deviation from a mass of 200. Our analysis proves, however, that it is not in disagreement with such a mass.

*4.* One of the tracks observed with the betatron of the General Electric Research Laboratory[5, 6] has the following characteristics:

Range 6.1 cm in argon of 80 cm pressure.

Curvature corresponding to a *negative* charge, observed radius of curvature 22 cm in field of $H = 800$ gauss.

This track was originally interpreted as caused by a meson, especially because of its apparent negative charge. However, it is quite compatible with a proton. Interpreting it in this way, we find for the midpoint of the track:

Residual range 3.1-cm air equivalent.
Proton energy 1.21 Mev.
$H\rho = 1.58 \cdot 10^5$, therefore calculated $\rho_H = 200$ cm.

To compensate for this magnetic curvature, and give the observed negative curvature, the scattering by itself must have had a radius of curvature $\rho = 20$ cm. The calculated $\rho_s = 43$ cm ($Z = 18$, $B = 1.23$). The actual scattering curvature is thus about twice the "average" value which is statistically sufficiently probable. The track is therefore compatible with a proton and cannot be used as evidence for production of a meson by the betatron. This conclusion must be reached in spite of the apparent negative curvature: If scattering predominates over magnetic deflection, the sign of the curvature clearly has no physical meaning.

Another track of range 5 cm showed a positive curvature of radius 27 cm. This is even closer to the theoretical curvature than the first example. In addition, there was definite evidence of single scattering in this track.

Tracks observed without a magnetic field in air[7] include one of 3.5-cm length and $\rho = 12$ cm and one of 13-cm length and 60-cm curvature. These are about three times the "expected" scattering for protons.

*5.* Another experiment which is affected by multiple scattering is the measurement by magnetic deflection of the energy of the proton arising from the photo-disintegration of the deuteron.[15] Fortunately, this experiment was done in deuterium gas at low pressure (8 cm Hg) and the main scattering is due to the water ($D_2O$) vapor in the cloud chamber whose partial pressure was probably about one-half of the total pressure. The magnetic field was also reasonably high (3762 gauss). The data:

Total range of protons (220 kev) in air 0.22 cm, in the cloud chamber about 3.2 cm (average of measurements).
At midpoint: residual range $x = 1.6$ cm.
Energy 110 kev.
Effective $Z\sqrt{P} = 1.9$.
Calculated from midpoint energy, $\rho_s = 46$ cm ($B = 1.02$), $\rho_H = 18$ cm.

This indicates a probable error of 40 percent in the determination of $H\rho$ which may be one cause of the large fluctuations in the measured energy release (although there are also other causes for such fluctuations).

*6.* We shall now discuss the method of Leprince-Ringuet and others[8, 16] for the measurement of the mass of cosmic-ray particles. The method consists of measuring the momentum and direction of emission of a recoil electron, and the momentum of the primary particle which has produced the recoil electron. The method is very free of theoretical assumptions such as the ionization as a function of energy. However, even this method is subject to errors due to multiple scattering. These come in four places: (a) the measurement of the momentum of the primary partlcle, (b) the momentum of the electron, (c) the angle $\chi_1$ between electron and primary particle in the plane perpendicular to the magnetic field, and (d) the angle $\chi_2$

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[14] D. R. Corson and R. B. Brode, Phys. Rev. **53**, 773 (1938).

[15] F. T. and M. M. Rogers, Phys. Rev. **55**, 263 (1939).

[16] L. Leprince-Ringuet and M. Lhéritier, J. de Phys. et rad. (8) **7**, 65 (1946).

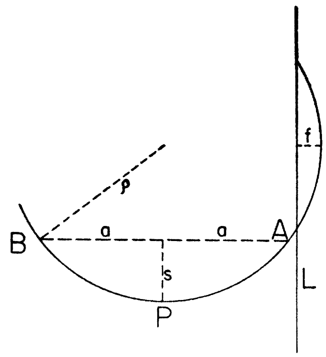


FIG. 2. Mass determination by the method of recoil electrons. $APB$ is the measured section of the track of the electron, $L$ represents the track of the primary.

between these particles in a plane parallel to the magnetic field.

The first of these is a standard curvature measurement to which our theory is directly applicable. The error is usually quite small because a long track is available for curvature measurements.

Errors $B$ and $C$ are related to each other. In order to determine the data on the recoil electron, one tries to fit a circle to the track of the electron as closely as possible. The fitting should be done over as large an arc as possible in order to make the radius of curvature as accurate as possible; on the other hand, multiple scattering will gradually make the track deviate from the circle so that the position of the center of the circle is shifted appreciably; for this reason, parts of the track too far from the point of emission should not be used. The practical compromise used by Leprince-Ringuet and coworkers is to make the fitting near the point at which the electron has been deflected through 90° by the magnetic field, and to use a track length of about a quarter circle. In effect then, the part of the electron track from $\theta=45°$ to $\theta=135°$ is used for fitting. (See Fig. 2.)

The error in the radius of curvature is slightly greater than that given by the theory of Section 2. The result in Eq. (13), for instance, should be multiplied by a factor

$$F(\alpha)=\alpha^2/2(1-\cos\alpha) \tag{17}$$

where $2\alpha$ is the angle subtended by the measured track length, in our case 90°. To find the angle of emission we have to extrapolate the electron track back to the track of the primary particle. A convenient measure of $\chi_1$ is the sagitta $f$ between electron track and primary track, the latter being substantially straight for our purposes. The extrapolation is done by assuming a circular path defined as the closest approximation to the track between $A$ and $B$ (Fig. 2). The error in $f$ due to multiple scattering is calculated in the Appendix; for our choice ($\alpha=45°$ and $\beta=45°$), the probable error in $f$ is

$$\Delta f=0.722\ \Delta\rho \tag{18}$$

where $\Delta\rho$ is the probable error of the radius of curvature. Now $f$ is related to the angle $\chi_1$ by the equation

$$f=\rho(1-\cos\chi_1). \tag{19}$$

The mass $M$ of the primary particle, neglecting the electron mass in comparison with it, is given by

$$Mc=p_0\left[\frac{E+mc^2}{E-mc^2}\cos^2\chi-1\right]^{\frac{1}{2}}, \tag{20}$$

where $p_0$ is the momentum of the primary, $E$ is the energy of the electron including rest mass, and

$$\cos\chi=\cos\chi_1\cos\chi_2 \tag{21}$$

provided the primary particle path lies in the plane perpendicular to the magnetic field as it normally does. From Eqs. (17) and (20) we can deduce the uncertainty in $M$.

However, the error in the radius (and therefore in $E$) and that in the sagitta $f$ (and therefore in $\chi_1$) are related. If the actual radius in the magnetic field is greater than the apparent radius of the track between $A$ and $B$, the true path will have a greater sagitta than $f$ and therefore a smaller $\cos\chi_1$ than the extrapolated circle. This means that the energy $E$ is greater than estimated and $\cos\chi_1$ smaller than estimated. Both errors tend in the same direction of making the actual mass of the primary less than the estimated one. The correlation between the errors in radius and in $f$ is

$$(\Delta\rho\Delta f)_{\text{Av}}=0.570(\Delta\rho_1{}^2)_{\text{Av}}, \tag{22}$$

where $\Delta\rho_1$ is the error in $\rho$ without the correction factor (17).

The measurement $D$ is independent of $B$ and $C$. The slope of the track in the direction of the magnetic field is measured approximately at the same point as the curvature; i.e., between points $A$ and $B$ when the track is approximately at right angles to the primary. Multiple scattering will cause a mean square deflection which is given by Eq. (1) with $x=\pi\rho/2$.

Two measurements by Leprince-Ringuet and co-workers are of special importance.

*A*. The meson track reported in reference 8 which gave a mass of 240 electron masses. The magnetic field was $H=2650$ gauss, the observed radius of curvature of the electron 1.07 cm and for the meson 117 cm. The errors according to the above formulae are as follows:

a. The meson momentum has a probable error of 3 percent.

b. The electron momentum has a probable error of 8.4 percent. This causes a probable error in the expression $(E+mc^2)/(E-mc^2)$ of 8.6 percent.

c. The error in $f$ is $0.061\rho$; therefore the error in $\cos\chi_1$ is 0.061. Including the correlation, the error in the meson mass due to causes $b$ and $c$ is 15 percent.

d. The probable angle of scattering is 4.6° leading to a probable error in $\cos\chi_2$ of 2.4 percent and in the meson mass of 3.6 percent.

Altogether the probable error of the meson mass (adding the squares of the individual errors) is 16 percent. The error due to the measurement itself was given by the authors as 10 percent. A mass of 200, which is 17 percent lower than the mass of 240 resulting from the experiment, is therefore within the probable error.

*B*. In reference 16 a track is reported which gives a mass for the primary particle of about 1000 electron masses. It can easily be seen that this primary cannot be a meson of mass 200. We shall show, however, that the measurements are compatible with a primary proton. The individual errors are as follows:

a. The measurement of the momentum of the primary particle has a probable error of 5 percent because of multiple scattering. Taking the upper limit of the measured radius of curvature, and adding the probable amount of scattering, we get a maximum probable momentum for the primary of 580 Mev/$c$. A proton of this momentum will emit recoil electrons of a momentum

$$p=\frac{2mv\cos\chi}{1-(v^2/c^2)\cos^2\chi}=(0.755-0.670\chi^2)\frac{\text{Mev}}{c}. \quad (23)$$

d. The measured angle $\chi_2$ is 20° to 24°. If we assume an electron momentum of 0.75 Mev as given by Eq. (23), we find that the probable angle of scattering in a track length of a quarter circle is 7.5°. We, therefore, get a minimum probable value of $\chi_2=12.5°$. Using this angle in Eq. (23), we get

$$p=(0.723-0.670\chi_1^2)\ \text{Mev}/c. \quad (24)$$

c. The observed sagitta $f$ is between 0 and 0.07 cm. Therefore, it is consistent with the experiments to assume $\chi_1=0$. This gives for the maximum probable value of the electron momentum

$$p_m=0.723\ \text{Mev}/c. \quad (25)$$

b. The measured value of the electron momentum is between 1.09 and 1.16 Mev/$c$. If we assume that the actual momentum is 0.72 Mev multiple scattering according to Eq. (13) will give a probable error for the radius of curvature of 14 percent. Since a 90° arc is used for determination of the radius of curvature, the error due to scattering should, according to Eq. (17), be increased by a factor 1.053 to 15 percent. The radius of curvature expected for an electron from a primary proton is 32 percent less than the observed radius of curvature. This difference is 2.1 times the probable error due to multiple scattering and is, therefore, statistically quite probable.

We see, therefore, that the observed track can be reconciled with the assumption that it is made by a primary proton. It is true that we had to assume all errors on the same side. However, the error in the radius of curvature of the electron was only 2.1 times the probable error due to multiple scattering. It is also true that a proton of a momentum of 580 Mev should ionize about twice as strongly as a particle at the minimum of the ionization curve, and the observed track appears not to have an ionization significantly above the minimum. But it is generally admitted that estimates of ionization without droplet counting are qualitative and often incorrect.

Even though considerable stretching of the probable errors was necessary, we still believe that this particle is most likely a proton. It seems to us that an entirely new particle could only be established by much more than one event and by measurements of much smaller probable error.

## 6

These examples show that multiple scattering disturbs very seriously all measurements on low energy particles in the cloud chamber. It leaves reasonably unaffected all measurements on particles of high velocity. The best methods to determine the meson mass by cloud-chamber techniques therefore seem to be:

a. Measurement of curvature and droplet counting for tracks of reasonably high velocity ($\beta \approx 0.5$).

b. Measurement of the change of magnetic curvature upon penetration of a (reasonably thick) solid plate, as discussed in detail by Wheeler and Ladenburg.[4]

c. Measurement of curvature, followed by measurement of the penetration through a set of thin solid plates. This method was used successfully by Baldwin and Klaiber[7] on the protons obtained by photo-disintegration of nuclei by $\gamma$-rays from the betatron, and by Fretter and Brode[17] on cosmic-ray mesons.

In all cases, the magnetic field should be high (at least 3000 gauss) and the cloud-chamber gas light (hydrogen or helium).

## 7

Our analysis of some of the direct measurements of the meson mass show that all those data which appear to give a very low mass are subject to error due to scattering. There is not a single measurement which cannot be reconciled with a unique mass in the neighborhood of 200 electron masses.

Indirect arguments[18] have also been given for the existence of mesons of low mass. These are mainly based on some evidence that there are very many low energy mesons in cosmic radiation at high elevations, and that very few mesons of high ionization appear in cloud-chamber photographs. However, the latter point is really quite understandable: In order to be easily recognized, a "heavy" track should have an ionization at least twice, and preferably four times, that of an "average" particle, i.e., of a high energy meson or of an electron of, say, 50 Mev. The ionization of the latter is about 30 percent higher than the minimum ionization.* A meson of twice this ionization must have a momentum $p<0.65\mu c$ and a kinetic energy below $0.15\mu c^2$. With a mass of 200 electron masses, its range in air or A is less than 3 g/cm$^2$. Therefore only about one percent of all mesons of energy below $6\cdot 10^8$ ev should have an ionization greater than twice that of a fast electron. If four times normal ionization is required for clear recognition, we must have $p<0.40\mu c$, $E_{\text{kin}}<0.08\mu c^2$, range$<0.6$ g/cm$^2$, so that only 0.2 percent of all mesons below $6\cdot 10^8$ ev will qualify.

As regards the number of slow mesons from counter experiments, this is very difficult to obtain below $3\cdot 10^8$ ev. Undoubtedly, in some analyses[19] many electrons have been considered as slow mesons. There does not seem to be much experimental basis for the (frequently made) assumption that, at about 4000 meters elevation, the number of mesons per unit energy interval increases rapidly below $3\cdot 10^8$ ev.

We, therefore, do not believe that there is any real evidence for an unexpectedly small number of mesons of observably high ionization. Even if there were, we should consider it more likely that this is caused by an unknown absorption of slow mesons (about whose properties, after all, we know very little) than to a spread in mass.

This brings us to a more philosophical point: All known fundamental particles have a definite rest mass. To discover a particle for which this is not the case would be a tremendous deviation from previous experience. For this reason the most unambiguous and accurate experimental evidence would have to be obtained before a non-unique rest mass could be accepted. It is obviously very easy to explain each new experiment by a new assumption, and this was done in the last few years in the case of the meson mass: It is much more difficult to explain all experiments by a minimum of assumptions, which was the way in which physics has progressed in the past. The burden of proof lies always with the discoverer of a *new* phenomenon, and must in this case lie with the advocates of the variable rest mass of the meson.

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[17] W. B. Fretter and R. B. Brode, Bull. Am. Phys. Soc. (September, 1946), Abstract C17.

[18] M. Schein, Carnegie Inst. of Wash. Yearbook **43**, 61 (1944).

* For this and the following, see reference 11, Figs. 1 to 4.

[19] E.g., P. V. Auger, Phys. Rev. **61**, 684 (1942).

## APPENDIX

### *Discussion of Tracks of Considerable Curvature*

In connection with the experiments of Leprince-Ringuet (Section 5, subsection 6), the problem of measuring electron tracks of considerable curvature arises. We shall assume in close accordance with the measuring procedure that the "best circle" is fitted to a section $AB$ of the track (see Fig. 2). This amounts essentially to measuring the distance $AB=2a$ and the length of the sagitta $s$ of the curve $AB$. If $AB$ were a circle, its radius would be

$$\rho = a^2/2s + s/2. \tag{51}$$

This equation will be used as the definition of the measured radius of the curve.

Let us assume that the magnetic field alone would deflect the electron by an angle $2\alpha$ over the distance from $B$ to $A$, and that the radius of curvature in the magnetic field alone is $\rho_0$. Then, the track length from the point $P$ to either $A$ or $B$ will be nearly $\rho_0\alpha$. The actual deflection from $P$ to $A$ will be $\alpha+\theta_1$, where the probable value of $\theta_1$ is calculated from Eq. (1) with $x=\rho_0\alpha$. The lateral displacement of the track with respect to the circle of radius $\rho_0$ and the point $A$ is given by Eq. (3) with $x=\rho_0\alpha$.

Figure 3 illustrates the situation. $BPA$ now represents the circle in the magnetic field alone. The lateral displacements at $A$ and $B$ are assumed to be $y_1$ and $y_2$, respectively. In order to have the same ordinates for the end points of the actual track, we increase the track length by $z_1$ and $z_2$, respectively, at its two ends giving the new end points $A'$ and $B'$ for the actual track. The vertical displacements of $A'$ with respect to $A$ and of $B'$ with respect to $B$ are then

$$\begin{aligned} V_1 &= y_1 \cos\alpha + z_1 \sin\alpha, \\ V_2 &= y_2 \cos\alpha + z_2 \sin\alpha. \end{aligned} \tag{52}$$

The condition that these be equal determines $z_1-z_2$; otherwise the $z$'s are arbitrary. It is a welcome check on the calculations to ascertain at each stage that the significant results are independent of $z_1+z_2$.

The sagitta $s$ of the "observed" track $B'PA'$ is now

$$s=\rho_0(1-\cos\alpha)+Y\cos\alpha+Z\sin\alpha, \tag{53}$$

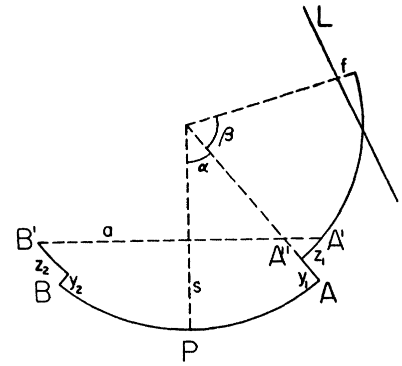


FIG. 3. Mass determination by the method of recoil electrons. $BPA$ is a section of the circle which the particle would describe in the magnetic field alone. The actual path goes from $B'$ *via* $P$ to $A'$. Its continuation beyond $A'$ is shown up to a point where the path is parallel to the straight line $L$ which represents the path of the primary.

where

$$Y=\tfrac{1}{2}(y_1+y_2), \quad Z=\tfrac{1}{2}(z_1+z_2). \tag{54}$$

The cord $A'B'$ is

$$2a=2\rho_0\sin\alpha-2Y\sin\alpha+2Z\cos\alpha. \tag{55}$$

Using Eq. (51) we find for the radius, after some algebra

$$\rho=\rho_0-\frac{y_1+y_2}{2(1-\cos\alpha)}. \tag{56}$$

It will be noted that this result is independent of $Z$.

The error in the measured curvature is now

$$\Delta\left(\frac{1}{\rho}\right)=\frac{y_1+y_2}{2\rho_0^2(1-\cos\alpha)}, \tag{57}$$

whereas the simple formula (6) would give

$$\Delta\left(\frac{1}{\rho}\right)=\frac{y_1+y_2}{x^2}=\frac{y_1+y_2}{\rho_0^2\alpha^2}. \tag{58}$$

Equation (57) becomes identical with (58) in the limit of small $\alpha$; in general the error given by the exact formula (57) is greater by a factor

$$F(\alpha)=\frac{\alpha^2}{2(1-\cos\alpha)}. \tag{59}$$

This result has been used in Eq. (17).

We shall now consider the error in the sagitta $f$ between the continuation of the track $B'A'$ and a straight line $L$ which makes an angle $\alpha+\beta$

with the cord $B'A'$. For this purpose we first calculate the direction of the track at $A'$. If we had no multiple scattering, we would have the circle $BPA$ whose direction at $A$ would be $\alpha$ (with respect to the tangent to the circle at $P$ which is parallel to $B'A'$ by construction). Actually, because of multiple scattering, the direction of the actual track at $A''$ will be $\alpha+\theta_1$, where $\theta_1$ is calculated from Eq. (1). Moreover, we have assumed that the length of the actual track is greater than that of the ideal circle by the amount $z_1$; this causes an additional deflection $z_1/\rho_0$. The actual direction of the track at $A'$ is therefore $\alpha+\theta_1+(z_1/\rho_0)$. On the other hand, in the measurement it is assumed that $B'PA'$ is a circle. From the sagitta $s$ and the cord $2a$ of this circle, one would conclude that the direction at $A'$ is $\alpha'$ where

$$\tan\frac{\alpha'}{2}=\frac{s}{a}=\tan\frac{\alpha}{2}+\frac{Y}{\rho_0\sin\alpha}+\frac{Z}{\rho_0(1+\cos\alpha)}. \quad (60)$$

The actual curve deviates from the fitted circle in direction by the amount

$$\delta=\alpha+\theta_1+z_1/\rho_0-\alpha' =\theta_1-\frac{y_2+y_1(1+2\cos\alpha)}{2\rho_0\sin\alpha}. \quad (61)$$

This expression is again independent of $Z$, Eq. (54).

We now proceed in two steps. First we consider a circle of the ideal radius $\rho_0$ which starts from $A'$ in the actual direction of the path. Then the actual path will be displaced with respect to this "ideal circle" by a lateral displacement $y_3$ which can be calculated from Eqs. (1) and (3) with the track length $x=\rho_0\beta$. This displacement is statistically independent of the displacement between the "ideal circle" and the assumed circle which we shall now calculate.

The angle between these two circles is given by Eq. (61). The radius of the ideal circle is $\rho_0$, for the assumed circle $\rho$. A simple calculation shows that the distance between the two circles at an angle $\beta$ from their common starting point $A'$ is

$$\rho\delta\sin\beta+(\rho_0-\rho)(1-\cos\beta). \quad (62)$$

Inserting Eq. (56) for $\rho-\rho_0$ and (61) for $\delta$, and adding $y_3$, we get for the change in the sagitta $f$ (measured $f$ minus true $f$)

$$\Delta f=\rho_0\theta_1\sin\beta -y_1\left(\frac{1+2\cos\alpha}{2\sin\alpha}\sin\beta+\frac{1-\cos\beta}{2(1-\cos\alpha)}\right) -y_2\left(\frac{\sin\beta}{2\sin\alpha}+\frac{1-\cos\beta}{2(1-\cos\alpha)}\right)+y_3. \quad (63)$$

In (63) the quantities $y_1$, $y_2$, and $y_3$ are statistically independent of each other but there is a correlation between $y_1$ and $\theta_1$. According to a theory of Fermi (see reference 11, page 265), the correlation is given by

$$(y_1\theta_1)_{\mathrm{Av}}=\tfrac{1}{2}x(\theta_1^2)_{\mathrm{Av}}. \quad (64)$$

Using also Eq. (3), we can find the mean square error for the sagitta $f$.

The change of the sagitta and the change of the radius are correlated, cf. Eqs. (56), (63), (64).

For the actual evaluation we shall make the simplifying assumption that $\alpha=\beta=45°$. Then the factor (59) in the probable error for the radius is 1.053. The error in the sagitta becomes from (63)

$$\Delta f=\frac{4}{\pi\sqrt{2}}x\theta_1-\left(1+\frac{1}{\sqrt{2}}\right)y_1-y_2+y_3. \quad (65)$$

Evaluation gives

$$(\Delta f^2)_{\mathrm{Av}}=2.735(y_1^2)_{\mathrm{Av}}=0.520(\Delta\rho_1)^2_{\mathrm{Av}}, \quad (66)$$

where $\Delta\rho_1$ is the error in radius calculated from Eq. (10). The correlation between $\rho$ and $f$ has the value

$$(\Delta f\Delta\rho)_{\mathrm{Av}}=(\tfrac{3}{2}+\sqrt{2})(y_1^2)_{\mathrm{Av}}-(\sqrt{2}+1)(y_1\rho_0\theta_1)_{\mathrm{Av}} +(1+1/\sqrt{2})(y_2^2)_{\mathrm{Av}}=0.570(\Delta\rho_1)^2_{\mathrm{Av}}. \quad (67)$$