From (2.9), (2.11), (2.14) we find $e^c=(m/2a^3)^{\frac{1}{2}}$. Similarly the values of A and B can be found to be

$$A=(m/2a^3)^{\frac{1}{2}}(4a-m)/(2a+m), \quad (2.16)$$

$$B=(2a^3/m)^{\frac{1}{2}}(2a-2m)/(2a+m). \quad (2.17)$$

As our final solution for the gravitational potentials we obtain

$$e^{\mu}=(1+m/2a)^6/(1+mr^2/2a^3)^2, \quad (2.18)$$

$$e^{\nu}=[2a-2m+m(4a-m)r^2/2a^3]^2/[(2a+m)(1+mr^2/2a^3)]^2. \quad (2.19)$$

III. CONCLUSION

In the Schwarzschild solution it was found[4] that a sphere of given density is bounded in mass and size by $m\leqslant 4a/9$, $a^2\leqslant 8R^2/9$. In our solution these bounds are smaller and are given by $m\leqslant 0.4a$, $a^2\leqslant 0.27R^2$. Thus an observer using an isotropic coordinate system would find smaller upper bounds than an observer using a coordinate system in which the line element takes the form (1). This shows that these upper bounds are definitely a property of the coordinate system used by the observer, and it is conceivable that coordinate systems may exist in which these upper bounds may be infinite.

Having obtained the interior solution it is easy to show that the complete solution is mathematically equivalent to the Schwarzschild solution. It is already known that the transformation

$$\bar{r}=(1+m/2r)^2r, \; r\geqslant a$$

takes the isotropic exterior solution into the Schwarzschild exterior solution. The transformation

$$\bar{r}=(1+m/2a)^3r/(1+mr^2/2a^3), \; 0\leqslant r\leqslant a$$

does the same for the interior solution. Moreover the two transformations piece together continuously at the boundary $r=a$.

[4] A. S. Eddington, reference 2, p. 170.

PHYSICAL REVIEW VOLUME 70, NUMBERS 1 AND 2 JULY 1 AND 15, 1946

Forced Vibrations of Piezoelectric Crystals

H. EKSTEIN
Armour Research Foundation, Chicago, Illinois
(Received March 6, 1946)

The vibrations of anisotropic bodies under the influence of sinusoidally variable volume forces and boundary stresses are investigated. The displacement components are represented as sums of a system of "zero-order" solutions which solve approximately the free-vibration problem. By using Betti's theorem, the problem is reduced to a system of inhomogeneous linear equations which, for the free-body case, further reduces to the homogeneous system derived in an earlier paper (reference 2). If the external forces are piezoelectric, the forces are no longer given explicitly because the electrical field distribution is known only if Maxwell's equations are solved simultaneously. However, if the pertinent piezoelectric constants are small, the field can be calculated approximately as if the crystal were not vibrating. The solutions can then be obtained by the above method, and the electric reaction of the crystal upon the driving system can be determined. As an example, forced vibrations of thin quartz plates between parallel electrodes are discussed.

I. INTRODUCTION

THE rigorous solution of vibration problems meets such great mathematical difficulties that even in comparatively simple cases only approximation methods can be used. The analogy with scalar vibration problems of the Schrödinger type suggests that the solution U_i might be represented as a linear combination of "zero-order" functions $u_i^{(m)}$. This method is straightforward if the $u_i^{(m)}$ satisfy the boundary conditions of the problem. But in most cases, the convenient system $u_i^{(m)}$ does not satisfy the

given boundary conditions. In this case, the sum

$$U_i = A_m u_i^{(m)}$$

cannot be differentiated term by term after being inserted in the differential equation of the problem. In the following, the self-adjoint property of the differential operator, as stated by Betti's theorem, is used to avoid this difficulty. It is then only necessary to assume that an infinite series involving $u_i^{(m)}$ may be integrated term by term, a condition which imposes fewer restrictions upon the series. In this manner, the coefficients A_m can be calculated after a secular equation has been solved. The usual methods of perturbation theory can be used to obtain approximate solutions.

The piezoelectric excitation of vibrations presents a more complicated problem. The stress depends upon the electric field which in turn depends upon the strain. It would be necessary to solve Maxwell's equations simultaneously with the elastic equations. The following approximation method can be used when the piezoelectric constants are small, as in most cases: the electric field is put equal to the external field (in absence of the crystal) so that the problem reduces to the case of given surface and volume stresses. Once this problem is solved, e.g., by the perturbation method described above, the polarization can be expressed in terms of the known strains. With this value of the polarization, the electric field equations are integrated again. As a result, one obtains the electric reaction of the crystal upon the driving system. Successive higher approximations can be obtained by inserting the improved values of the electric field into the elastic equation, etc.

It is convenient to represent the reaction of the crystal upon the driving system by an equivalent electric network. It will be shown that this is possible if the network contains a series of parallel resonant circuits and a series of resonant circuits which are coupled inductively and capacitatively to the parallel circuits. The values of the circuit elements can be expressed completely in terms of the crystal constants. This equivalence will be discussed for a particular case.

Very little is known about the energy dissipation in crystals. Although Voigt has developed a formal theory of friction, in which the frictional stresses are proportional to the rate of change of the strains, experiments have shown that the process is more complicated. In the case of quartz, e.g., the friction depends mainly upon the minute surface irregularities[1] rather than upon volume constants. Under these conditions, it seems preferable to renounce an introduction of damping in the general equations and to insert simply a damping constant into the final formulas, or to insert a resistance into each resonant circuit of the equivalent electrical network.

II. FORCED VIBRATIONS WITH GIVEN EXTERNAL FORCES

In this section the forced vibrations of a non-piezoelectric crystal, driven by sinusoidally variable volume forces and boundary stresses, are treated. Only cartesian coordinates are used, so that we do not need to distinguish between covariant and contravariant tensors.

In the steady state, all mechanical quantities have a time-part $\exp(i\omega t)$ where $\nu = \omega/2\pi$ is the frequency of the driving forces. In the following, this time-part will be omitted. Let U_i be the displacement components, and

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{1}$$

the strain tensor. With the usual convention on summation, the stress tensor is

$$T_{ij} = -c_{ijkl} S_{kl} = -c_{ijkl}\partial U_k/\partial x_l, \tag{2}$$

where

$$c_{ijkl} = c_{jikl} = c_{jilk} = c_{ijlk} = c_{klij}. \tag{3}$$

The dynamical equations are:

$$f_i^{(e)} + \rho\omega^2 U_i = \frac{\partial T_{ij}}{\partial x_j} = -c_{ijkl}\frac{\partial^2 U_k}{\partial x_j \partial x_l}, \tag{4}$$

where $f_i^{(e)}$ are the components of the volume force. If $\varphi_i^{(e)}$ are the components of the boundary stresses, then the boundary conditions are:

$$T_{ij} n_j = -n_j c_{ijkl}\partial U_k/\partial x_l = \varphi_i^{(e)} \tag{5}$$

where n_j are the components of the normal to the surface, directed inward. The solution U_i is now represented as a linear combination of

[1] K. S. Van Dyke, Proc. I.R.E. **23**, 386 (1935).

functions $u_i^{(m)}$.

$$U_i = A_m u_i^{(m)}, \tag{6}$$

where in general the number of functions $u_i^{(m)}$ will be infinite. It is not required that $u_i^{(m)}$ satisfy any specific boundary conditions. Practically, one will choose solutions of a reasonably similar but sufficiently simple problem for $u_i^{(m)}$.

We multiply Eq. (4) by $u_i^{(n)}$, sum over the index i, and integrate over the crystal volume:

$$\int u_i^{(n)}(f_i^{(e)} + \rho\omega^2 U_i)dv = -\int u_i^{(n)} c_{ijkl} \frac{\partial^2 U_k}{\partial x_j \partial x_l} dv. \tag{7}$$

Through integration by parts, the second member becomes

$$-\int u_i^{(n)} c_{ijkl} \frac{\partial^2 U_k}{\partial x_j \partial x_l} dv = \int u_i^{(n)} n_j c_{ijkl} \frac{\partial U_k}{\partial x_l} ds + \int c_{ijkl} \frac{\partial u_i^{(n)}}{\partial x_j} \frac{\partial U_k}{\partial x_l} dv, \tag{8}$$

the first integral on the right-hand side being extended over the surface.

The second integral of the second member of Eq. (8) does not change when the functions $u_i^{(n)}$ and U_i are interchanged, on account of the symmetry of c_{ijkl}. Therefore

$$\begin{aligned} -\int c_{ijkl} \frac{\partial u_i^{(n)}}{\partial x_j} \frac{\partial U_k}{\partial x_l} dv &= \int u_i^{(n)} c_{ijkl} \frac{\partial^2 U_k}{\partial x_j \partial x_l} dv + \int u_i^{(n)} n_j c_{ijkl} \frac{\partial U_k}{\partial x_l} ds \\ &= \int U_i c_{ijkl} \frac{\partial^2 u_k^{(n)}}{\partial x_j \partial x_l} dv + \int U_i n_j c_{ijkl} \frac{\partial u_k^{(n)}}{\partial x_l} ds. \end{aligned} \tag{9}$$

Equation (9) expresses Betti's theorem.

Equation (7) may now be written

$$\int u_i^{(n)}(f_i^{(e)} + \rho\omega^2 U_i)dv + \int U_i c_{ijkl} \frac{\partial^2 u_k^{(n)}}{\partial x_j \partial x_l} dv + \int U_i n_j c_{ijkl} \frac{\partial u_k^{(n)}}{\partial x_l} ds + \int u_i^{(n)} \varphi_i^{(e)} ds = 0. \tag{10}$$

In the last integral Eq. (5) has been used.

We substitute Eq. (6) for U_i and assume that the integration can be carried out term by term. If we define

$$N_{nm} = N_{mn} = \int u_i^{(n)} u_i^{(m)} dv, \tag{11}$$

$$-H_{nm} = -H_{mn} = \int u_i^{(m)} c_{ijkl} \frac{\partial^2 u_k^{(n)}}{\partial x_j \partial x_l} dv + \int u_i^{(m)} n_j c_{ijkl} \frac{\partial u_k^{(n)}}{\partial x_l} ds, \tag{12}$$

$$F_n = \int f_i^{(e)} u_i^{(n)} dv + \int \varphi_i^{(e)} u_i^{(n)} ds, \tag{13}$$

then Eqs. (10) and (6) yield

$$A_m(\rho\omega^2 N_{mn} - H_{mn}) + F_n = 0. \tag{14}$$

The fact that $H_{mn} = H_{nm}$ in Eq. (12) results from Betti's theorem (Eq. (9)).

In the case of free vibrations, $F_n = 0$, and Eq. (14) becomes a homogeneous system of linear equations which has non-vanishing solutions only for certain characteristic values of $\rho\omega^2$. This case has been discussed in a previous paper.[2]

Let us suppose that the functions $u_i^{(m)}$ are exact solutions of the free problem where $f_i^{(e)} = \varphi_i^{(e)} = 0$. Then it can be proved that the "non-orthogonality" integrals N_{nm} vanish for $n \neq m$.[3] The non-diagonal elements H_{nm} also

[2] H. Ekstein, Phys. Rev. 66, 108 (1944). Further referred to as I.

[3] A. E. H. Love, *Mathematical Theory of Elasticity* (Dover Publications, New York, 1944), p. 180. This is not necessarily true for degenerate solutions but it can be so postulated.

vanish in this case, because by Eq. (4) (for $f_i^{(e)}=0$) the volume integral in Eq. (12) becomes proportional to N_{nm} and, therefore, vanishes, while the surface integral in Eq. (12) vanishes in virtue of Eq. (5) (for $\varphi_i^{(e)}=0$). Then Eq. (14) reduces to

$$A_n(\rho\omega^2 N_{nn}-H_{nn})+F_n=0, \tag{15}$$

or

$$A_n=-\frac{F_n}{\rho\omega^2 N_{nn}-H_{nn}}. \tag{16}$$

By Eq. (6) we have

$$U_i=\frac{-F_n u_i^{(n)}}{\rho\omega^2 N_{nn}-H_{nn}}, \tag{17}$$

which is, of course, equal to the solution of the analogous scalar problem. If we want to take into account the effect of energy dissipation, we simply add an imaginary term to the denominator:

$$U_i=\frac{-F_n u_i^{(n)}}{\rho\omega^2 N_{nn}-H_{nn}+i\eta_n}, \tag{18}$$

where η_n may be a function of the driving frequency, but not necessarily proportional to it. In any case, when the dissipation is small, only one term of the series (18) will be important, *viz.* the term for which

$$\rho\omega^2\cong H_{nn}/N_{nn}. \tag{19}$$

On the other hand, if the functions $u_i^{(n)}$ are not exact solutions of the "free" problem under consideration, but solutions of a similar but simpler problem, then the above statements are, in general, true approximately, except when several $u_i^{(n)}$ are degenerate. One case of this particular type was discussed in I.

III. PIEZOELECTRIC EXCITATION OF MECHANICAL VIBRATIONS AND REACTION UPON THE DRIVING SYSTEM

When a piezoelectric crystal is placed between two electrodes with sinusoidally variable potential difference, it is excited to mechanical vibrations. It will be assumed that the crystal is otherwise free from mechanical forces. According to Voigt, the general equations of piezoelectricity are

$$-T_{ij}=c_{ijkl}S_{kl}-e_{ijk}E_k, \tag{20}$$

$$P_i=e_{kli}S_{kl}+\kappa_{ik}E_k. \tag{21}$$

In these equations, E_k are the components of the electric field intensity, e_{ijk} the piezoelectric constants, κ_{ik} the (clamped) susceptibilities, and P_i the components of the electric polarization.

We consider again the steady state, where all variables have a time-dependent factor exp $(i\omega t)$ which will be omitted in the following equations.

The dynamical equations are again

$$\rho\omega^2 U_i=\partial T_{ij}/\partial x_j, \tag{4a}$$

as in Eq. (4) in absence of external mechanical forces, and the boundary conditions are

$$T_{ij}n_j=0. \tag{5a}$$

These equations do not yet fully determine the problem, because the field intensity is in turn dependent upon the strains.

In the absence of true charges, we have

$$\frac{\partial D_i}{\partial x_i}=\frac{\partial(E_i+4\pi P_i)}{\partial x_i}=0, \tag{22}$$

and

$$\text{curl } \mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{23}$$

We shall consider only comparatively low frequencies so that the second member in Eq. (23) may be disregarded. We then have the equation of the static field

$$\partial E_i/\partial x_j-\partial E_j/\partial x_i=0, \tag{24}$$

and the field intensity can be represented as a potential gradient:

$$-E_i=\partial V/\partial x_i. \tag{25}$$

On the conducting electrodes

$$V=\text{constant}, \tag{26}$$

$$E_t=0, \tag{27}$$

$$D_n=4\pi q, \tag{28}$$

where q is the surface charge density, D_n the normal component of the displacement, and E_t the tangential component of the field intensity.

According to the method discussed in the introduction, we first calculate the field as if the crystal were absent, i.e., we integrate the electrostatic problem defined by Eqs. (22), (24) to (26) as if P_i were zero. In this manner, we obtain a field distribution described by the function E_{oi}.

TABLE I. Comparison of notation.

Voigt	x	y	z	x_x	y_y	z_z	y_z	x_z	x_y
This paper	x_1	x_2	x_3	S_{11}	S_{22}	S_{33}	$2S_{23}$	$2S_{13}$	$2S_{12}$
Voigt	X_x	Y_y	Z_z	Y_z	X_z	X_y	u	v	w
This paper	T_{11}	T_{22}	T_{33}	T_{23}	T_{13}	T_{21}	U_1	U_2	U_3

TABLE II. Correlation of subscripts.

Voigt	1	2	3	4	5	6
This paper	11	22	33	23 or 32	13 or 31	12 or 21

Inserting this into Eqs. (2) and (4a) and taking into account the definition of S_{kl} (Eq. (1)) we obtain

$$-e_{ijk}\frac{\partial E_{ok}}{\partial x_j}+\rho\omega^2 U_i=-c_{ijkl}\frac{\partial^2 U_k}{\partial x_j \partial x_l}, \tag{29}$$

and by Eq. (5a)

$$n_j c_{ijkl}\frac{\partial U_k}{\partial x_l}=n_j e_{ijk}E_{ok}. \tag{30}$$

Equations (29) and (30) are analogous to Eqs. (4) and (5) which can be solved by the method discussed in Section II. Let the solution thus obtained be U_{oi}. If this is inserted into Eq. (21) we obtain

$$P_{1i}=e_{kli}\frac{\partial U_{ok}}{\partial x_l}+\kappa_{ik}E_{1k} \tag{31}$$

as a first approximation for the polarization. E_{1k} are here the components of the field intensity in first approximation, which can be determined by integrating the electrostatic Eqs. (25) to (27) together with Eq. (31). When this is done, Eq. (28) yields the first approximation of the charge density, i.e., the reaction of the crystal upon the driving system.

This method follows closely the one which Voigt uses for the static problems of piezoelectricity. The same method has been used by Cady[4] for a particular vibration problem in piezoelectricity.

It seems necessary to establish the correspondence between the abridged notation used in this paper and Voigt's notation (Table I).

In order to correlate the constants c_{ijkl} used here with the constants c_{mn} used by Voigt, it suffices to correlate each pair of subscripts (ij) and (kl) with one of Voigt's subscripts n and m, respectively. Table II gives this correlation. e.g., our constant $c_{1232}=c_{1223}$ equals Voigt's constant $c_{64}=c_{46}$. The physical definitions are identical with those of Voigt, e.g., the stresses are positive when they are compressive.

The piezoelectric constants e_{ijk} must satisfy the following symmetry requirements:

$$e_{ijk}=e_{jik}, \quad e_{iik}=e_{kki}, \tag{32}$$

if they are to agree with Voigt's piezoelectric constants e_{nm}. The correlation between our constants e_{ijk} and Voigt's e_{nm} is found by replacing the first pair of subscripts (ij) by Voigt's corresponding subscript n according to Table II; the third subscript k of our notation is identical with Voigt's second subscript m, e.g.,

$$e_{123}=e_{63}, \quad e_{232}=e_{42}.$$

IV. THIN QUARTZ PLATES OF THE Y_θ TYPE

The general method will be applied to the case of thin rectangular quartz plates which have one of their major edges parallel to the crystallographic twofold axis. The AT and BT crystals on which most of the experimental work concerning coupling of modes has been done, belong to this type. The crystals under consideration have metal coating on both major surfaces but otherwise they are free. The weight of this metal will be disregarded.

In this section the explicit notation of Voigt will be used rather than the abridged notation of the preceding sections. The edges will be assumed to be parallel to the x, y, z direction and to have the lengths $2a$, $2b$, $2c$, respectively, and the thickness $2b$ to be small as compared to either $2a$ or $2c$. The origin of the system of reference is at the center of the plate.

The first step is to select the zero-order modes. Of course, this selection is largely arbitrary, but the nature of the problem suggests the convenient choice. As pointed out by Atanasoff and Hart[5] the vibrational spectrum of thin plates becomes simple as the lateral dimensions tend toward infinity. The complex spectrum actually found

[4] W. G. Cady, Physics **7**, 237 (1936).

[5] F. V. Atanasoff and P. F. Hart, Phys. Rev. **59**, 85 (1941).

on thin plates can be attributed to the existence of the small lateral surfaces. It is logical, therefore, to consider as zero-order functions those which satisfy the *volume equations of free vibrations (Eq. (4) with $f_i^{(e)}=0$) and the boundary conditions (5) of free vibrations on the major, but not on the lateral surfaces.* These zero-order functions will be closer to exact solutions as the lateral surfaces are smaller in comparison to the major ones.

It has been shown in a previous paper[6] that, if the solution is assumed under the form

$$u, v, w = \left.\begin{matrix}\cos \alpha x \\ \sin \alpha x\end{matrix}\right\} f(y), \tag{33}$$

it is possible to satisfy the above-mentioned requirements with elementary functions of y. Furthermore, it is possible to satisfy approximately the boundary conditions of free vibrations on the lateral boundary, by convenient choice of α.

One particular type of zero-order mode was treated explicitly in (II), namely,

$$u_t = \begin{matrix}\cos \dfrac{m\pi}{2a} x \sin \dfrac{n\pi}{2b} y + \cdots & (m \text{ odd}, n \text{ odd}) \\ \sin \dfrac{m\pi}{2a} x \sin \dfrac{n\pi}{2b} y + \cdots & (m \text{ even}, n \text{ odd})\end{matrix} \tag{34}$$

and the components v and w are proportional to b/a. The frequency belonging to this mode is

$$\nu_t = \frac{1}{2}\left(\frac{c_{66}}{\rho}\right)^{\frac{1}{2}}\left[\left(\frac{n}{2b}\right)^2 + k\left(\frac{m}{2a}\right)^2\right]^{\frac{1}{2}}. \tag{35}$$

If a tends towards infinity, this "thickness shear mode" becomes identical with the thickness-shear solution for an infinite plate

$$u = \sin \frac{n\pi}{2b} y, \quad v = w = 0 \qquad \text{with } \nu = \frac{1}{2}\left(\frac{c_{66}}{\rho}\right)^{\frac{1}{2}} \frac{n}{2b}. \tag{36}$$

From the equations given in (II) the existence of other types of zero-order modes of the form (33) could be inferred. For these "lateral" modes the frequency is mainly determined by the dimensions a and c, and the wave number α is of the order of magnitude of the inverse thickness $1/2b$.

We shall apply the method developed above to this case. If the presence of the crystal is disregarded, the electric field between the two plane-parallel electrodes is

$$E_{oy} = -V/2b, \quad E_{ox} = E_{oz} = 0, \tag{37}$$

disregarding, as usual, the small curvature of the lines of force in the neighborhood of the lateral edges.

According to Eq. (29) the dynamical equation is that of the free body:

$$\rho\omega^2 U_i = -c_{ijkl}\frac{\partial^2 U_k}{\partial x_j \partial x_l}. \tag{38}$$

On the major surfaces, we have $n_1 = n_3 = 0$ and $n_2 = \mp 1$ for $y = \pm b$, so that Eq. (30) becomes:

$$-n_j c_{ijkl}\frac{\partial U_k}{\partial x_l} = \pm e_{i22} E_{oy}. \tag{39}$$

As the subscript i runs from 1 to 3, Voigt's corresponding subscript n takes the values 6, 2, and 4, so that the piezoelectric constants in Eq. (39) are e_{62}, e_{22}, and e_{42}. But according to Voigt, the constants e_{22} and e_{24} vanish, since the x axis coincides with a twofold axis. Therefore, we have

$$-n_j c_{ijkl}\frac{\partial U_k}{\partial x_l} = \pm e_{62} E_{oy}, \tag{40}$$

$$-n_j c_{2jkl}\frac{\partial U_k}{\partial x_l} = -n_j c_{3jkl}\frac{\partial U_k}{\partial x_l} = 0.$$

If we introduce the symbols $\varphi_i^{(e)}$ of Eq. (5) for the first members of Eq. (39), we obtain

$$\varphi_1^{(e)} = \pm e_{62} E_{oy}, \quad \varphi_2^{(e)} = \varphi_3^{(e)} = 0. \tag{41}$$

The quantities F_n defined by Eq. (13) are:

$$F_p = e_{62} E_{oy} \int [u_p(b) - u_p(-b)]\, dx\, dz, \tag{42}$$

where p stands for one of the zero-order modes. According to the discussion in Section 2, only those zero-order modes whose frequencies are near the driving frequency $\omega/2\pi$, will be of importance. We shall discuss here the case where

[6] H. Ekstein, Phys. Rev. **68**, 11 (1945). Further referred to as II.

one of the high frequency shear modes is excited, i.e., when ν is close to one of the values given by Eq. (35). When the index p in Eq. (42) refers to one of the "lateral" modes, then the quantity α in Eq. (33) is of the order of $1/b$ so that u changes its sign many times between $x=-a$ and $x=+a$. Therefore, most of the integral (42) will cancel out, and these quantities F_p will be very small. The "lateral" modes are only slightly "driven" by the exciting field. These values F_p will be disregarded.

In general, F_p vanishes for each mode u_p which is an even function of y or an uneven function of x or z, as can be seen from Eq. (42). Therefore, the thickness-shear modes u_l with m even, according to Eq. (34), are not "driven."

For the thickness modes with m and n odd, we find by Eqs. (34) and (42):

$$F_p=e_{62}E_{oy}2\cdot 2c\int_{-a}^{+a}\cos\frac{m\pi}{2a}x\,dx=\frac{16}{m\pi}cae_{62}E_{oy}. \quad (43)$$

If all the zero-order functions are chosen, the coefficients of Eq. (14) are known. After solving for A_p, the solution appears under the form (6):

$$U_i=A_pu_i^{(p)}.$$

We now have to calculate the charge q on the electrodes by Eq. (31). The quantities U_{ok} there are identical with the approximate solution used here. Because of the linear character of all the equations, we can determine separately the contribution to q of the applied voltage V and those of the zero-order modes $u_i^{(p)}$.

The contribution of the voltage V alone, disregarding the crystal motion, to the surface charge density is the same as for a non-piezoelectric crystal plate in a condenser:

$$q_v=(1+4\pi\kappa_{22})\frac{V}{8\pi b} \quad (44)$$

on the plane $y=b$ and the opposite at $y=-b$. The total charge due to the applied voltage is

$$Q_v=\int q\,dx\,dz=\epsilon_{22}\frac{ac}{2\pi b}V, \quad (45)$$

where

$$\epsilon_{22}=1+4\pi\kappa_{22} \quad (46)$$

is the dielectric constant of the plate in the y direction.

The separate contribution of each zero-order mode is determined by putting $V=0$ on both electrodes. The "lateral" modes will give a negligible contribution to the charge Q because the integration over x cancels most of the half-waves against each other, just as for the excitation.

For the thickness modes, the displacements are slowly variable functions of x. We may expect, therefore, that the x and z-derivatives of the electric field will also be small. In the following, all terms proportional to b/a will be disregarded when they are in addition to terms of the order of unity. Under these conditions, we can write for Eqs. (22) and (24):

$$\operatorname{div}\mathbf{D}=\partial D_y/\partial y=0, \quad (47)$$

$$\partial E_z/\partial y=0, \quad (48)$$

$$\partial E_x/\partial y=0, \quad (49)$$

so that D_y, E_z, and E_x are constant. Since, by Eq. (27) E_z and E_x vanish at the boundaries, they must vanish everywhere.

From Eqs. (31) and (47) we obtain

$$\frac{\partial D_y}{\partial y}=\frac{\partial}{\partial y}\left[4\pi e_{kl2}\frac{\partial u_k^{(p)}}{\partial x_l}+(1+4\pi\kappa_{22})E_y\right]=0 \quad (50)$$

for the contribution of one single zero-order mode of the shear type. From Eq. (34) it is clear that of the derivatives $\partial u_k/\partial x_l$ only $\partial u/\partial y$ is large; all others are smaller by a factor of order b/a and can be omitted. Integrating Eq. (50) we have:

$$D_y=4\pi e_{62}\frac{\partial u^{(p)}}{\partial y}+\epsilon_{22}E_y=\phi(x,z), \quad (51)$$

where ϕ is a function of x and z only. We integrate Eq. (51) over the thickness remembering that

$$-V=\int E_y dy=0$$

for this calculation:

$$4\pi e_{62}[u^{(p)}(b)-u^{(p)}(-b)]=2b\phi. \quad (52)$$

Eliminating ϕ from Eqs. (51) and (52) and

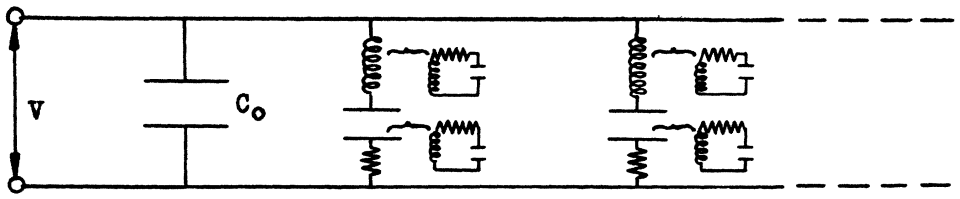

FIG. 1. Equivalent network.

inserting D_y into Eq. (28) we have

$$Q^{(p)} = \int q^{(p)} dxdz$$

$$= -\frac{4\pi e_{62}}{4\pi 2b} \int [u^{(p)}(b) - u^{(p)}(-b)] dxdz. \quad (53)$$

The minus sign in the third member is caused by the fact that at the electrode $y=b$, the normal n has the $(-y)$ direction.

Inserting the functions $u^{(p)}$ from Eq. (34) into Eq. (53) we can carry out the integration:

$$Q^{(p)} = -\frac{8ace_{62}}{m\pi b} \quad (m \text{ odd}),$$
$$Q^{(p)} = 0 \quad (m \text{ even}). \quad (54)$$

The total charge will be the sum of Q_v and of the contribution of each mode, $Q^{(p)}$, multiplied by their respective coefficients A_p:

$$Q = \frac{ac\epsilon_{22}V}{2\pi b} - \sum_p A_p \frac{8ace_{62}}{m\pi b}. \quad (55)$$

The subscript p stands for any combination of odd integers m and n. The current flowing through the crystal is

$$I = i\omega Q = i\omega\left(\frac{ac\epsilon_{22}V}{2\pi b} - \sum A_p \frac{8ace_{62}}{m\pi b}\right). \quad (56)$$

V. ELECTRIC NETWORK ANALOGY

In most cases of piezoelectric excitation, the reaction of the crystal upon the input voltage is measured. Therefore, it seems convenient to substitute for the crystal an analogous electric network.

However, the representation of crystal vibrations by equivalent networks has played a role in the recent research which is not limited to the case of electric excitation. Giebe and Scheibe,[7] and Giebe and Blechschmidt[8] used a semi-empirical equivalence to derive vibrational frequencies which could not be calculated. They replaced the crystal by a number of coupled linear oscillators to which they attributed certain characteristic frequencies and coupling constants which were partly surmised and partly determined empirically.

Lack[9] found that the variation of resonant frequencies of a quartz crystal subjected to temperature change can be represented qualitatively if it is assumed that the crystal is equivalent to a system of two coupled resonant circuits, one of which is being driven by an external force. A great number of papers dealing with crystal vibrations have used this empirical analogy, but none of them has derived the equivalent network from the equations of elasticity.

The following network analogy is based upon the approximate solution of the piezoelectric equations.

The individual terms of Eq. (56) can be considered as partial currents flowing through impedances which are all connected in parallel.

The first term

$$i\omega\frac{ac\epsilon_{22}V}{2\pi b}$$

is equivalent to the current through a condenser of capacity

$$C_0 = ac\epsilon_{22}/2\pi b \quad (57)$$

which is the static capacity of the quartz plate.

The other terms are equivalent to partial

[7] E. Giebe and A. Scheibe, Ann. d. Physik [5] **9**, 93 (1931).
[8] E. Giebe and E. Blechschmidt, Ann. d. Physik **18**, 417 (1933).
[9] F. R. Lack, Bell Sys. Tech. J. **8**, 515 (1929).

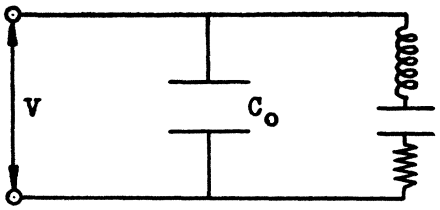

FIG. 2. Simplified equivalent circuit.

currents I_p flowing through a number of parallel branches, each of which represents an uncoupled thickness mode.

It follows that

$$i\omega A_p \frac{e_{26} 8ac}{m\pi b} = I_p \tag{58}$$

if m refers to any parallel branch.

The general network representing the crystal has the equation

$$\sum_p I_p i\left(\omega M_{pq} - \frac{1}{\omega C_{pq}}\right) - V_q = 0. \tag{59}$$

We take $V_q = V$ for all subscripts referring to thickness modes (m odd) and $V_q = 0$ for all other equivalent meshes. In Eq. (59) M_{pq} is the mutual inductance and C_{pq} the mutual capacitance of two meshes.

In particular

$$M_{qq} = L_q, \quad C_{qq} = C_q, \tag{60}$$

are the self-inductance and the capacity of a single mesh. If the Eqs. (15) which determine the values A_p are to be identical with Eq. (59) then it follows from Eqs. (43) and (58) that

$$\begin{aligned} \rho N_{pq}/\gamma = M_{pq}; \quad L_q = \rho N_{qq}/\gamma, \\ H_{pq}/\gamma = 1/C_{pq}; \quad 1/C_q = H_{qq}/\gamma, \end{aligned} \tag{61}$$

where

$$\gamma = 8e_{26}ac/m\pi b. \tag{62}$$

If energy dissipation is to be taken into account, each mesh can be assigned a resistance R_p. As pointed out in the introduction, these resistance values are not determined by crystal constants, but mainly by small irregularities variable from one crystal to another.

Figure 1 illustrates the network.

If a suitable analytical form for the lateral zero-order modes were known then Eqs. (61) in conjunction with the definitions (11) and (12) would allow a determination of the reactive components of the equivalent network. As this is not the case, we limit ourselves to a qualitative discussion.

If the exciting frequency ω is close to the natural frequency of one of the thickness modes so that

$$\rho\omega^2 \cong H_{qq}/N_{qq} = \rho/L_q C_q$$

then the other parallel branches will offer a very high impedance and can be disregarded.

In general, one or several lateral zero-order modes will have values H_{qq}/N_{qq} which are close to $\rho\omega^2$. This corresponds to a state where one or several coupled circuits are tuned to the parallel branch which is at near-resonance. In this case, even small values of the corresponding constants H_{pq} and N_{pq} will give rise to a considerable effective coupling. The resonance curve will exhibit the familiar "double-hump" shape.

However, it will be possible to find crystal dimensions such that the quasi-degeneracy described above does not appear. In this case, the coupled circuits may be disregarded, and the whole equivalent system reduces to the well-known simplified circuit (Fig. 2).

If it is desired, as in most practical applications, to have a resonator with the least possible energy dissipation, then the crystal dimensions will be chosen so as to avoid the coupling as far as possible. This circumstance was empirically recognized by Sykes[10] who also measured a number of desirable crystal dimensions.

[10] R. A. Sykes, Bell Sys. Tech. J. **23**, 52 (1944).