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On the Expansibility of Solutions in Powers of the Interaction Constants

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IT is usual in quantum theory to treat the interaction of particles with fields as a small perturbation which gives rise to transitions from one state to another of the unperturbed system. Even when all approximations higher than the first one in which a given effect appears are infinite on the present method of calculation, it is usual to assume that this first approximation is a good approximation to the exact solution for sufficiently small values of the interaction constant, and that it is increasingly good the smaller its value. Underlying this procedure is the tacit assumption that the exact solution is a continuous function of the interaction constant for vanishingly small values of it, and can therefore be expanded as a series in ascending powers of the interaction constant. The purpose of this note is to show that in the classical theory, where the equations can be solved exactly in certain circumstances, there are exact solutions of the fundamental equations which do not possess this property, and their study leads to a deeper understanding of the whole problem.

As an example, consider the interaction of an electron of mass m and charge e with the electromagnetic field. The exact classical equations of motion of a point electron have been given by Dirac (1938) and are

$$m\dot{v}_k-\tfrac{2}{3}e^2\left(\frac{d\dot{v}_k}{d\tau}+v_k\dot{v}^2\right)=eF_{kl}{}^{in}v^l, \tag{1}$$

where $F_{kl}{}^{in}$ is the ingoing field acting on the electron, v_k is the four-velocity of the particle and a dot denotes differentiation with respect to the proper time τ. The important point about Eq. (1) is that it contains higher derivatives of v_k than the first, and hence the solution of the equation is not uniquely determined if only the position and velocity of the electron are specified at any given instant of time. Taking the simplest case of a free electron for which the right side of (1) is zero, Dirac has shown that the equation has two types of solutions, the first type being of the well-known form

$$v_k=\text{constant}, \tag{2}$$

and the second type of the form

$$v_0=\cosh\,(e^{a\tau}+b), \tag{3}$$

where $a=3m/2e^2$ and b is an arbitrary constant. In solutions of the second type the velocity of the electron tends extremely rapidly to infinity even in the absence of an external field, and we have therefore to exclude such solutions as non-physical ones. Solutions of the first type may be called the physical solutions. The distinction between physical and non-physical solutions persists in the presence of an external field. Equation (1) possesses solutions of both types, however small e may be, provided it is finite. But if e is exactly zero, then clearly Eq. (1) possesses only solutions of the type (2). This circumstance leads one to suspect that the non-physical solutions (3) must depend in a singular way on e as $e\rightarrow 0$. An inspection of (3) immediately confirms this, for as $e\rightarrow 0$, $a\rightarrow\infty$ and the solution (3) contains an essential singularity at $e=0$. It is for this reason that (3) cannot be expanded as a series in ascending powers of e. We have therefore found a new criterion for distinguishing the physical from the non-physical solutions. The physical solutions are those which are continuous functions of e as $e\rightarrow 0$ and can therefore be expanded as series in ascending powers of e; the non-physical solutions are essentially singular functions of e as $e\rightarrow 0$, and cannot be expanded as series in ascending powers of e.

It can be seen at once from dimensional considerations that this criterion is still valid when the point particle possesses a dipole or higher multipole moment, exact equations for which can always be found as proved by Bhabha and Harish-Chandra (1944, 1946). Suppose the particle possesses a multipole of order 2^n with an interaction constant g_n having the dimension charge times length to the power n. The radiation reaction terms in the translational equations of motion are quadratic in g_n and hence contain derivatives of order $2n+2$ with respect to τ. Similarly, the radiation reaction terms in the rotational equations of motion of the spin contain $2n+1$ derivatives with respect to τ. The appearance of these higher derivatives in the equations again has the consequence that they possess non-physical solutions which are essentially singular functions of g_n at $g_n=0$ since no such solutions exist if $g_n=0$. The exact classical equations for a point dipole illustrate this point (Bhabha, 1940, Eqs. (30), (36); Bhabha and Corben, 1941, Eqs. (101), (103), and (106)). We are therefore led to the following general theorem:

The exact equations of motion of point particles possess two types of solutions; the first type, called the physical solutions, are continuous functions of the interaction constants at the point where the values of these constants is zero, and hence can be expanded as series in ascending powers of the constants; the second type, called the non-physical solutions, have an essential singularity at the point where the values of the interaction constants is zero, and hence cannot be expanded as series in ascending powers of the interaction constants.

Hitherto it has been customary to exclude the non-physical solutions by postulating that the allowed solutions are those which do not become infinite in the infinitely distant future. The theorem stated above now permits one to exclude the non-physical solutions by postulating that the allowed solutions shall be continuous functions of the interaction constants as the value of these constants tends to zero. This implies *ipso facto* that it must be possible to expand any allowed solution as a series in ascending powers of the interaction constants.

The situation mentioned above is of so general a nature that we should also expect it to hold in any quantum theory which is a quantization of a classical theory of the type mentioned. Thus, one could exclude the non-physical solutions in quantum theory by simply postulating that the allowed solutions shall be continuous functions of the interaction constants as the values of the latter tend to zero. Indeed, the usual perturbation method of solving problems in quantum mechanics as a series in ascending powers of the interaction constants automatically excludes the non-physical solutions. Our discussion shows that the expansibility of solutions as series in ascending powers of the interaction constants is not so much a property susceptible of proof or disproof, but rather a postulate which allows one to select the physical and exclude the non-physical solutions.

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