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Schwarzschild Interior Solution in an Isotropic Coordinate System

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The relativistic equations for the case of a sphere of perfect fluid of constant density are solved when an isotropic coordinate system is used. It is again found that a sphere of given density has upper bounds on its mass and radius but that these upper bounds are smaller than those given by the ordinary Schwarzschild solution.

INTRODUCTION

IN determining the gravitational potentials for a spherical mass by means of relativity theory it is customary to use a coordinate system for which the line element takes the form

$$ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2, \quad (1)$$

where ν, λ are functions of r alone. If we denote by m the gravitational mass of the body as measured by its external gravitational field, it is well known that the Schwarzschild exterior solution is

$$e^{\nu}=e^{-\lambda}=1-2m/r. \quad (2)$$

If the density of the sphere is ρ and the radius is a then it is also well known[1] that the interior solution is

$$e^{\nu}=\tfrac{1}{4}[3h(a)-h(r)]^2,\ e^{-\lambda}=[h(r)]^2, \quad (3)$$

where $h(r)=(1-r^2/R^2)^{\frac{1}{2}}$ and $R^2=3/8\pi\rho$. The gravitational potentials (3) are valid everywhere inside the sphere. This solution is based on the following physical assumptions: (1) the density is constant; (2) the pressure is zero at the surface of the sphere and is finite and positive inside the sphere; (3) the matter comprises a perfect fluid at rest; (4) the gravitational potentials of the exterior and interior solutions are continuous at the boundary of the sphere; (5) de^{ν}/dr is continuous at the boundary.

When one uses an isotropic coordinate system[2] the line element takes the form

$$ds^2=e^{\nu}dt^2-e^{\mu}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2) \quad (4)$$

and the corresponding exterior solution is known to be

$$e^{\nu}=[q-m/r]^2/q^2,\ e^{\mu}=q^4, \quad (5)$$

where $q=(1+m/2r)$. However as far as the author knows the corresponding interior solution has never been derived. It is the object of this paper to obtain this solution on the same physical bases as given above.

I. SOLUTION OF THE FIELD EQUATIONS

Since the matter in the sphere comprises a a perfect fluid at rest, the components of the energy-momentum tensor T_j^i satisfy the following equations: $T_1^1=T_2^2=T_3^3=-p$, $T_j^i=0$, $i\neq j$, where p is the proper pressure of the fluid. Since the matter is also of constant proper density we have $T_4^4=\rho=$constant. For the line element (4) the field equations imply:[3]

$$8\pi p=e^{-\mu}\left(\frac{\mu'^2}{4}+\frac{\mu'\nu'}{2}+\frac{\mu'+\nu'}{r}\right), \quad (1.1)$$

$$8\pi p=e^{-\mu}\left(\frac{\mu''+\nu''}{2}+\frac{\nu'^2}{4}+\frac{\mu'+\nu'}{2r}\right), \quad (1.2)$$

$$8\pi\rho=e^{-\mu}\left(\mu''+\frac{\mu'^2}{4}+\frac{2\mu'}{r}\right). \quad (1.3)$$

In the above equations the prime notation means differentiation with respect to r. If we substitute

$$\mu=4\log w-2\log r,\ x=\log r, \quad (1.4)$$

Eq. (1.3) becomes

$$d^2w/dx^2=\tfrac{1}{4}w-2\pi\rho w^5. \quad (1.5)$$

This can be integrated to give

$$\left(\frac{dw}{dx}\right)^2=\frac{w^2}{4}-2\frac{\pi\rho}{3}w^6+k. \quad (1.6)$$

[1] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, Oxford, 1934), pp. 245–247.
[2] A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge University Press. New York, 1923), pp. 93–95.
[3] R. C. Tolman, reference 1, pp. 242–243.

In order to avoid a singularity at the origin it is not difficult to show that $k=0$. An infinite singularity in this case implies the pressure would be infinite at the origin. Taking $k=0$ we can integrate (1.6) to obtain

$$w=[R/\cosh(x+c)]^{\frac{1}{2}}, \tag{1.7}$$

where $R^2=3/8\pi\rho$ and "c" is an arbitrary constant. The corresponding solution for μ is given by

$$e^{\mu}=4R^2/(e^c r^2+e^{-c})^2. \tag{1.8}$$

In order to determine ν, we note that the equality of (1.1) and (1.2) implies

$$\nu''+\nu'^2/2-\nu'/r-\mu'\nu'+\mu''-\mu'^2/2-\mu'/r=0. \tag{1.9}$$

By use of (1.8) Eq. (1.9) can be reduced to

$$\nu''/\nu'+\nu'/2-1/r-\mu'=0. \tag{1.10}$$

This can be integrated to give

$$e^{\nu}=\left\{\frac{Ar^2+B}{e^c r^2+e^{-c}}\right\}^2, \tag{1.11}$$

where A, B are both arbitrary constants.

Thus, three arbitrary constants remain in our solution which are evaluated by means of the boundary conditions.

II. THE BOUNDARY CONDITIONS

By means of (1.1), (1.8) (1.11) the pressure is determined by

$$8\pi p=[A(e^{-2c}-2r^2)+B(e^{2c}r^2-2)]/R^2(Ar^2+B). \tag{2.1}$$

Thus $p=0$ at $r=a$ implies

$$A(e^{-2c}-2a^2)+B(e^{2c}a^2-2)=0. \tag{2.2}$$

Further if we let $v=(1-m/2a)(1+m/2a)$, then the continuity of e^{ν}, e^{μ} at $r=a$ implies

$$Aa^2+B=(e^c a^2+e^{-c}), \tag{2.3}$$

$$2R=(1+m/2a)^2(e^c a^2+e^{-c}). \tag{2.4}$$

Solving (2.2), (2.3) for A, B, we obtain

$$A=v(2-e^{2c}a^2)/(e^{-c}-a^2e^c), \tag{2.5}$$

$$B=v(e^{-2c}-2a^2)/(e^{-c}-a^2e^c). \tag{2.6}$$

Since the pressure at $r=0$ is given by

$$(p)_{r=0}=3a^2/8\pi R^2(e^{-2c}-2a^2), \tag{2.7}$$

and since this pressure is positive, we must have

$$e^{-c}-e^c a^2>0. \tag{2.8}$$

Equation (2.4) can be written in the form

$$e^c a^2+e^{-c}=2R/(1+x)^2, \tag{2.9}$$

where $x=m/2a$. By squaring both sides of (2.9) and subtracting $4a^2$ we obtain

$$(e^{-c}-e^c a^2)^2=4R^2(1-y(1+x)^4)/(1+x)^4, \tag{2.10}$$

where $y=a^2/R^2$. Taking the positive square root of both sides we have, because of (2.8), that

$$e^{-c}-e^c a^2=2R[1-y(1+x)^4]^{\frac{1}{2}}/(1+x)^2. \tag{2.11}$$

The final boundary condition to be satisfied is the continuity of de^{ν}/dr at $r=a$. This leads to the equation

$$2a^2(Ae^{-c}-Be^c)/(e^c a^2+e^{-c})^2=2x/(1+x)^2; \tag{2.12}$$

by means of (2.5), (2.6), (2.9), (2.11) this last equation can be put into the form

$$y/[1-y(1+x)^4]^{\frac{1}{2}}=4x/(1+x)^5(1-x). \tag{2.13}$$

Solving this for y we obtain

$$y=4x/(1+x)^6. \tag{2.14}$$

Equation (2.14) represents the determination of the mass "m" in terms of the density ρ and radius a of the sphere. If x is small we have as an approximate equation $y=4x$. This implies $m=4\pi\rho a^3/3$, which is of course the ordinary Newtonian expression. Physically we require that m be uniquely determined if a and ρ are specified. This means that x must be a single valued function of y. By finding the maximum value of y we find that this condition implies $x\leqslant 0.2$, $y\leqslant 0.27$. This means $m\leqslant 0.4a$, $a^2\leqslant 0.27R^2$.

Hence we find, as was found in the Schwarzschild solution, that a sphere of given density has an upper bound on its size and mass. More will be said on this point in the conclusion of this paper.

Because it is not easy to express x in terms of y we find it better to consider m and a as the known constants for the sphere. We can then explicitly express all other constants in terms of these two. The density ρ is given by (2.14) to be

$$\rho=\frac{3m}{4\pi a^3(1+m/2a)^6}. \tag{2.15}$$

From (2.9), (2.11), (2.14) we find $e^c=(m/2a^3)^{\frac{1}{2}}$. Similarly the values of A and B can be found to be

$$A=(m/2a^3)^{\frac{1}{2}}(4a-m)/(2a+m), \quad (2.16)$$

$$B=(2a^3/m)^{\frac{1}{2}}(2a-2m)/(2a+m). \quad (2.17)$$

As our final solution for the gravitational potentials we obtain

$$e^{\mu}=(1+m/2a)^6/(1+mr^2/2a^3)^2, \quad (2.18)$$

$$e^{\nu}=[2a-2m+m(4a-m)r^2/2a^3]^2/[(2a+m)(1+mr^2/2a^3)]^2. \quad (2.19)$$

III. CONCLUSION

In the Schwarzschild solution it was found[4] that a sphere of given density is bounded in mass and size by $m\leqslant 4a/9$, $a^2\leqslant 8R^2/9$. In our solution these bounds are smaller and are given by $m\leqslant 0.4a$, $a^2\leqslant 0.27R^2$. Thus an observer using an isotropic coordinate system would find smaller upper bounds than an observer using a coordinate system in which the line element takes the form (1). This shows that these upper bounds are definitely a property of the coordinate system used by the observer, and it is conceivable that coordinate systems may exist in which these upper bounds may be infinite.

Having obtained the interior solution it is easy to show that the complete solution is mathematically equivalent to the Schwarzschild solution. It is already known that the transformation

$$\bar{r}=(1+m/2r)^2r,\ r\geqslant a$$

takes the isotropic exterior solution into the Schwarzschild exterior solution. The transformation

$$\bar{r}=(1+m/2a)^3r/(1+mr^2/2a^3),\ 0\leqslant r\leqslant a$$

does the same for the interior solution. Moreover the two transformations piece together continuously at the boundary $r=a$.

[4] A. S. Eddington, reference 2, p. 170.

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Forced Vibrations of Piezoelectric Crystals

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The vibrations of anisotropic bodies under the influence of sinusoidally variable volume forces and boundary stresses are investigated. The displacement components are represented as sums of a system of "zero-order" solutions which solve approximately the free-vibration problem. By using Betti's theorem, the problem is reduced to a system of inhomogeneous linear equations which, for the free-body case, further reduces to the homogeneous system derived in an earlier paper (reference 2). If the external forces are piezoelectric, the forces are no longer given explicitly because the electrical field distribution is known only if Maxwell's equations are solved simultaneously. However, if the pertinent piezoelectric constants are small, the field can be calculated approximately as if the crystal were not vibrating. The solutions can then be obtained by the above method, and the electric reaction of the crystal upon the driving system can be determined. As an example, forced vibrations of thin quartz plates between parallel electrodes are discussed.

I. INTRODUCTION

THE rigorous solution of vibration problems meets such great mathematical difficulties that even in comparatively simple cases only approximation methods can be used. The analogy with scalar vibration problems of the Schrödinger type suggests that the solution U_i might be represented as a linear combination of "zero-order" functions $u_i^{(m)}$. This method is straightforward if the $u_i^{(m)}$ satisfy the boundary conditions of the problem. But in most cases, the convenient system $u_i^{(m)}$ does not satisfy the