

Resonance Reactions

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(Received August 1, 1946)

The considerations of a previous note are extended to include the possibility of several resonance levels. It is shown, in the case of resonance scattering, that R , which is the tangent of the phase shift divided by the wave number, is the sum of the reciprocals of linear functions of the energy (Eq. (12)), each term corresponding to one resonance level. All the coefficients in this expression for R are real, energy independent constants. As a result, it appears most natural to write the cross section (Eq. (12a)) as the square of a ratio of two expressions which are themselves fractional expressions of the energy. It is possible to write the cross section also as the square of a single fractional expression of the energy (Eq. (13)). However, the coefficients of the fractional expression are then slowly varying functions of the energy and are not real but subject to other, more involved limitations. The results are quite similar if, in addition to scattering, a reaction is possible also. The cross sections can be represented, most naturally, by means of the squares of the elements of a matrix (Eq. (33)). However, this matrix is the quotient of two matrices which involve the matrix \mathfrak{R} and only \mathfrak{R} is a simple function (Eq. (33a)) of the energy. The cross sections can be evaluated in a simple closed form only if either there is

only one pair of reaction products possible (in addition to the reacting pair) (Eqs. (35), (35a)), or if there are only two resonances present. It is shown, however, that the cross sections can be represented also in the usual form (Eq. (42)) but the "constants" of this form are not strictly independent of energy and not real any more but subject to more involved restrictions. It turns out that the cross section becomes zero between consecutive resonances if only elastic scattering is possible. The elastic scattering cross section does not become zero in general for any value of the energy if a nuclear reaction or inelastic scattering is also possible. If the collision can yield, instead of the colliding pair of particles, only another pair, the cross section for the production of this pair will become zero between successive resonances if the product of certain real quantities has the same sign for both resonances. If the collision may result in any of three or more pairs of particles (e.g., $H^2+H^2 \rightarrow H^1+H^3$, or $He^3+\text{neutron}$, or $He^4+\gamma$) no cross section will vanish, in general, for any value of the energy. The considerations of the present paper are restricted to the case in which the relative angular momenta of the reaction products as well as of the reacting particles vanishes.

INTRODUCTION

IT was attempted, in a previous article,¹ to give a derivation of the resonance formula using a minimum of arbitrary assumptions. It was found useful to consider the wave function in that part of the configuration space in which all particles of the two colliding nuclei are close together. This part of the wave function may be called internal wave function. A reasonable generalization of the usual resonance formulae could be obtained by assuming that the internal wave function is, within the resonance region, in first approximation independent of energy and also the same, no matter whether the system was formed by the collision of the two particles appearing on the left side or of the two particles appearing on the right side of the equation describing the nuclear reaction (thus, e.g., that the internal wave function is the same no matter whether the compound state is obtained by the

collision of Li^6 with H^2 , or of Li^7 with H^1 or of He^4 with He^4). On the basis of this assumption, which is supposed to be valid in the first approximation, the second approximation could be calculated and gave the formulae for scattering and collision cross sections in question. An *a posteriori* justification of the procedure could be obtained by estimating the third approximation. This was found not to affect the results obtained in the second approximation to any appreciable extent if the constants appearing in the second approximation have values similar to those which seem to be, on the present experimental evidence, the usual ones and if the inaccuracy of the wave function in the inside of the internal region is not much greater than on the surface of it. This last assumption restricts the validity of the formulae obtained to a region which is limited by the resonance levels nearest to the one considered.

The present note will deal with the case of several resonance levels with a view, in particular, to obtaining the behavior of the cross sections, etc. in the region between two resonance

¹ E. P. Wigner, Phys. Rev. **70**, 15, 1946. This paper will be referred to as "previous note."

energies.² The greater generality of the conditions, from the physical point of view, will be reflected, mathematically, in the assumption that the internal wave function is, in the first approximation, a linear combination, with energy dependent coefficients, of several energy independent wave functions, corresponding to the several resonance levels which play a role. In order to simplify the analysis, it will be further assumed that there are energy values, the so-called resonance levels, for which the phase shift is just π , i.e., for which the outgoing wave has just opposite sign to what would be the outgoing wave if there was no nuclear interaction present between the colliding particles. This assumption will be formulated more sharply later for the case in which the compound state can be obtained from more than one pair of nuclear particles, i.e., for the case in which a nuclear reaction is possible. This assumption was not made in the note referred to above and inasmuch as the final formulae show the existence of an energy value as postulated here, the assumption has been proved in the case of a single resonance level.

RESONANCE SCATTERING

We now proceed with the consideration of the case in which the compound nucleus can disintegrate only in one way which is then the same

$$\begin{aligned} & -\frac{\hbar^2 i k_1}{2Mu_1u_2}(\exp(ik_2a) - U_2^* \exp(-ik_2a))(-\exp(-ik_1a) - U_1 \exp(ik_1a)) \\ & + \frac{\hbar^2 i k_2}{2Mu_1u_2}(\exp(ik_2a) + U_2^* \exp(-ik_2a))(\exp(-ik_1a) - U_1 \exp(ik_1a)) = (E_1 - E_2) \int \Psi_2^* \Psi_1. \end{aligned} \quad (3)$$

The last integral, as all integrals of Ψ , has to be extended over the internal region. It was pointed out in the previous note that (3) can be derived also by considering the material balance in the internal region for the wave function

$$\varphi_1 \exp(-iE_1 t/\hbar) + \varphi_2 \exp(-iE_2 t/\hbar),$$

which is a non-stationary solution of Schrödinger's equation.

² P. L. Kapur and R. Peierls, Proc. Roy. Soc. **A166**, 277 (1938); H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937); F. Kalckar, J. R. Oppenheimer, and R. Serber, Phys. Rev. **52**, 273 (1937); H. A. Bethe, Rev. Mod. Phys. **9**, 71 (1937) (pages 101-117); G. Breit, Phys. Rev. **58**, 1068 (1940); **69**, 472 (1946).

way in which it was formed. No reaction can take place in this case and the only phenomenon which occurs is that of scattering. We shall assume that the relative angular momentum of the colliding particles is zero. The wave function in the peripheral region of the configuration space (outside the internal region) is then

$$\varphi_1 = (4\pi)^{-\frac{1}{2}} r^{-1} u_1^{-1} [\exp(-ik_1 r) - U_1 \exp(ik_1 r)] \psi(i). \quad (1)$$

The same convention is being used here as before: φ_1 is the wave function of the stationary state with energy E_1 and unit flux of the incoming wave, k_1 is the wave vector $v_1 = u_1^2 = \hbar k_1 / M$ the relative velocity of the particles for energy E_1 . The $\psi(i)$ is the normalized, real wave function of the internal coordinates of the colliding particles, r is their distance. U_1 is the quantity to be determined; it gives by the equation $U_1 = \exp(2i\delta_1)$ the phase shift and hence the scattering cross section at energy E_1 . Equation (1) is valid in the peripheral region, i.e., if $r > a$. In the internal region, we define

$$\varphi_1 = \Psi_1 \quad (r < a). \quad (2)$$

Integration of the equation $\varphi_2^* H \varphi_1 - \varphi_1 (H \varphi_2)^* = (E_1 - E_2) \varphi_1 \varphi_2^*$ (where H is the Hamilton operator) over the internal region gives by means of Green's theorem

One can eliminate the exponentials from (3) by the substitution

$$\bar{U} = U e^{2ika}; \quad \bar{\Psi} = \Psi e^{ika}, \quad (4)$$

after which (3) becomes

$$\begin{aligned} & k_1(1 - \bar{U}_2^*)(1 + \bar{U}_1) + k_2(1 + \bar{U}_2^*)(1 - \bar{U}_1) \\ & = -2iMu_1u_2(E_1 - E_2)\hbar^{-2} \int \bar{\Psi}_2^* \bar{\Psi}_1. \end{aligned} \quad (5)$$

This is quite analogous to Eq. (8) of the preceding note.¹ The reality condition is obtained by noting that the conjugate imaginary of (1) and

(2) must be, apart from a constant, identical with (1) and (2). The constant is easily seen to be $-U_1^*$. This gives

$$U_1^*U_1 = |U_1|^2 = 1; \quad \Psi_1^* = -U_1^*\Psi_1. \quad (6)$$

Both equations are valid, of course, not only for the quantities with the index 1 but for any other index, corresponding to any other value of the energy, as well. The first shows that, in a stationary state, the intensities of incoming and outgoing waves are equal. Both equations hold equally for the unbarred quantities for which they are written and for the barred quantities of (4).

It was assumed in the previous note that all Ψ_1, Ψ_2, \dots , etc. are multiples of one definite Ψ . This assumption will be generalized now to the assumption that the Ψ are linear combinations, with energy dependent coefficients, of a set of energy independent functions. Evidently, there is a certain arbitrariness in the choice of these functions. It will be assumed that these functions $\Psi_\lambda, \Psi_\mu, \dots$, etc., are the internal wave functions for energy the values E_λ, E_μ, \dots for which U becomes -1 , i.e., for the various centers of resonance levels. We have, hence,

$$\Psi_1 = \alpha_{1\lambda}\Psi_\lambda + \alpha_{1\mu}\Psi_\mu + \dots; \quad (7)$$

$$U_\lambda = U_\mu = \dots = -1; \quad \alpha_{\lambda\mu} = \delta_{\lambda\mu}. \quad (7a)$$

Because of (7a) and (6), the Ψ_λ are real. The assumptions embodied in (7), (7a) do not follow

$$\begin{aligned} \frac{1}{k_2} \frac{1-U_2^*}{1+U_2^*} + \frac{1}{k_1} \frac{1-U_1}{1+U_1} &= -\frac{2i\hbar^2(E_1-E_2)}{M} \left\{ \frac{1}{v_\lambda c_\lambda(E_1-E_\lambda)(E_2-E_\lambda)} + \frac{1}{v_\mu c_\mu(E_1-E_\mu)(E_2-E_\mu)} + \dots \right\} \\ &= 2i\hbar \left\{ \frac{1}{k_\lambda c_\lambda} \left(\frac{1}{E_1-E_\lambda} - \frac{1}{E_2-E_\lambda} \right) + \frac{1}{k_\mu c_\mu} \left(\frac{1}{E_1-E_\mu} - \frac{1}{E_2-E_\mu} \right) + \dots \right\}. \quad (11) \end{aligned}$$

It then follows that the R_1 defined by (12) becomes

$$\begin{aligned} \frac{i}{k_1} \frac{1-U_1}{1+U_1} &= R_1 \\ &= \frac{2\hbar}{k_\lambda c_\lambda(E_\lambda-E_1)} + \frac{2\hbar}{k_\mu c_\mu(E_\mu-E_1)} + \dots + R_\infty, \quad (12) \end{aligned}$$

where R_∞ is independent of energy. The real

from simple physical postulates. A comparison with the results of the previous note will show, however, that they are reasonable generalizations of the results obtained there. It is also believed that they could be derived, by a little more algebra, in a way similar to that given there, from the same postulates, and that they could be verified afterwards in an entirely similar fashion to that used there.

Applying (5) for $E_1 = E_\lambda, E_2 = E_\mu$ and neglecting the difference between barred and unbarred quantities, the left side of (5) vanishes because of (7a) and one has

$$\int \Psi_\mu^* \Psi_\lambda = \delta_{\mu\lambda} c_\lambda, \quad (8)$$

i.e., that the internal wave functions of the various resonance levels are orthogonal. It then follows that, in general,

$$\int \Psi_2^* \Psi_1 = \alpha_{2\lambda}^* \alpha_{1\lambda} c_\lambda + \alpha_{2\mu}^* \alpha_{1\mu} c_\mu + \dots \quad (9)$$

Applying now (5) for $E_2 = E_\lambda$ gives

$$2k_1(1+U_1) = -2iMu_\lambda u_1(E_1-E_\lambda)\hbar^{-2}\alpha_{1\lambda}c_\lambda$$

or

$$\alpha_{1\lambda} = i\hbar^2 k_1(1+U_1)/Mu_\lambda u_1(E_1-E_\lambda)c_\lambda. \quad (10)$$

Similar equations hold, of course, for $\alpha_{1\mu}$, etc. so that all the α are determined by (10) in terms of the U .

Dividing now (5) by $k_1 k_2(1+U_1)(1+U_2^*)$ gives because of (9) and (10)

quantities R defined in (12) are M/\hbar times the quantities S defined in the previous note; (12) itself is the generalization of (14a) given there. The c_λ, c_μ , etc., are, because of (8), all positive real quantities, they correspond to $(2/b)^2$ of the previous note. The notation R has been chosen for the quantity appearing in (12) because it has the dimension of a length and because its square is closely related by (13) to the scattering cross section. However, both R and also the

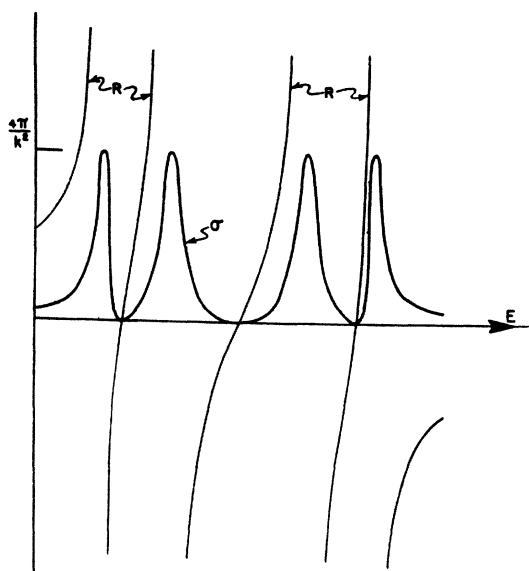


FIG. 1. Resonance scattering. R , i.e., tangent of phase shift divided by the wave number, in arbitrary units and scattering cross section σ in units of maximum possible cross section $4\pi/k^2$ (schematic).

constant R_∞ can be negative as well as positive. From $U = e^{2i\delta}$ and the formula for the cross section in terms of the phase shift δ one obtains for the scattering cross section

$$\sigma(E) = \frac{4\pi}{k^2} \sin^2 \delta = \frac{4\pi R^2}{1 + k^2 R^2}. \quad (12a)$$

This shows that the cross section is $4\pi R^2$ as long as $kR \ll 1$. It assumes its maximum possible value of $4\pi/k^2$ if $R = \infty$.

Figure 1 shows the general trend of R as function of energy and also σ as function of energy. It shows, in the instance represented, the familiar pattern of sharp lines. As long as the lines are as well separated as in Fig. 1, the width of the line at E_λ from half-maximum to half-maximum is given by

$$\Gamma_\lambda = 4\hbar/c_\lambda. \quad (12b)$$

Equation (12a) can also be written in the form

$$\sigma(E) = 4\pi |S|^2; \quad S = R/(1 - ikR). \quad (13)$$

S is the quotient of two rational functions of E , i.e., itself a rational function. Since the degree of its numerator is not higher than the degree of

its denominator, it can be written in the form

$$S = S_0 + \sum_v \frac{s_v}{F_v - E}. \quad (13a)$$

The sum of (13a) contains as many terms as are resonance levels, i.e., as there are terms in the sum of (12). If R_∞ of (12) vanishes, S_0 will vanish also. Although (13), (13a) are, mathematically, completely equivalent to (12) and (12a) and although they give a simpler expression for the cross section than (12) and (12a) do, the latter seem to me preferable for three reasons. First, the symbols s and F which occur in (13a) are in general complex while those in (12) are all real. Second, there are no restrictions on the constants E_λ , c_λ , except that they are real and the latter ones positive. The s_v and F_v of (13c) satisfy complicated equations (not to be given here) which express the fact that $R = S(1 + ikS)^{-1}$ is real. Finally, the E_λ , c_λ are strictly independent of energy while s_v and F_v depend on the energy through k . They are, of course, slowly varying functions of the energy and their energy dependence is really significant only for very low k , e.g., for neutrons near the thermal region.

A rather good approximation for S is given by

$$S_{\text{appr}} = \sum_\lambda \frac{\frac{1}{2}k\Gamma_\lambda/k_\lambda}{E_\lambda - \frac{1}{2}ik\Gamma_\lambda/k_\lambda - E}, \quad (14)$$

which is an expression often given in the literature.³ Equation (14) is accurate as long as the energy E is not in the neighborhood of more than one resonance level E_λ , neighborhood meaning a distance of the order Γ_λ . It is always accurate if the energy actually coincides with one of the resonance levels and a good approximation for all E if the resonance levels are all distinct, i.e., their distance is greater than their width. As a result, (14) is, under ordinary conditions, a very good approximation for every E . One

³ Cf. e.g., H. A. Bethe and G. Placzek, reference 2. The development leading to (13), (13a) is very similar to sections 4 and 5 of G. Breit's last paper (cf. reference 2). Breit already pointed out the limitations of this representation of the resonance formula as they manifest themselves in his model. In particular, he noted that although, in first approximation, the absolute value of our $k s_v$ is oppositely equal to the imaginary part of F_v (cf. (14)), this does not hold rigorously and that, in general, the absolute value of $k s_v$ is greater than the imaginary part of F_v . This is an inequality which follows from the equations to which the s_v and F_v are subject.

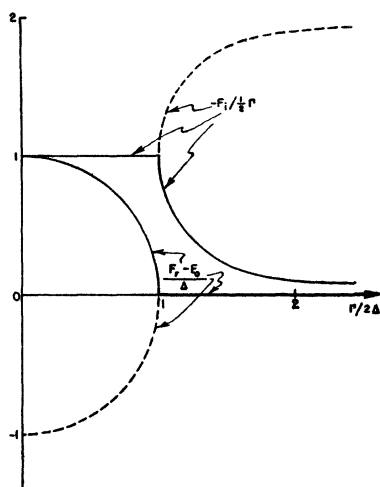


FIG. 2a

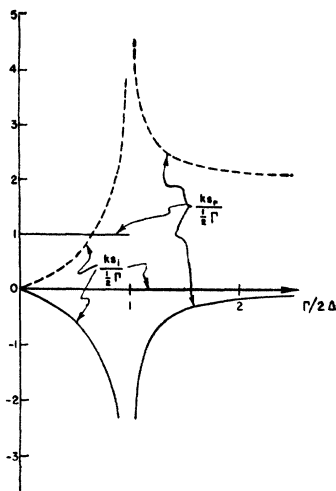


FIG. 2b

FIG. 2. Relation between two descriptions of resonance scattering. Two resonances at $E_0 - \Delta$ and $E_0 + \Delta$, each of width Γ , give apparent resonances at F . The real parts F_r of the two F are given in Fig. 2a in units of Δ (one of them by a broken line), the negative imaginary parts in units of $\frac{1}{2}\Gamma$. The real parts of the two F are the same for $\Gamma > 2\Delta$, the imaginary parts for $\Gamma < 2\Delta$. Figure 2b gives the real and imaginary parts s_r and s_i of the strengths of the two apparent resonances in units $\frac{1}{2}\Gamma$.

sees this also by calculating

$$S - S_{\text{appr}} = (1 - ikR)^{-1} \times \sum_{\lambda \neq \mu} \frac{\frac{1}{2}k\Gamma_\lambda/k_\lambda}{E_\lambda - \frac{1}{2}ik\Gamma_\lambda/k_\lambda - E} \frac{\frac{1}{2}k\Gamma_\mu/k_\mu}{E_\mu - E}. \quad (14a)$$

Because of the $\lambda \neq \mu$ condition, one of the factors after the summation sign is always small if all the resonances are distinct. The second factor can become very large if $E = E_\mu$ but is, in this case, compensated by the factor before the summation sign. In spite of all this, (14) can become grossly inaccurate, e.g., between two levels the width of which is comparable with their distance. In this case (13) with S_{appr} substituted for S can even become larger than the maximum possible width $4\pi/k^2$. The reason is, of course, that the approximate values of s_r and F_r (i.e., $\frac{1}{2}k\Gamma_\nu/k_\nu$ and $E_\nu - \frac{1}{2}ik\Gamma_\nu/k_\nu$) which were adopted in (14) do not satisfy the conditions to which the actual s_r and F_r are subject.

Figures 2a and 2b illustrate the behavior of the quantities s_r and F_r in the particularly simple case of only two resonance levels. These are assumed to be at $E_0 - \Delta$ and $E_0 + \Delta$ and to have both the same width Γ . The energy E_0 is supposed to be so high as compared with Δ or Γ that the variation of k with energy can be neglected. Figure 2a gives the real part and the negative

imaginary part of the two $F - E_0$, in units of Δ and $\frac{1}{2}\Gamma$, respectively, both as function of $\Gamma/2\Delta$. The former is ± 1 in the approximation in which (14) is valid, the latter 1. One sees that the imaginary part of both F is indeed $-\frac{1}{2}\Gamma$ as long as $\Gamma < 2\Delta$ but that this does not hold if $\Gamma > 2\Delta$. In the latter case, both F are purely imaginary, the real parts of both having gone to zero as Γ approached Δ . The s are shown in Fig. 2b. They are complex for $\Gamma < 2\Delta$ but real for $\Gamma > 2\Delta$. The absolute value of ks is always greater than the imaginary part of the corresponding F , a fact which has been recognized already by Breit.³ It is worth while to note that, in spite of this, the absolute value of kS never exceeds 1. Inasmuch as we believe that the Γ and E_λ are the physically significant quantities, these figures illustrate the somewhat artificial nature of the representation (13), (13a).

RESONANCE REACTIONS

The treatment of the case in which, in addition to elastic scattering a real nuclear reaction (or, at least, inelastic scattering) is possible, differs from the above one in that the peripheral wave function has, instead of (1), the form

$$\varphi_{1j} = \sum_l (4\pi)^{-\frac{1}{2}} r_l^{-1} u_{1l}^{-1} [\delta_{jl} \exp(-ik_{1l}r_l) - U_{1il} \exp(ik_{1l}r_l)] \psi(i_l). \quad (15)$$

In this as in the following formulae, the indices j, l, \dots etc. denote pairs of nuclei which can react with each other or appear as products of the reaction. The $\psi(i_i)$ is the product of the real normalized wave functions of the pair l . (E.g., j can denote the pair $\text{Li}^6 + \text{H}^2$; l the pair $\text{Li}^7 + \text{H}^1$; m the pair $\text{He}^4 + \text{He}^4$ etc.) The r_l is the relative distance of the nuclei forming the pair l , the relative velocity of this pair for the energy value E_1 of the total system is denoted by $v_{1l} = u_{1l}^2 = \hbar k_{1l}/M_l$, the M_l being the relative mass of the pair l . In (15), because of the δ_{jl} , only nuclei of the pair j approach each other; the U_{1jl} are the elements of the collision matrix.

The square $|U_{1jl}|^2$ gives the number of pairs l formed if one pair of the kind j collides.

Equation (15) is valid in the peripheral region. In the internal region, i.e., in the part of the configuration space in which all particles are close together, we have instead of (2)

$$\varphi_{1j} = \Psi_{1j}. \quad (15a)$$

In order to calculate the second approximation we need an equation analogous to (3). This can be obtained again by integration of the equation

$$\varphi_{2j}^* H \varphi_{1l} - \varphi_{1l} (H \varphi_{2j})^* = (E_1 - E_2) \varphi_{1l} \varphi_{2j}^*$$

over the internal region. This yields by Green's theorem

$$\begin{aligned} & - \sum_m \frac{\hbar^2 i k_{1m}}{2M_m u_{1m} u_{2m}} [\delta_{jm} \exp(ik_{2m}a) - U_{2jm}^* \exp(-ik_{2m}a)] [-\delta_{lm} \exp(-ik_{1m}a) - U_{1lm} \exp(ik_{1m}a)] \\ & + \sum_m \frac{\hbar^2 i k_{2m}}{2M_m u_{1m} u_{2m}} [\delta_{jm} \exp(ik_{2m}a) + U_{2jm}^* \exp(-ik_{2m}a)] [\delta_{lm} \exp(-ik_{1m}a) - U_{1lm} \exp(ik_{1m}a)] \\ & = (E_1 - E_2) \int \Psi_{2j}^* \Psi_{1l}. \quad (16) \end{aligned}$$

In order to eliminate the exponentials, one can again introduce barred quantities

$$\bar{\Psi}_l = \Psi_l \exp(ik_l a); \quad \bar{U}_{lm} = U_{lm} \exp[i(k_m + k_l)a]. \quad (17)$$

With this substitution, and after multiplication with $\exp[i(k_{1l} - k_{2j})a]$, (16) goes over into

$$\begin{aligned} & i\hbar \sum_m (\delta_{lm} + \bar{U}_{1lm}) u_{1m} u_{2m}^{-1} (\delta_{jm} - \bar{U}_{2jm}^*) + i\hbar \sum_m (\delta_{lm} - \bar{U}_{1lm}) u_{1m}^{-1} u_{2m} (\delta_{jm} + \bar{U}_{2jm}^*) \\ & = 2(E_1 - E_2) \int \bar{\Psi}_{2j}^* \bar{\Psi}_{1l}. \quad (18) \end{aligned}$$

This is analogous to Eq. (22) of the previous note. The bars will be left off again forthwith.

We shall now make the assumption mentioned at the end of the Introduction more precise. It will be assumed that there are "resonance energies" E_λ, E_μ, \dots , etc. for which the matrices $\mathbb{U}_\lambda = \|U_{\lambda lm}\|$ have a characteristic value -1 . The corresponding normalized characteristic vectors will be denoted by \mathfrak{g}_λ with components $\beta_{\lambda m}$ so that $\mathbb{U}_\lambda \mathfrak{g}_\lambda = -\mathfrak{g}_\lambda$, or, more in detail,

$$\sum_m U_{\lambda lm} \beta_{\lambda m} = -\beta_{\lambda l}. \quad (19)$$

This amounts to the assumption that, at the resonance energies, there are superpositions of incident waves

$$\sum_l \beta_{\lambda l} (4\pi)^{-\frac{1}{2}} r_l^{-1} u_{\lambda l}^{-1} \exp(-ik_{\lambda l} r_l) \psi(i_l), \quad (20a)$$

for which the outgoing wave has, apart from the sign, the same value which it would have if there was no nuclear reaction or scattering, but that the sign of the outgoing wave is opposite. This means that the outgoing wave for the superposition of incoming waves given by (20a) is

$$\sum_l \beta_{\lambda l} (4\pi)^{-\frac{1}{2}} r_l^{-1} u_{\lambda l}^{-1} \exp(ik_{\lambda l} r_l) \psi(i_l). \quad (20b)$$

The internal wave function for the incoming wave (20a) (and the outgoing wave (20b)) will be denoted by Ψ_λ and it will further be assumed that these are the $\Psi_\lambda, \Psi_\mu, \dots$ which permit one to express, in first approximation, the internal wave functions linearly for all energies and all incident waves:

$$\Psi_{1j} = \alpha_{1\lambda} \Psi_\lambda + \alpha_{1\mu} \Psi_\mu + \dots \quad (21)$$

The coefficients α depend both on the energy (E_1 in (21)) and also on the pair of particles (j in (21)) the collision of which is represented by the wave function Ψ_{1j} of (15a). The fact that the internal wave function is Ψ_λ for energy E_λ if the incoming wave is given by (20a) means that

$$\Psi_\lambda = \sum_j \beta_{\lambda j} \Psi_{\lambda j} \\ = \sum_j \beta_{\lambda j} (\alpha_{\lambda\lambda}^i \Psi_\lambda + \alpha_{\lambda\mu}^i \Psi_\mu + \dots). \quad (22)$$

This is equivalent with

$$\sum_j \beta_{\lambda j} \alpha_{\lambda\mu}^j = \delta_{\lambda\mu}, \quad (23)$$

which is the analogon of the second equation of (7a). Since the $\Psi_\lambda, \Psi_\mu, \dots$, etc. can be considered as solutions of the same characteristic value problem with the same homogeneous boundary conditions embodied in (20a), (20b), they are mutually orthogonal

$$\int \Psi_\mu^* \Psi_\lambda = \delta_{\mu\lambda} c_\lambda; \quad c_\lambda = c_\lambda^* > 0, \quad (24)$$

which is the analogon of (8). Equation (24) follows also from the more general equation (18) if one inserts λ for 1, μ for 2 and multiplies it with $\beta_{\lambda i} \beta_{\mu j}^*$, sums over l and j and considers (19), the symmetric nature of \mathfrak{U} and the first part of (22).

No assumption analogous to the one contained in (21) was made in the previous note and it is, in fact, believed that this assumption follows from the other more general assumptions made in the Introduction. This was explicitly demonstrated in the previous note for the case of a single resonance level: the energy E_λ was denoted there by E_0 (there was only one E_λ), the corresponding vector \mathfrak{B}_λ was denoted there by $u_0 \mathfrak{B}$ (with components $u_{0i} \beta_i$), and all internal wave functions were the same apart from constant factors. The existence of both the E_0 and the vector \mathfrak{B} was derived there while it is assumed here because it simplifies the analysis to such a great extent.

Since (20b) is the conjugate complex of (20a), (the vector \mathfrak{B} will be shown in (26) to be real) the peripheral wave function is real and the same must hold then for Ψ_λ . This also follows from the more general equations which are the generaliza-

tions of (6).

$$\sum_l U_{1jl}^* U_{1lm} = \delta_{jm}; \quad -\sum_l U_{1jl}^* \Psi_{1l} = \Psi_{1j}^*. \quad (25)$$

One can easily convince oneself that (25) holds both for the unbarred and also for the barred quantities defined in (17). The first equation of (25) reads, in matrix notation $\mathfrak{U}_1^* \mathfrak{U}_1 = 1$ and is, because of the unitary nature of \mathfrak{U} , equivalent with the statement that \mathfrak{U} is symmetric. Both (25) and (16) or (18) are rigorous, i.e., do not involve any approximations. However, (21) has only approximate validity.

Introducing now (21) into (18), the right side of this goes over into

$$2(E_1 - E_2) \sum_\lambda \alpha_{2\lambda}^{j*} \alpha_{1\lambda}^l c_\lambda. \quad (18')$$

We can define the matrix product of two vectors \mathbf{x} and \mathbf{y} as the matrix $\mathbf{x} \times \mathbf{y}$ the j, l element of which is $(\mathbf{x} \times \mathbf{y})_{jl} = x_j y_l$; (18) then becomes in the matrix-vector notation which has been used also in the previous note

$$i\hbar(1 + \mathfrak{U}_1) \mathfrak{U}_1 \mathfrak{U}_2^{-1} (1 - \mathfrak{U}_2^\dagger) \\ + i\hbar(1 - \mathfrak{U}_1) \mathfrak{U}_1^{-1} \mathfrak{U}_2 (1 + \mathfrak{U}_2^\dagger) \\ = 2(E_1 - E_2) \sum_\lambda c_\lambda (\alpha_{1\lambda} \times \alpha_{2\lambda}^*). \quad (18a)$$

All matrices (and vectors) have as many dimensions as there are ways of disintegration for the compound state, the indices 1, 2 give the energy values E_1, E_2 to which the quantities refer, $\mathfrak{U}_1, \mathfrak{U}_2$ are diagonal matrices with diagonal elements u_{1i}, u_{2i} , respectively, $\alpha_{1\lambda}$ is a vector with components $\alpha_{1\lambda}^l$, etc. Equation (19) then becomes

$$\mathfrak{U}_\lambda \mathfrak{B}_\lambda = -\mathfrak{B}_\lambda, \quad (19a)$$

while (23) goes over into

$$\mathfrak{B}_\lambda \cdot \alpha_{\lambda\mu} = \delta_{\lambda\mu}. \quad (23a)$$

It may be worth while to remark that because of (19a) and the symmetric nature of \mathfrak{U}_λ , the vectors \mathfrak{B}_λ can be chosen to be real and that they can be assumed to be normalized

$$\mathfrak{B}_\lambda = \mathfrak{B}_\lambda^*; \quad \mathfrak{B}_\lambda \cdot \mathfrak{B}_\lambda = \sum_l \beta_{\lambda l}^2 = 1. \quad (26)$$

One can substitute E_μ for E_2 in (18a) and apply it to the vector \mathfrak{B}_μ^* , i.e., substitute E_μ for E_2 in (18), multiply it with $\beta_{\mu j}^*$, and sum over j . Because of the symmetry of \mathfrak{U}_μ and (19), the second term on the left side drops out, the first

simplifies considerably and one has

$$\begin{aligned} 2i\hbar(1+\mathbb{U}_1)u_1u_\mu^{-1}\mathfrak{B}_\mu^* \\ = 2(E_1 - E_\mu) \sum_\lambda c_\lambda (\alpha_{1\lambda} \times \alpha_{\mu\lambda}^*) \mathfrak{B}_\mu^* \\ = 2(E_1 - E_\mu) c_\mu \alpha_{1\mu}. \end{aligned} \quad (27)$$

The last part follows from (23) or (23a) and the general equation

$$(\mathbf{x} \times \mathbf{y}) \mathbf{z} = (\mathbf{y} \cdot \mathbf{z}) \mathbf{x}. \quad (\text{I})$$

From (27), the vector $\alpha_{1\mu}$ becomes

$$\alpha_{1\mu} = \frac{i\hbar}{c_\mu(E_1 - E_\mu)} (1 + \mathbb{U}_1) u_1 u_\mu^{-1} \mathfrak{B}_\mu. \quad (27a)$$

As was pointed out before, \mathfrak{B}_μ can be assumed to

$$\begin{aligned} u_2^{-1}(1 - \mathbb{U}_2^\dagger)(1 + \mathbb{U}_2^\dagger)^{-1} u_2^{-1} + u_1^{-1}(1 + \mathbb{U}_1)^{-1}(1 - \mathbb{U}_1) u_1^{-1} \\ - \sum_\lambda \left[\frac{2i\hbar}{c_\lambda(E_1 - E_\lambda)} - \frac{2i\hbar}{c_\lambda(E_2 - E_\lambda)} \right] u_\lambda^{-1} (\mathfrak{B}_\lambda \times \mathfrak{B}_\lambda) u_\lambda^{-1} = 0. \end{aligned} \quad (29)$$

It then follows that the terms which depend on E_1 (and similarly the terms which depend on E_2) give an energy independent matrix. One can clearly replace in (29) all the diagonal matrices $u^{-1} = \mathbf{v}^{-\frac{1}{2}}$ by the diagonal matrices $q^{-1} = \mathbf{f}^{-\frac{1}{2}}$ and obtain for the matrix \mathfrak{R}_1

$$\begin{aligned} \mathfrak{R}_1 = iq_1^{-1}(1 + \mathbb{U}_1)^{-1}(1 - \mathbb{U}_1) q_1^{-1} \\ = \mathfrak{R}_\infty + \sum_\lambda \frac{2\hbar}{c_\lambda(E_\lambda - E_1)} q_\lambda^{-1} (\mathfrak{B}_\lambda \times \mathfrak{B}_\lambda) q_\lambda^{-1}. \end{aligned} \quad (30)$$

This is the analogon of the final result (12) for scattering and represents the general solution of our problem just as (36), (36a) of the previous note represented the general solution in the case of a single resonance level. \mathfrak{R}_∞ is an arbitrary real symmetric matrix, the E_λ are arbitrary real energy values, the c_λ are positive real numbers the \mathfrak{B}_λ real vectors and they can be assumed to be normalized, or, if not, $c_\lambda = 1$ can be assumed.

The matrix $\mathbb{U}_1 - 1$ can be expressed in terms of \mathfrak{R}_1 as

$$\mathbb{U}_1 - 1 = 2iq_1\mathfrak{R}_1q_1 / (1 - iq_1\mathfrak{R}_1q_1) \quad (31)$$

and the cross section for the transformation of

be real. Inserting (27a) into (18a) one obtains for the right side of the latter

$$\begin{aligned} \sum_\lambda \frac{2(E_1 - E_2)\hbar^2}{c_\lambda(E_1 - E_\lambda)(E_2 - E_\lambda)} (1 + \mathbb{U}_1) u_1 \\ \times (u_\lambda^{-1} \mathfrak{B}_\lambda \times u_\lambda^{-1} \mathfrak{B}_\lambda) u_2 (1 + \mathbb{U}_2^\dagger) \end{aligned} \quad (28)$$

because of the general equation

$$(\mathbf{a} \mathbf{x} \times \mathbf{b} \mathbf{y}) = \mathbf{a} (\mathbf{x} \times \mathbf{y}) \mathbf{b}' \quad (\text{II})$$

valid for all matrices \mathbf{a} , \mathbf{b} (\mathbf{b}' being the transposed of \mathbf{b}) and all vectors \mathbf{x} , \mathbf{y} . Thus (18a) becomes after multiplication with $u_1^{-1}(1 + \mathbb{U}_1)^{-1}$ from the left, $(1 + \mathbb{U}_2^\dagger)^{-1} u_2^{-1}$ from the right and division by $i\hbar$

pair j into pair l is

$$\begin{aligned} \sigma_{jl}(E_1) = \frac{\pi}{k_{1j}^2} |(\mathbb{U}_1 - 1)_{jl}|^2 \\ = \frac{4\pi}{k_{1j}^2} \left| \left(\frac{q_1 \mathfrak{R}_1 q_1}{1 - iq_1 \mathfrak{R}_1 q_1} \right)_{jl} \right|^2. \end{aligned} \quad (31a)$$

Although both this expression and that for \mathfrak{R}_1 (30) is quite simple, it does not seem possible to express in general the $\sigma_{jl}(E)$ in closed form in terms of the E , k_j , k_l , the energy independent quantities E_λ , β_λ , c_λ , and \mathfrak{R}_∞ .

HIGHER SPINS

It is easy to generalize (30), (31a) for the case that the nuclei of the pair j , say, have spins j and j' , i.e., angular momenta $j\hbar$ and $j'\hbar$, respectively. If our assumption remains valid, that only those states need to be taken into account in which the angular momentum of the motion of the pair around their common center of mass vanishes, one can consider the collision system formed by the pair j as a mixture of states for which the total angular momentum has one of the values

$$J = |j - j'|, |j - j'| + 1, \dots, j + j' - 1, j + j'. \quad (32)$$

This total angular momentum consists of the vector sum of the spins of the particles making up the pair j since, according to assumption, the relative motion of the particles has no angular momentum. From the wave functions with a definite J only such compound states Ψ can be reached which have the same J . As a consequence, one must consider as many sets of resonance levels as are J in (32) and define for each set an \mathfrak{R}^J according to (30) in which, then, the summation is to be extended only over the compound states with spin J . The total cross section then becomes the weighted sum of expressions (31a), corresponding to the components of the mixture (32)

$$\sigma_{jl}(E) = \frac{4\pi k_l}{k_j} \sum_{J=|j-j'|}^{j+j'} g_J \left| \left(\frac{\mathfrak{R}^J}{1-iq\mathfrak{R}^Jq} \right)_{jl} \right|^2, \quad (33)$$

$$g_J = \frac{2J+1}{(2j+1)(2j'+1)}.$$

The k_j , q , \mathfrak{R}^J all refer⁴ to the energy E in (33). It is, probably, unnecessary to remark that (30) can be written in a somewhat simpler form

$$\mathfrak{R}^J(E) = \mathfrak{R}_\infty^J + \sum_\lambda \frac{\gamma_\lambda \times \gamma_\lambda}{E_\lambda - E}, \quad (33a)$$

where the J on the summation sign indicates that the summation is to be extended only over those λ for which the angular momentum of the compound state Ψ_λ is $J\hbar$. The real vectors γ_λ are, as are the β_λ and the \mathfrak{R}_∞^J , independent of energy

$$\begin{aligned} \gamma_\lambda &= (2\hbar/c_\lambda)^{1/2} q_\lambda^{-1} \beta_\lambda, \\ \gamma_{\lambda j} &= (2\hbar/c_\lambda)^{1/2} k_{\lambda j}^{-1} \beta_{\lambda j}. \end{aligned} \quad (33b)$$

They are, of course, not normalized.

The \mathfrak{R}_∞^J of (33a) could be, off hand, arbitrary real symmetric matrices. They describe that part of the reaction or scattering which would be present even in the absence of resonance levels.

⁴ Strictly speaking, the matrix in (33) is undefined because \mathfrak{R} and $q\mathfrak{R}q$ do not commute in general. A comparison with (32) shows that the proper definition is

$$\frac{\mathfrak{R}}{1-iq\mathfrak{R}q} = q^{-1} \frac{q\mathfrak{R}q}{1-iq\mathfrak{R}q} q^{-1}. \quad (33')$$

The fraction on the right side is completely defined since it is a function of a single matrix $q\mathfrak{R}q$. The right side of (33') will be meant always when the left side is written for brevity.

It has been argued, in the previous note, that these \mathfrak{R}_∞ can be considered to be diagonal matrices as long as we can speak of pure resonance reactions. Even if this should not be the case, two remarks will be applicable to the matrix elements $R_{\infty jl}$. First, since they correspond to non-resonance reactions, their order of magnitude will not be greater than the nuclear radius. As a result, their product with $q_j q_l = (k_j k_l)^{1/2}$ will be small compared to 1 except if the energy of both pairs j and l is quite high. Second, $R_{\infty jl}(k_j k_l)^{1/2}$ will be particularly small if either the pair j , or the pair l , consists of a light quantum and a nucleus. If both j and l are such pairs, the process for which $R_{\infty jl}(k_j k_l)^{1/2}$ is responsible is essentially a Compton effect on a nucleus or a similar straight scattering term. Even if only one pair, say j , contains a light quantum, $q_j R_{\infty jl} q_l$ will be a high order correction term which is, e.g., consistently neglected in the usual treatments of light absorption or emission. It therefore appears to be justified to neglect $R_{\infty jl}$ if either of the pairs j or l contains a light quantum.

Since every symmetric matrix can be written as a sum of matrices ($\gamma_\rho \times \gamma_\rho$), it is possible, formally, to set the \mathfrak{R}_∞^J equal to zero if one admits a few terms in the sum of (33a) for which E_λ is infinite but the γ_λ also infinite in such a way that the corresponding terms give a finite contribution to (33a), replacing \mathfrak{R}_∞^J . This is particularly tempting in the case of the (n, γ) reaction. For these reactions, all pairs l, \dots , etc. contain a light quantum (in addition to the normal or excited state of the product nucleus), except one, the initial state j . Because of the remarks made before, all the matrix elements of \mathfrak{R}_∞^J will vanish except the jj element and \mathfrak{R}_∞^J can be replaced by a single term ($\gamma_\infty \times \gamma_\infty$) with a vector γ_∞ all the components of which vanish, except the j component. Although the elimination of \mathfrak{R}_∞^J , which has just been described, is a very formal one, it will be adopted later because it simplifies the formulae at least from a formal point of view. We shall write then

$$\mathfrak{R}^J(E) = \sum_\lambda \frac{\gamma_\lambda \times \gamma_\lambda}{E_\lambda - E}. \quad (33c)$$

However, when using (33c) we must keep in

mind that it has a few (or, in the case of an (n, γ) reaction, one) finite terms with $E_\lambda = \infty$.

EVALUATION OF (33)

Although the preceding formulae give, in principle, a complete solution of our problem, they are unsuited not only to practical calculations but even for obtaining a qualitative picture of the variation of the cross sections. The difficulty in using (33) (or (31a) and (30)) consists in the evaluating of the elements of the matrix $q\mathfrak{R}q(1-iq\mathfrak{R}q)^{-1}$ if only the matrix elements of \mathfrak{R} are given.

This difficulty can be easily overcome if there are only two states involved, i.e., if only one pair of reaction products is possible. On the whole, this is an exceptional case as even the (n, γ) reaction has several possible end products corresponding to the different excited states of the product nucleus. Examples in question may be, however, some (n, γ) reactions of light elements, and, perhaps, reactions of the kind $Li^7 + n = Be^7 + H^1$.

If there is, in addition to the original pair of nuclei, only one pair of reaction products possible, the matrix $q\mathfrak{R}q$ becomes two-dimensional and its reciprocal can be found easily. One obtains for

$$\frac{q\mathfrak{R}q}{1-iq\mathfrak{R}q} = (1-D-i(S_{jj}+S_{ll}))^{-1} \times \begin{vmatrix} S_{jj}-iD & S_{jl} \\ S_{jl} & S_{ll}-iD \end{vmatrix}, \quad (34)$$

in which

$$\begin{aligned} S_{jj} &= k_j \left(R_{jj\infty} + \sum_{\lambda} \frac{\gamma_{\lambda j}^2}{E_{\lambda} - E} \right), \\ S_{jl} &= (k_j k_l)^{\frac{1}{2}} \left(R_{jl\infty} + \sum_{\lambda} \frac{\gamma_{\lambda j} \gamma_{\lambda l}}{E_{\lambda} - E} \right), \\ S_{ll} &= k_l \left(R_{ll\infty} + \sum_{\lambda} \frac{\gamma_{\lambda l}^2}{E_{\lambda} - E} \right), \\ D &= S_{jj} S_{ll} - S_{jl}^2. \end{aligned} \quad (34a)$$

It is assumed that all compound states have the same J . As a result of (34), the cross section for scattering becomes

$$\sigma_{jj} = \frac{4\pi}{k_j^2} \frac{S_{jj}^2 + D}{(1-D)^2 + (S_{jj} + S_{ll})^2} \quad (35)$$

while the reaction cross section is

$$\sigma_{jl} = \frac{4\pi}{k_j^2} \frac{S_{jl}^2}{(1-D)^2 + (S_{jj} + S_{ll})^2}. \quad (35a)$$

The remarkable feature of this last expression is that it goes through zero at least between any two consecutive resonances for which $\gamma_{\lambda j} \gamma_{\lambda l}$ has the same sign since, evidently, S_{jl} goes through zero between two such points. It is similar, in this respect, to the scattering cross section of Fig. 1 which applies if no reaction is possible. However, the scattering cross section given by (35), which applies if a reaction is possible in addition to the scattering, does not exhibit this feature any more, since it can vanish only if both S_{jj} and D vanish for the same E . Even the reaction cross section does not become zero between consecutive maxima if either more than one pair of reaction products are possible or if there are compound states with more than one value of J . The former case will be investigated below. In the latter case the cross section is, according to (33), a weighted average of expressions of the form (35a). This could vanish only if the expression (35a) would vanish for the same E for every J . Such an occurrence has "zero probability."

If several pairs of reaction products are possible, the matrices \mathfrak{R} and $q\mathfrak{R}q$ have more than two dimensions and the reciprocal of $1-iq\mathfrak{R}q$ becomes an involved expression. It seems worth while, in that case, to adopt the convention of (33c) to eliminate the constant matrix \mathfrak{R}_{∞}^J which occurs in (33a). The J will again be omitted in the following transformation.

Because of (II), one can write for the denominator of the expression in (33)

$$1-iq\mathfrak{R}q = 1 - i \sum_{\lambda} \frac{(q\gamma_{\lambda} \times q\gamma_{\lambda})}{E_{\lambda} - E}. \quad (36)$$

We shall try to write for

$$(1-iq\mathfrak{R}q)^{-1} = 1 + i \sum_{\mu\nu} A_{\mu\nu} (q\gamma_{\mu} \times q\gamma_{\nu}). \quad (36a)$$

The $A_{\mu\nu}$ are, of course, functions of the energy. The product of the right sides of (36a) and (36) must give 1. It can be evaluated by means of the general equation

$$(\mathbf{x} \times \mathbf{y})(\mathbf{z} \times \mathbf{w}) = (\mathbf{y} \cdot \mathbf{z})(\mathbf{x} \times \mathbf{w}). \quad (\text{III})$$

One obtains for the $A_{\mu\nu}$ the equations

$$A_{\mu\lambda} - \frac{\frac{1}{2}i}{E_\lambda - E} \sum_\nu A_{\mu\nu} \Gamma_{\nu\lambda} = \frac{\delta_{\mu\lambda}}{E_\lambda - E}, \quad (37)$$

where the $\Gamma_{\nu\lambda}$ are the scalar products

$$\frac{1}{2}\Gamma_{\nu\lambda} = (\mathfrak{q}\gamma_\nu \cdot \mathfrak{q}\gamma_\lambda) = \sum_j k_j \gamma_{\nu j} \gamma_{\lambda j}. \quad (37a)$$

The Γ with two indices are, in contrast to the Γ of (13a), functions of the energy because of the k which enters (37a). They are, however, slowly varying functions of the energy. The $A_{\mu\nu}$ will turn out to be rapidly varying functions of E .

Both the $\Gamma_{\nu\lambda}$ and the $A_{\mu\lambda}$ are symmetric in their two indices: $\Gamma_{\nu\lambda} = \Gamma_{\lambda\nu}$; $A_{\mu\lambda} = A_{\lambda\mu}$. As a result of (36a), (37), and (III), the matrix in (33) can also be written as⁴

$$\frac{\mathfrak{R}}{1 - iq\mathfrak{R}q} = \sum_{\mu\lambda} A_{\mu\lambda} (\gamma_\mu \times \gamma_\lambda). \quad (38)$$

This is how far the transformation of (33) can be carried easily without making approximations.

At this point, the possibility of a transformation similar to that given by (13) should be mentioned. Equation (37) is evidently a matrix equation although the rows and columns of the matrices occurring in it do not refer to the different reaction products as in the case of the matrices \mathfrak{q} , \mathfrak{R} , etc., but refer to the different resonance levels λ , μ , ν , etc. After multiplication with $E_\lambda - E$, (37) can be written as

$$\mathcal{Q}(\mathcal{E} - E1) - \frac{1}{2}i\mathcal{Q}\mathcal{G} = 1. \quad (39)$$

In this, \mathcal{E} is the diagonal matrix with the diagonal elements E_λ , E_μ , etc., \mathcal{Q} and \mathcal{G} are symmetric matrices with elements $A_{\mu\nu}$ and $\Gamma_{\mu\nu}$. It follows from (39) that

$$\mathcal{Q} = (\mathcal{E} - \frac{1}{2}i\mathcal{G} - E1)^{-1}. \quad (39a)$$

If the symmetric matrix $\mathcal{E} - \frac{1}{2}i\mathcal{G}$ has no double characteristic values—which would be, after all, an exceptional case—it will be possible to bring it into the diagonal form \mathfrak{F}

$$\mathfrak{F} = \mathcal{T}(\mathcal{E} - \frac{1}{2}i\mathcal{G})\mathcal{T}^{-1}. \quad (39b)$$

Because of the symmetry of $\mathcal{E} - \frac{1}{2}i\mathcal{G}$, one can assume that \mathcal{T} is a complex orthogonal matrix

$$\mathcal{T}^{-1} = \mathcal{T}'. \quad (39c)$$

One has because of (39b)

$$\mathcal{E} - \frac{1}{2}i\mathcal{G} - E1 = \mathcal{T}^{-1}(\mathfrak{F} - E1)\mathcal{T} \quad (40)$$

and

$$\mathcal{Q} = \mathcal{T}'(\mathfrak{F} - E1)^{-1}\mathcal{T}. \quad (40a)$$

The diagonal elements F_ν of \mathfrak{F} , as well as the elements $T_{\mu\nu}$ of \mathcal{T} will still be slowly varying functions of the energy. However, the elements of \mathcal{Q} will be

$$A_{\mu\lambda} = \sum_\nu T_{\nu\mu}(F_\nu - E)^{-1}T_{\nu\lambda} \quad (40b)$$

at least in the neighborhood of the real part of F_ν , rapidly varying functions of E .

Because of (40b), (38) becomes

$$\begin{aligned} \frac{\mathfrak{R}}{1 - iq\mathfrak{R}q} &= \sum_{\mu\lambda} \sum_\nu \frac{T_{\nu\mu}T_{\nu\lambda}}{F_\nu - E} (\gamma_\mu \times \gamma_\lambda) \\ &= \sum_\nu \frac{(\mathbf{s}_\nu \times \mathbf{s}_\nu)}{F_\nu - E}, \end{aligned} \quad (41)$$

wherein the vector \mathbf{s}_ν is defined by

$$s_{\nu l} = \sum_\lambda T_{\nu\lambda} \gamma_{\lambda l}. \quad (41a)$$

Finally, (33) gives for the cross section

$$\sigma_{jl}(E) = \frac{4\pi k_l}{k_j} \sum_J g_J \left| \sum_\nu \frac{s_{\nu j} s_{\nu l}}{F_\nu - E} \right|. \quad (42)$$

There is, of course, a separate set of F_ν and $s_{\nu j}$ for every value of J and the sum of (42) should be taken in this sense. Equation (42) is the analogon of (13), (13a) and is subject to the same kind of limitations as are those equations: the quantities $s_{\nu l}$, $s_{\nu j}$, F_ν are in general complex; they are subject to rather complicated equations which express the fact that the matrices \mathcal{E} and \mathcal{G} are real; they are functions of the energy although slowly varying functions. Except for these limitations (42) is a very concise form of the resonance formula.⁵ It should be remembered, however, that the expressions (34), (35) give the cross sections in terms of the $R_{jj\infty}$, $R_{jl\infty}$, $R_{ll\infty}$, E_λ , γ_λ , etc. which are all real, subject to no equations and independent of energy. Of course, they apply only in the case of the simplest re-

⁵ Again, Breit's work anticipates many of the results derived here. Although Breit's paper deals with a rather special model, a comparison of his formulae with ours shows that the model used by him already exhibits practically all the features which prevail in the general case.

actions in which only one pair of reaction products is possible.

In order to derive an expression for the cross sections which is not subject to the limitations to which (42) is subject, one will try to avoid the doubtful diagonalization (39b) and try to solve (37) for the $A_{\mu\lambda}$ more directly. If several reaction products are possible, one will obtain a manageable expression only if one resorts to approximations, and it will be assumed that the energy differences are larger than the widths Γ . It will then turn out that the $A_{\mu\mu}$ are, in general, considerably larger than the $A_{\mu\lambda}$ with $\mu \neq \lambda$. One can, therefore, obtain a first approximation for $A_{\mu\mu}$ by using the equation $\lambda = \mu$ of (37) and neglecting the $A_{\mu\nu}$ terms with $\mu \neq \nu$. This gives

$$A_{\mu\mu} = \frac{1}{E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu}}. \quad (43)$$

One then obtains for $A_{\mu\lambda}$ from (37) by keeping, from the sum, only the terms $\nu = \lambda$ and $\nu = \mu$ and using (43) for the latter

$$A_{\mu\lambda} = \frac{\frac{1}{2}i\Gamma_{\mu\lambda}}{(E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu})(E_{\lambda} - E - \frac{1}{2}i\Gamma_{\lambda\lambda})}, \quad (\mu \neq \lambda). \quad (43a)$$

In the next approximation, it is necessary to take the $A_{\mu\mu}$ with $\mu \neq \lambda$ into account. Using (43a) for these, the matrix needed for (33) becomes

$$\frac{\Re}{1 - i\mathfrak{q}\mathfrak{H}\mathfrak{q}} = \sum_{\mu} \frac{(\gamma_{\mu} \times \gamma_{\mu})}{E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu}} + \sum_{\lambda \neq \mu} \frac{\frac{1}{2}i\Gamma_{\mu\lambda}(\gamma_{\mu} \times \gamma_{\lambda})}{(E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu})(E_{\lambda} - E - \frac{1}{2}i\Gamma_{\lambda\lambda})}. \quad (45)$$

In this approximation, it is still easy to write (45) in the form (41). The F_{μ} are the same as in (44)

$$F_{\mu} = E_{\mu} - \frac{1}{2}i\Gamma_{\mu\mu} \quad (45a)$$

and the s_{μ} becomes

$$\mathbf{s}_{\mu} = \gamma_{\mu} + \sum_{\lambda \neq \mu} \frac{\frac{1}{2}i\Gamma_{\mu\lambda}}{F_{\lambda} - F_{\mu}} \gamma_{\lambda}. \quad (45b)$$

One sees that neither the F nor the \mathbf{s} are entirely independent of the energy. The \mathbf{s} are not real any more as they were in first approximation. However, in the present approximation, i.e., up to terms of the order $\Gamma/(E_{\lambda} - E_{\mu})$, the imaginary part of F_{ν} is still equal to the square of the length of $q\gamma_{\nu}$.

The above is intended to show that if one uses equations of the form (42) without taking the restrictions on the F and \mathbf{s} into consideration, one automatically restricts oneself to the case in which the level width is small compared with the spacing of the levels. Under such conditions, the matrix elements are at least approximately real. As one approaches the case in which the Γ are of the same order as the spacing of the levels, the \mathbf{s} become complex in general. However, there remain relations which replace the reality condition which is valid for $\Gamma \ll E_{\lambda} - E_{\mu}$. It does not appear justifiable to make statistical statements on the cross sections without taking these conditions into account. Of

This can be considered to be small as compared with (43) because the energy differences are, in general, larger than the Γ . For this reason, the terms with $\lambda \neq \mu$ may be neglected in first approximation in (38) and one obtains with (33)

$$\sigma_{ji}(E) = \frac{4\pi k_l}{k_j} \sum_{\mathcal{J}} g_{\mathcal{J}} \left| \sum_{\mu}^{\mathcal{J}} \frac{\gamma_{\mu j} \gamma_{\mu l}}{E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu}} \right|^2. \quad (44)$$

This equation actually has the form of (42) and constitutes an approximation to it in the same sense as (14) is an approximation to (13c). One sees the analogy between (44) and the customary one level formula, perhaps, most easily if one denotes

$$(q_j \gamma_{\mu j})^2 = k_j \gamma_{\mu j}^2 = \frac{1}{2} \Gamma_{\mu j}, \quad (44a)$$

so that, because of (37a), $\sum_j \Gamma_{\mu j} = \Gamma_{\mu\mu}$. One then sees that $\Gamma_{\mu\mu}$ is what is usually denoted by Γ_{μ} and that the matrix elements, which are the $q_j \gamma_{\mu j}$, are real. This shows that $s_{\nu j}$ of (42) are also approximately real, i.e., their complex phases are small. However, if the Γ are of the same order of magnitude as the energy differences between resonances, (44) may become grossly inaccurate, just as (14) was inaccurate under similar conditions.

course, (33) always remains valid, but it does not seem to be easy to make statistical statements on the basis thereof without resorting to somewhat crude assumptions.

One can calculate the $A_{\mu\mu}$ in the following approximation by taking into account all terms in the Eq. (37) with $\lambda = \mu$ but using for $A_{\mu\nu}$ the approximation (43a). This gives

$$(E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu})A_{\mu\mu} = 1 - \frac{1}{4} \sum_{\nu \neq \mu} \frac{\Gamma_{\nu\mu}^2}{(E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu})(E_{\nu} - E - \frac{1}{2}i\Gamma_{\nu\nu})}.$$

Since the last term herein is a correction term one may write for the right side $1/(1 + \frac{1}{4}\sum \dots)$ where $\sum \dots$ is the sum on the right side. This gives then

$$A_{\mu\mu} = \frac{1}{E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu} + \frac{1}{4} \sum_{\nu} \Gamma_{\nu\mu}^2 / (E_{\nu} - E - \frac{1}{2}i\Gamma_{\nu\nu})}. \quad (46)$$

The term with $\nu = \mu$ has to be omitted in the summation over ν in the denominator. In general, (46) is still approximate. However, one can easily convince oneself that (46) is actually accurate if there is, in addition to μ , only one more resonance level. The approximation corresponding to (46) for $A_{\mu\lambda}$ with $\mu \neq \lambda$ is

$$A_{\mu\lambda} = \frac{\frac{1}{2}i\Gamma_{\mu\lambda}^2}{(E_{\mu} - E - \frac{1}{2}i\Gamma_{\mu\mu})(E_{\lambda} - E - \frac{1}{2}i\Gamma_{\lambda\lambda}) \left(\Gamma_{\mu\lambda} - \frac{1}{2}i \sum_{\nu} \frac{\Gamma_{\nu\lambda}\Gamma_{\nu\mu}}{E_{\nu} - E - \frac{1}{2}i\Gamma_{\nu\nu}} \right) + \frac{1}{4}\Gamma_{\mu\lambda}^3}. \quad (46a)$$

In the summation over ν in the denominator the terms $\nu = \mu$ and $\nu = \lambda$ have to be omitted so that the sum over ν vanishes if there are only two resonance levels, μ and λ . Under this condition (46a) is accurate and differs only by the last term in the denominator from (43a).

Evidently, the last equations are too complicated for being of great practical value. This can hardly be expected otherwise since our result is quite general except that we assumed only short range forces and considered only collisions in which the angular momentum of the motion of the colliding particles vanishes. Hence (33) should be able to represent the variation of the various cross sections with energy under almost arbitrary conditions. In most cases, if special conditions prevail which are different from those assumed above, it may be easiest to refer back to (33). One property of the cross sections that is of fairly great generality can be, however, easily checked by means of the above formulae, that is that the reaction cross section goes through zero between successive maxima, at least if $\gamma_{\lambda i} \gamma_{\lambda j}$ has the same sign for both, if only two pairs of products are possible but that this does not occur if several products can be formed.