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The Stability of Synchrotron Orbits*

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The stability of electron orbits in a synchrotron with a frequency modulated r-f has been investigated. The method consists in expanding the equations of motion and in solving them to zeroth and to first order of approximation. Attention is centered on the relatively low voltages —up to 200 Mev—and consequently the effects of radiation damping are of minor importance and may be neglected. With the correct frequency modulation, the electrons, in zeroth order, move in circular orbits with a constant radius. In first order their r and θ coordinates oscillate with the two frequencies ω_1 and ω_2 while the axial coordinate z oscillates with the frequency ω_3. It is shown that as the energy of the electrons increases, the amplitudes of the ω_1 and ω_3 motions decrease approximately as $1/E^{\frac{1}{2}}$ while the ω_2 amplitude decreases as $1/E^{\frac{3}{4}}$. Thus the electron orbits are stable. A numerical example is calculated in detail for a machine where the injection energy is $\frac{1}{2}$ Mev and the final energy 150 Mev. The accuracy of the solution is discussed and it is concluded that it will be correct to better than one percent provided the frequencies ω_1, ω_2, and ω_3 are not commensurable. In the latter case, under certain conditions, secular changes in the orbits could occur which would destroy their stability.

THE purpose of the present paper is to investigate the stability of electron orbits in a synchrotron accelerator. The type of synchrotron to be considered is similar to that described by Veksler[1] and by McMillan,[2] but contains a variation proposed by H. R. Crane; namely, the radiofrequency which furnishes the electron acceleration is to be frequency modulated in such a manner that the equilibrium electron orbits consist of circles with a constant radius.

The equations of motion for the electrons are very complex and there is little hope of finding an exact solution of them. It is, however, possible to develop the variables and to solve by the method of successive approximation. The present work carries the problem through the zeroth and first orders of approximation. It is found that, providing the injection velocity of the electrons is fairly high, say 25,000 ev or more, the orbits are definitely stable and that any oscillational motions which are initially present will be damped out as the electrons attain higher energy.

Cylindrical coordinates will be employed. The z axis is the symmetry axis of the synchrotron, r, the radius of the orbit, and θ is its azimuthal angle. The electron orbits lie in a ring shaped region defined by the coordinates $z=\pm z_0$ and $r=a\pm r_0$. Within this region the magnetic field has its greatest component in the z direction but, except in the median plane, it also possesses an r component. It is assumed that the field varies inversely as r^n and that it changes sinusoidally in time. The electron motion is to take place close to the median plane, $z=0$, and to the equilibrium orbit where $r=a$. It is therefore possible to develop

$$\frac{1}{r^n} \quad \text{as} \quad \frac{1}{a^n}-\frac{n(r-a)}{a^{n+1}}+\cdots$$

and, since the calculation will not go beyond first order, to retain only the first two terms. The magnetic field has neither curl nor divergence in the region and these conditions impose relationships between the components of the field. (Strictly speaking, curl H is not zero since H is assumed to vary with the time. The extra terms, however, are proportional to $a^2\Omega^2/c^2$ and are negligibly small.) One thus obtains,

$$H_z=H_0(1-n(r-a)/a)\sin\Omega t,$$
$$H_r=-(nz/a)H_0\sin\Omega t,$$
$$H_\theta=0.$$

The electron will be subjected to two types of electric fields both of which are directed mainly in the θ, that is, in a tangential direction. The first of these is caused by the betatron effect and is equal to,

$$(\epsilon_\theta)_\beta=-\frac{1}{2\pi rc}\int_0^r\frac{\partial H_z}{\partial t}2\pi r\,dr.$$

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[1] V. Veksler, J. Phys. U.S.S.R. 9, 153 (1945).

[2] E. M. McMillan, Phys. Rev. 68, 143 (1945).

The integral may be divided into two parts, (1) the contribution from $r=0$ to $r=a$ and (2) the contribution from $r=a$ to $r=r$. The former may be designated by means of a constant coefficient, L, while the latter may be developed in powers of $(r-a)/a$ and only the terms through the first order of approximation retained.

$$(\epsilon_\theta)_\beta = -(\Omega/cr)[L+H_0a(r-a)]\cos\Omega t.$$

The second type of electric field is that caused by the radiofrequency. Let there be N gaps each with an oscillating potential of frequency ω_0 and magnitude V_0. While the accelerations occur at the gaps, it will be convenient to consider that the electric field is spread out over the entire ring. The field which the electron feels depends upon the phase of its motion relative to the phase of the radiofrequency. Thus

$$(\epsilon_\theta)_{\text{rf}} = -\frac{V_0N}{2\pi r}\sin\left(\theta - \int^t \omega_0\,dt\right).$$

The radiofrequency, ω_0, is to be made a function of the time in order that the equilibrium orbit may have the constant radius $r=a$, and for that reason it will prove convenient to express the argument of the radiofrequency as $\int^t\omega_0dt$.

The equations of motion of an electron in an electric and magnetic field are well known and in cylindrical coordinates have the following form,

$$\frac{d}{dt}(m\dot r) - mr\dot\theta^2 = \frac{e}{c}[VH]_r + e\epsilon_r, \tag{1}$$

$$\frac{d(r^2m\dot\theta)}{dt} = \frac{er}{c}[VH]_\theta + er\epsilon_\theta, \tag{2}$$

$$\frac{d}{dt}(m\dot z) = \frac{e}{c}[VH]_z + e\epsilon_z. \tag{3}$$

The velocity V and the mass m of the electron are further related,

$$m = \frac{m_0}{[1-(V^2/c^2)]^{\frac{1}{2}}}, \tag{4}$$

and

$$V^2 = \dot r^2 + r^2\dot\theta^2 + \dot z^2. \tag{5}$$

These equations may be put in a somewhat more suitable form. Multiply Eqs. (1)–(3) by $m\dot r$, $m\dot\theta$, and $m\dot z$ and add. The result may be written,

$$\frac{1}{2}\frac{d}{dt}[m^2\dot r^2 + m^2r^2\dot\theta^2 + m^2\dot z^2] = em(\dot r\epsilon_r + r\dot\theta\epsilon_\theta + \dot z\epsilon_z).$$

The left-hand side of the equation is, however, from (4) and (5) equal to

$$\frac{1}{2}\frac{d}{dt}(m^2c^2 - m_0{}^2c^2) = m\dot m c^2.$$

Thus the four equations which define r, θ, z, and m may be expressed as,

$$\frac{d}{dt}(m\dot r) - mr\dot\theta^2 = \frac{e}{c}[VH]_r + e\epsilon_r, \tag{6}$$

$$c^2\frac{dm}{dt} = e(\dot r\epsilon_r + r\dot\theta\epsilon_\theta + \dot z\epsilon_z), \tag{7}$$

$$\frac{d}{dt}(m\dot z) = \frac{e}{c}[VH]_z + e\epsilon_z, \tag{8}$$

$$m^2\dot r^2 + m^2r^2\dot\theta^2 + m^2\dot z^2 = m^2c^2 - m_0{}^2c^2. \tag{9}$$

The magnetic and electric field components may be substituted into the above equations. It will be convenient to replace e by $-e$, corresponding to the negative electronic charge. Two simplifying assumptions will be made. (1) The growth of the magnetic field is exceedingly slow in comparison with the electronic motions and, therefore, in any study of the stability of the orbits it will be possible to replace $\sin\Omega t$ by Ωt and $\cos\Omega t$ by unity without materially changing the problem. This approximation is strictly valid only for the initial portion of the cycle but it will be shown later that it may be adapted for any portion. (2) It will appear that for reasonable values of the radiofrequency potential V_0, the phase $\int^t\omega_0dt-\theta$ will be small and consequently $\sin(\int^t\omega_0dt-\theta)$ will be simplified to $\int^t\omega_0dt-\theta$. An auxiliary study has been made which apparently shows that the assumed simplification, while it may affect the motion somewhat, cannot have any essential influence upon the orbit stability.

With these simplifications and substitutions the equations of motion (6) through (9) become,

$$m\frac{d\dot{r}}{dt}+\dot{r}\dot{m}-mr\dot{\theta}^2=-\frac{e}{c}r\dot{\theta}\left[1-\frac{n(r-a)}{a}\right]H_0\Omega t, \quad (10)$$

$$c^2\dot{m}=\frac{e\Omega\dot{\theta}}{c}[L+H_0a(r-a)]+\frac{eV_0N\dot{\theta}}{2\pi}\left(\theta-\int^t\omega_0dt\right), \quad (11)$$

$$m\frac{d\dot{z}}{dt}+\dot{m}\dot{z}=-\frac{enr\dot{\theta}z}{ca}H_0\Omega t, \quad (12)$$

$$m^2\dot{r}^2+m^2r^2\dot{\theta}^2+m^2\dot{z}^2=m^2c^2-m_0{}^2c^2. \quad (13)$$

It will be noted that terms arising from the radiation damping have been omitted. Two remarks may be made. (1) The most critical portion of the electron motion from the point of view of orbit stability is the initial portion and here the radiation is of negligible importance. (2) At the later times when the radiation becomes larger, it will act as a very slowly varying frictional force. Its effect will be merely to displace the phase of the electron relative to the radiofrequency acceleration so that on each cycle the electron may receive sufficient energy to compensate for the radiation losses.

In attacking the equations of motion, it will be fruitful to investigate those orbits (a) which lie in the median plane and (b) for which the radius is constant. These will be defined as the equilibrium orbits and will represent the zeroth approximation to the general motion. Setting $r=a$ and $z=0$ into the equations, one obtains,

$$m=m_0(1+K^2t^2)^{\frac{1}{2}},$$

$$\theta=\int^t\omega_0dt+(a^2H_0-L)\frac{2\pi\Omega}{V_0Nc},$$

$$\omega_0=\frac{c}{a}\frac{Kt}{(1+K^2t^2)^{\frac{1}{2}}},$$

where

$$K=aeH_0\Omega/m_0c^2.$$

Since for the equilibrium orbit, the electron velocity is $a\dot{\theta}$, the radiofrequency may be expressed as,

$$\omega_0=\dot{\theta}=V/a=c\beta/a.$$

It will now be assumed that the actual orbits of the electron differ from the equilibrium orbits by only small quantities. Thus, let,

$$r=a+\lambda\rho,$$

$$m=m_0(1+K^2t^2)^{\frac{1}{2}}+\lambda\mu,$$

$$z=\lambda\zeta,$$

$$\theta=\int^t\omega_0dt+(a^2H_0-L)\frac{2\pi\Omega}{V_0Nc}+\lambda\varphi,$$

$$\omega_0=\frac{c}{a}\frac{Kt}{(1+K^2t^2)^{\frac{1}{2}}},$$

where λ is a parameter of smallness and ρ, μ, φ, and ζ are functions of the time.

These values for r, m, z, and θ are substituted in the equations of motion (10), (11), (12), and (13). The zeroth-order terms are automatically satisfied and all terms in λ^2 and higher are neglected since only an approximation through first order is desired.

$$\frac{d\dot{\rho}}{dt}+\frac{K^2t}{1+K^2t^2}\dot{\rho}+\frac{c^2}{a^2}\frac{(1-n)K^2t^2}{(1+K^2t^2)}\rho-\frac{c^2\mu}{m_0a(1+K^2t^2)^{\frac{1}{2}}}=0, \quad (14)$$

$$\rho+\frac{a^2}{c}\frac{(1+K^2t^2)^{\frac{1}{2}}}{Kt}\dot{\varphi}-\frac{a\mu}{m_0K^2t^2(1+K^2t^2)^{\frac{1}{2}}}=0, \quad (15)$$

$$\dot{\mu}-\frac{eNV_0Kt}{2\pi ac(1+K^2t^2)^{\frac{1}{2}}}\varphi-\frac{\mu}{t(1+K^2t^2)}=0, \quad (16)$$

$$\frac{d\dot{\zeta}}{dt}+\frac{K^2t}{1+K^2t^2}\dot{\zeta}+\frac{c^2nK^2t^2}{a^2(1+K^2t^2)}\zeta=0. \quad (17)$$

The straightforward solution of these equations is still very difficult although perhaps not impossible. It will, however, be possible to obtain a rather simple solution which holds to a high order of approximation except in the immediate neighborhood of $t=0$. According to our equations $t=0$ represents $m=m_0$ and consequently a zero initial injection energy. It is, however, planned to inject at, say, $\frac{1}{2}$ Mev and at this, or higher energies, t is much greater than the critical region just referred to. (This point will be discussed later.)

The method of making the approximation lies in the realization that the magnetic field is changing very slowly indeed compared with the orbital motion of the electrons. Let us suppose that at a time $t=T$, the magnetic field is frozen. The values of r, m, z, and θ in this case are, to first order of approximation,

$$r=a+\lambda\rho_0,$$

$$m=m_0(1+K^2T^2)^{\frac{1}{2}}+\lambda\mu_0,$$

$$z=\lambda\zeta_0,$$

$$\theta=\omega_0 t+\lambda\varphi_0,$$

$$\omega_0=\frac{c}{a}\frac{KT}{(1+K^2T^2)^{\frac{1}{2}}}.$$

Since the magnetic field is frozen, the betatron terms $a^2H_0\Omega$ and $L\Omega$ of course vanish. The equations for ρ_0, μ_0, ζ_0 and φ_0 may be obtained by the method used earlier and are,

$$\frac{d\dot{\rho}_0}{dt}+\omega_0^2(1-n)\rho_0-\frac{c^2\mu_0}{m_0a(1+K^2T^2)^{\frac{1}{2}}}=0, \quad (18)$$

$$\rho_0+\frac{a}{\omega_0}\dot{\varphi}_0-\frac{a\mu_0}{m_0K^2T^2(1+K^2T^2)^{\frac{1}{2}}}=0, \quad (19)$$

$$\dot{\mu}_0-\frac{eNV_0\omega_0}{2\pi c^2}\varphi_0=0, \quad (20)$$

$$\frac{d\dot{\zeta}_0}{dt}+n\omega_0^2\zeta_0=0. \quad (21)$$

Since in these equations ω_0 and T are constants, it is readily possible to integrate them exactly. The results are as follows where the C_S and B_S are integration constants and where ω_1 and ω_2 are the positive roots of the equation.

$$\omega^4-\left[(1-n)\omega_0^2-\frac{\alpha}{(1+K^2T^2)^{\frac{3}{2}}}\right]\omega^2+\frac{\alpha\omega_0^2(K^2T^2+n)}{(K^2T^2+1)^{\frac{3}{2}}}=0,$$

where $\alpha=eNV_0/2\pi a^2m_0$.

$$\rho_0=C_1\cos(\omega_1t+B_1)+C_2\cos(\omega_2t+B_2),$$

$$\mu_0=\frac{m_0a\alpha}{c^2(1-n)}C_1\cos(\omega_1t+B_1)+\frac{m_0(1-n)}{a}\frac{K^2T^2}{(1+K^2T^2)^{\frac{1}{2}}}C_2\cos(\omega_2t+B_2),$$

$$\varphi_0=-\frac{C_1}{a(1-n)^{\frac{1}{2}}}\sin(\omega_1t+B_1)-\frac{c(1-n)^{\frac{1}{2}}}{a^2\alpha^{\frac{1}{2}}}\frac{KT(K^2T^2+n)^{\frac{1}{2}}}{(K^2T^2+1)^{\frac{3}{4}}}C_2\sin(\omega_2t+B_2),$$

$$\zeta_0=C_3\cos(\omega_3t+B_3),$$

where $\omega_3=n^{\frac{1}{2}}\omega_0$.

This solution for the case of the constant magnetic field is very interesting and contains in it many of the essential features of the motion. The coordinates r, m, and z oscillate harmonically around their average values, namely, a $m_0(1+K^2T^2)^{\frac{1}{2}}$ and zero, respectively. The motion of θ consists of a uniform increase with the time upon which is superimposed a small harmonic oscillation. The average value of the electron mass divided by m_0 will be defined as $f=(1+K^2T^2)^{\frac{1}{2}}$. The equation defining ω_1 and ω_2 is, therefore,

$$\omega^4-\left[(1-n)\omega_0^2-\frac{\alpha}{f^3}\right]\omega^2+\frac{\alpha\omega_0^2(f^2-1+n)}{f^3}=0,$$

and has the solution,

$$\omega^2=\frac{(1-n)\omega_0^2}{2}-\frac{\alpha}{2f^3}\pm\left[\left(\frac{(1-n)\omega_0^2}{2}-\frac{\alpha}{2f^3}\right)^2-\frac{\alpha\omega_0^2(f^2-1+n)}{f^3}\right]^{\frac{1}{2}}. \quad (22)$$

The numerical order of magnitude of the quantities involved in the above equation may be readily estimated for the two limiting cases $f=2$; that is, the time of injection, and $f=200$, the time when the electron will have acquired approximately 100 Mev. Let $a=100$ cm and $n=\frac{1}{2}$. (n determines the fall off law of the static magnetic field.) Let the radiofrequency potential and the number of gaps N be such that $V_0N=3000$ volts $=10$ e.s.u.

Quantity	$f=2$	$f=200$
$\frac{(1-n)\omega_0^2}{2}$	1.7×10^{16}	2.2×10^{16}
$\frac{\alpha}{2f^3}$	5.3×10^{12}	5.3×10^{6}
$\frac{\alpha\omega_0^2(f^2-1+n)}{f^3}$	2.5×10^{30}	3.8×10^{28}

A study of these figures shows that ω_1 and ω_2 may be expressed as,

$$\omega_1=(1-n)^{\frac{1}{2}}\omega_0, \qquad (23)$$

$$\omega_2=\frac{\alpha^{\frac{1}{2}}(f^2-1+n)^{\frac{1}{2}}}{(1-n)^{\frac{1}{2}}f^{\frac{3}{2}}}, \qquad (24)$$

with an accuracy equal to one part in a thousand when $f=2$ and far better for $f=200$.

The description of the electron motion in the constant magnetic field may now be completed.

1. The average mass of the electron is fm_0.
2. The principal frequency is that associated with the uniform increase of the azimuthal angle θ. It is

$$\omega_0=\frac{c}{a}\frac{(f^2-1)^{\frac{1}{2}}}{f}.$$

3. The electron oscillates in the z direction with a constant amplitude (determined by the initial conditions) and with a circular frequency $n^{\frac{1}{2}}\omega_0$. Evidently this motion is stable only if $n>0$.
4. The radial distance r is equal to a constant plus the sum of two harmonic terms. The first of these has a frequency $\omega_1=(1-n)^{\frac{1}{2}}\omega_0$ which is independent of the accelerating electric field. It will be spoken of as the radial frequency. Clearly this motion will be stable provided $n<1$.
5. The second harmonic term has a frequency

$$\omega_2=\frac{\alpha^{\frac{1}{2}}}{(1-n)^{\frac{1}{2}}}\frac{(f^2-1+n)^{\frac{1}{2}}}{f^{\frac{3}{2}}}.$$

This will be called the frequency of the phase oscillation (phase of θ relative to the radio-frequency field). It is proportional to the square root of the accelerating potential and, for large energies, inversely to the square root of the electron energy. By substituting the numerical constants $a=100$ cm, $n=\frac{1}{2}$, and $V_0N=10$ e.s.u. one finds that ω_2/ω_0 varies from 1/30 to 1/330 as f increases from 2 to 200.

6. The amplitudes of oscillation of r, θ, and m are interrelated for both of the harmonic oscillations ω_1 and ω_2 but otherwise are constant and are determined by the initial conditions.

The nature of the solution for the case where the magnetic field increases slowly may be inferred from the solutions just obtained for the frozen field problem. It is to be expected that the amplitudes C_1, C_2, and C_3 will be slowly varying functions of the time. The frequencies ω_0, ω_1, ω_2, and ω_3 are functions of f, and hence will depend upon the time. These changes will be brought about by the fact that Eqs. (14)–(17) differ from (18)–(21). The differences are of two sorts, (a) the coefficients of the dependent variables and their derivatives are functions of the time in the first group of equations while they are constants in the second group. (b) Equation (14) has the extra term

$$\frac{K^2t}{1+K^2t^2}\dot{\rho},$$

Eq. (16) the extra term

$$-\frac{\mu}{t(1+K^2t^2)},$$

and Eq. (17) the term

$$\frac{K^2t}{1+K^2t^2}\dot{\zeta}.$$

It is to be expected that the influence of these extra terms will be small and will contribute to the slow time variation of the amplitudes and frequencies. It will be convenient to divide the various terms of Eq. (14) through (17) into principal terms and extra terms, the latter being those which have just been defined.

It is now assumed that the solution of Eqs. (14)–(17) may be written in the following form:

$$\rho=D_1\cos\left(\int^t\omega_1dt+B_1\right)+D_2\cos\left(\int^t\omega_2dt+B_2\right),$$

$$\mu=D_3\cos\left(\int^t\omega_1dt+B_1\right)+E_3\sin\left(\int^t\omega_1dt+B_1\right),+D_4\cos\left(\int^t\omega_2dt+B_2\right)+E_4\sin\left(\int^t\omega_2dt+B_2\right),$$

$$\varphi = D_5 \sin\left(\int^t \omega_1 dt + B_1\right) + E_5 \cos\left(\int^t \omega_1 dt + B_1\right) + D_6 \sin\left(\int^t \omega_2 dt + B_2\right) + E_6 \cos\left(\int^t \omega_2 dt + B_2\right),$$

$$\zeta = D_7 \cos\left(\int^t \omega_3 dt + B_3\right).$$

It is understood that the above expressions constitute a solution of the equations of motion subject to the following conditions.

1. B_1, B_2, and B_3 are integration constants.

2. D_1, D_2, D_3, D_4, D_5, D_6, and D_7 are slowly varying functions of the time. Thus when they occur in the principal terms of the differential equations, D_i and $\dot{D}_i$ are to be retained but $d\dot{D}_i/dt$ is to be neglected. In the case of the extra terms (whose influence is assumed to be small), D_i will be retained, but $\dot{D}_i$ and $d\dot{D}_i/dt$ are discarded.

3. E_3, E_4, E_5, and E_6 are small, slowly varying, functions of the time. In the principal terms E_i is retained but $\dot{E}_i$ and $d\dot{E}_i/dt$ are not. In the extra terms, E_i, as well as its derivatives, will be neglected.

4. ω_1 and ω_2 are assumed to have the form given by Eqs. (23) and (24) rather than by the more exact relation (22). This means that in any of the equations, a term involving α is to be neglected with respect to a term involving ω_0^2. This last assumption is merely a matter of convenience. It would be quite possible to carry through the work using the exact relation (22), but the anticipated changes from this procedure would amount to only one part in a thousand even for $f=2$.

The above conditions governing the solution of the equations of motion may appear to limit the validity of the results. Actually this is not the case. The basic ideas behind them are very clear and are, (a) an exact solution for the problem of a constant magnetic field has been found, and (b) in the actual problem the magnetic field changes very slowly indeed compared with the motion of the electron. Consequently, the solution for the actual problem cannot differ greatly from the constant field case and must represent a slow unfolding of that motion.

The complete solution for r, θ, z, and m through the zeroth and first order of approximation will now be given. It will prove convenient to express the frequencies in terms of f, the ratio of the average electron mass to its rest mass.

$$f = (1+K^2t^2)^{\frac{1}{2}}, \quad K = \frac{aeH_0\Omega}{m_0c^2}, \quad \alpha = \frac{eNV_0}{2\pi a^2 m_0}, \quad \omega_0 = \frac{c(f^2-1)^{\frac{1}{2}}}{af}, \quad \omega_1 = (1-n)^{\frac{1}{2}}\omega_0,$$

$$\omega_2 = \frac{\alpha^{\frac{1}{2}}(f^2-1+n)^{\frac{1}{2}}}{(1-n)^{\frac{1}{2}} f^{\frac{3}{2}}}, \quad \omega_3 = n^{\frac{1}{2}}\omega_0,$$

A_1, A_2, A_3, B_1, B_2, B_3 are integration constants.

$$r = a + \frac{A_1}{(f^2-1)^{\frac{1}{4}}} \cos\left(\int^t \omega_1 dt + B_1\right) + A_2\left[\frac{f^3}{(f^2-1+n)(f^2-1)^2}\right]^{\frac{1}{4}} \cos\left(\int^t \omega_2 dt + B_2\right),$$

$$\theta = \int^t \omega_0 dt + \frac{2\pi\Omega(a^2H_0-L)}{V_0Nc} - \frac{A_1}{a(1-n)^{\frac{1}{2}}(f^2-1)^{\frac{1}{4}}}\left[\sin\left(\int^t \omega_1 dt + B_1\right) - \frac{aKf}{2c(1-n)^{\frac{1}{2}}(f^2-1)} \cos\left(\int^t \omega_1 dt + B_1\right)\right] - \frac{c(1-n)^{\frac{1}{2}}}{a^2\alpha^{\frac{1}{2}}} A_2\left[\frac{f^2-1+n}{f^3}\right]^{\frac{1}{4}}$$
$$\times\left[\sin\left(\int^t \omega_2 dt + B_2\right) + \frac{K(1-n)^{\frac{1}{2}} f^{\frac{3}{2}}(f^2-1)^{\frac{1}{2}}}{4\alpha^{\frac{1}{2}}(f^2-1+n)^{\frac{3}{2}}}\left(\frac{3(1-n)}{f^2} - 1\right) \cos\left(\int^t \omega_2 dt + B_2\right)\right],$$

$$z=\frac{A_3}{(f^2-1)^{\frac{1}{4}}}\cos\left(\int^t \omega_3 dt+B_3\right),$$

$$m=fm_0+\frac{m_0 a\alpha A_1}{c^2(1-n)(f^2-1)^{\frac{1}{4}}}\left[\cos\left(\int^t \omega_1 dt+B_1\right)+\frac{aK(f^2+1)}{c(1-n)^{\frac{1}{2}}f(f^2-1)}\sin\left(\int^t \omega_1 dt+B_1\right)\right]$$

$$+\frac{m_0(1-n)A_2}{a}\left[\frac{(f^2-1)^2}{(f^2-1+n)f}\right]^{\frac{1}{4}}\left[\cos\left(\int^t \omega_2 dt+B_2\right)+\frac{a^2K\alpha^{\frac{1}{2}}(f^2+n-1)^{\frac{1}{2}}(f^2+1)}{c^2(1-n)^{\frac{3}{4}}f^{\frac{3}{4}}(f^2-1)^{\frac{3}{4}}}\sin\left(\int^t \omega_2 dt+B_2\right)\right].$$

The solution just given is very satisfactory. It shows that the orbits are stable since the amplitude of the oscillations of z and of r and θ for the ω_1 vibration decrease as $1/f^{\frac{1}{2}}$ for large values of f. The amplitude of r and θ for the ω_2 motion (phase oscillation) decreases for large f as $1/f^{\frac{1}{4}}$. The phase differences between the motions (represented earlier by the quantities E_3, E_4, E_5, and E_6) also decrease as f increases. The amplitude of the oscillations in the electron mass for the vibration ω_2 will increase as $f^{\frac{1}{4}}$. This does nor represent any instability in the electron orbits, however, and merely gives rise to a slight spread in the final electron energies.

The expressions for r, θ, z, and m have been derived for the case of a uniformly increasing magnetic field. The whole point of the derivation, however, was that the magnetic field changes so slowly that the motion may be represented by slow changes in the constant magnetic field orbits. The particular manner in which the magnetic field changes is immaterial provided the rate of change is small enough. The expressions for r, θ, z, m, ω_0, ω_1, ω_2, and ω_3 given above may be readily modified for a sinusoidally changing magnetic field by making the following changes only.

(1) Replace

$$f=(1+K^2t^2)^{\frac{1}{2}} \text{ by } f=\left(1+\frac{K^2\sin^2\Omega t}{\Omega^2}\right)^{\frac{1}{2}}.$$

(2) Replace the constant phase angle of the equilibrium orbit,

$$\frac{2\pi\Omega}{V_0Nc}(a^2H_0-L), \text{ by } \frac{2\pi\Omega}{V_0Nc}(a^2H_0-L)\cos\Omega t.$$

(3) Wherever K occurs explicitly in the expressions for θ and m, replace it by $K\cos\Omega t$. (This is *not* to be construed as any change in the definition of K itself, however.)

The physical meaning of the constants K and α may be visualized in a number of ways. If the final mass attained by the electron at the end of the magnetic quarter cycle is m^*, then $K=\Omega[(m^*/m_0)^2-1]^{\frac{1}{2}}$. A second method of interpreting K consists in calculating the increase in the average electron energy per revolution divided by its rest mass. It may be readily shown that this quantity,

$$\frac{2\pi}{\omega_0 E_0}\left\langle\frac{dE}{dt}\right\rangle_{\text{Av}} \text{ is equal to } \frac{2\pi a}{c}K\cos\Omega t.$$

The magnetic field has been assumed to increase very slowly and hence the numerical values chosen for any actual case must be such that $K\ll c/(2\pi a\cos\Omega t)$. The constant α is related to the radiofrequency potential. If the electron should make one revolution in such a manner as to pick up the maximum amount of energy from the radiofrequency field, the phase angle would necessarily be 90 degrees. The energy it would obtain, divided by the rest energy, would be

$$\frac{(\Delta E)_{\max}}{E_0}=\frac{eNV_0}{m_0c^2}.$$

This quantity is, however, equal to

$$\frac{2\pi a^2}{c^2}\alpha=\frac{2\pi(f^2-1)\alpha}{f^2\omega_0^2}.$$

$$\alpha=\frac{f^2\omega_0^2}{2\pi(f^2-1)}\frac{(\Delta E)_{\max}}{E_0}.$$

If $NV_0=3000$ volts, then

$$\frac{(\Delta E)_{\max}}{E_0}\cong\frac{3}{500} \quad\text{and}\quad \frac{\alpha}{\omega_0^2}=\frac{3f^2\times 10^{-3}}{\pi(f^2-1)}.$$

It is therefore clear that the approximation which has been used, namely, one in which α

TABLE I. Numerical example.

t	No. of turns	$\omega_0/2\pi$	f	Equilibrium phase	z or r amplitude due to ω_3 or ω_1	r amp. due to ω_2	θ amp. due to ω_2	m amp. due to ω_2	ω_2/ω_0
1.57×10^{-5}	0	41.4	2	28.3°	1.0 cm	1.32 cm	20°	0.010 m_0	1/30
2.56×10^{-5}	425	45.1	3	28.3°	0.78	0.88	18.4°	0.012 m_0	1/39
4.44×10^{-5}	1,278	46.8	5	28.3°	0.59	0.57	16.4°	0.014 m_0	1/51
9.00×10^{-5}	3,410	47.5	10	28.3°	0.42	0.33	13.8°	0.016 m_0	1/73
1.81×10^{-4}	7,660	47.7	20	28.2°	0.29	0.20	11.5°	0.020 m_0	1/103
5.60×10^{-4}	16,230	47.8	40	28.0°	0.21	0.12	9.8°	0.023 m_0	1/146
2.01×10^{-3}	42,500	47.8	100	26.6°	0.13	0.06	7.8°	0.029 m_0	1/230
4.16×10^{-3}	198,000	47.8	298	0.0°	0.08	0.03	5.9°	0.039 m_0	1/400

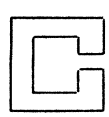
may be neglected in comparison to ω_0^2, will be very good except in the region where f is close to unity. Even for $f=1.05$, corresponding to an electron energy of 25 kev, $\alpha/\omega_0^2=1.02\times 10^{-2}$.

It may be of interest to construct a numerical example which will further bring out the order of magnitude of the quantities involved. We shall choose a sinusoidally varying magnetic field with an Ω corresponding to 60 cycles/sec. Let $a=100$ cm, $H_0=5000$ gauss, $V_0N=3000$ volts, $n=\frac{1}{2}$, and let the injection voltage correspond to $\frac{1}{2}$ Mev or $f=2$. The derived constants become, $K=1.105\times 10^5$, $\alpha=8.43\times 10^{13}$. Let the integration constants be chosen so that the initial amplitude of the z oscillation is 1 cm, the initial amplitude of the radial motion (that is, the ω_1 motion) is 1 cm while the initial amplitude of the phase oscillation is 20 degrees. These conditions mean that

$$A_1=A_3=3^{\frac{1}{4}} \quad \text{and} \quad A_2=\frac{\pi}{9}\frac{a^2\alpha^{\frac{1}{2}}\sqrt{2}}{c}\left[\frac{8}{3.5}\right]^{\frac{1}{4}}.$$

Let $B_1=B_2=B_3=0$.

The quantity, L, depends upon the flux of the magnetic field inside the equilibrium orbit, $r=a$, and will be a function of the magnet design. For this numerical example we will assume that the cross section of the iron has the form



where the return magnetic circuit is inside the electron orbits. One may estimate that the return flux in the iron has twice the value of the flux in the air gap at the pole faces and that, moreover, the equilibrium orbit lies midway in the air gap. If the radial extent of the pole pieces is 100 cm$\pm$7 cm the problem is determined and one finds that $L=-10^7$.

In Table I the first column gives the time, beginning with the injection time $t=1.57\times 10^{-5}$ second and extending to the end of a quarter-cycle of the magnetic field, that is, $t=1/240$. The second column lists the number of turns the electron has made since injection. The third column is the radiofrequency in megacycles, while the fourth gives the ratio of the average mass of the electron to its rest mass. The fifth column shows the phase of the equilibrium orbit, namely

$$\frac{2\pi\Omega}{V_0Nc}(a^2H_0-L)\cos\Omega t,$$

expressed in degrees.

The sixth column gives an amplitude in centimeters which is that, either of the z oscillation, or, of that r oscillation which is due to the radial motion (that is, to ω_1). The seventh, eighth, and ninth columns list the amplitudes of r, θ, and m which arise from the phase oscillation ω_2. The last column gives the ratio of the phase oscillation frequency to the orbital frequency ω_0. The ratio of the frequency of the z oscillation as well as that of the radial vibration to the orbital frequency is, of course, constant and equal to $n^{\frac{1}{2}}=(1-n)^{\frac{1}{2}}=1/\sqrt{2}$ for the case $n=\frac{1}{2}$.

The total displacement of the electron in the radial direction will be equal to the sum of the two radial amplitudes and for the numerical example just given has a maximum value of 2.32 cm at injection, which decreases to 0.11 cm at the conclusion of the magnetic quarter cycle. The position of the electron relative to the radiofrequency, that is, the phase, oscillates in the above example between 48.3° and 8.3° at the time of injection, decreasing to an oscillation between +5.9° and −5.9° at the conclusion. This extreme value of the phase angle, namely 48.3°, seems rather large since it has been assumed that the sine of the phase angle can be replaced by the angle itself in the equations of motion. It is not believed that this constitutes any essential limitation on the validity of the predicted motion however. The reasons for this belief will appear in the following general discussion of the accuracy of the solution which has been obtained.

1. The solution of the equations of motion which has been given is correct through zeroth and first order of approximation but neglects second and higher terms. (The fact that it becomes invalid for very small values of t will be discussed later.)

2. Throughout the discussion the decrease in the amplitudes of the r, θ, and z vibrations has been spoken of as a damping. This is not a proper description since the energy of these vibrations is not disappearing although the amplitudes do grow smaller. For example, if the motion is allowed to continue through the second quarter cycle of the magnetic field, the motion is strictly reversible and the amplitudes of the vibrations will increase as the energy of the electrons decreases. The situation may be visualized in a very direct fashion for the case of the z vibration. Here, the mass of the electron is growing with time but also the restoring force grows at almost precisely the same rate so that, for large values of f, the frequency ω_3 is independent of f. This motion is at right angles to the other (r and θ) motions and its energy is conserved. Thus at the points where $\dot{z}=0$, the potential energy $=\frac{1}{2}Kz^2=$ constant. Since the force constant K is proportional to f, the amplitude of the motion will be inversely proportional to $f^{\frac{1}{2}}$. This is precisely the result obtained earlier when f is large. It shows that the decrease in the amplitude cannot be attributed to a damping process but rather is merely a consequence of the simultaneous increase in the mass and in the restoring force constant. The situation for the other vibrations, ω_1 and ω_2, is more complicated but the mechanism by which the amplitudes are decreased appears to be of the same general nature.

3. The phase angle $(\theta-\int^t\omega_0 dt)$ has been assumed to be sufficiently small so that the sine of the phase may be replaced by the phase itself. This may not always be the case as has been illustrated in the numerical example. A study of the motion for the frozen magnetic field indicates that the phase oscillation (that is, the ω_2 oscillation) for large amplitudes bears the same relationship to the motion for small amplitudes that ordinary pendulum motion for large amplitudes bears to simple harmonic motion. Thus it is expected that, in the case of the slowly growing magnetic field, the period of the oscillation will be larger for those electrons with large phase amplitudes but that this circumstance should have no essential influence upon the stability of the orbits. The stability depends upon the growth of the electron mass as has been explained above.

4. There do exist some mechanisms affecting the orbits which have not been included in the analysis. Among these are, (a) the scattering of electrons by gas molecules, (b) space charge, and (c) radiation damping. The first two of these effects will reduce the final yield of high energy electrons. The third, radiation damping, will tend to increase the equilibrium phase angle during the latter portion of the magnetic quarter cycle. This would only become important if it became so large that the phase angle approached 90 degrees. The magnitude of the equilibrium phase angle may of course be reduced by using a higher value of the radiofrequency potential V_0.

5. The zeroth order, or equilibrium orbit, solution is valid for all values of t but the first-order approximation breaks down when t is sufficiently small; that is, for low values of the injection energy. This comes about in two ways: (a) the quadratic equation (22) which determines ω_1 and ω_2 can no longer be well approximated by the simpler relations (23) and (24); (b) The so-called extra terms in the equations of motion no longer have a small influence upon the motion as compared with the principal terms. One manifestation of this latter effect will be that the amplitudes E_3 to E_6 will not be small compared with the amplitudes D_3 to D_6, when taken in respective pairs. From both of these tests one

may estimate that for the numerical example the solution which has been given will be good to one part in a thousand for an injection energy of $\frac{1}{2}$ Mev or $f=2$. It will be good to around 1 percent for an injection energy of 25 kev or $f=1.05$.

The first-order equations of motion resemble in many respects the differential equations defining Bessel functions although they are more complex. When the Bessel equation is treated by the methods used here, one obtains the standard asymptotic form for the function. The argument of the function is the angle θ counting from an initial time when the electron was at rest. At $f=1.05$, $\theta=136$ (using the constants of the numerical example), and for such a large value of the argument the asymptotic expressions for the Bessel functions are not in error by more than 1 percent.

6. The estimates which have just been made apply only to the zeroth- and first-order approximations and are not an indication of the magnitude of the second- and higher order approximations. In general, the ratio of second- to first-order terms is roughly equal to the ratio of first-to zeroth-order terms. In the numerical example the first-order terms were of the order of one or two percent of the zeroth-order quantities so it would seem that the first-order approximation is adequate. There is an exception to the general rule, however, in the case where there exist commensurable relationships between the various fundamental frequencies of the system. Secular changes may then set in, and the amplitudes may slowly build up to large values. For example, the second-order terms have been examined, and it is found that one, and only one, divergent situation may occur. The condition for this is that $2\omega_3-\omega_1=\omega_0[2(n)^{\frac{1}{2}}-(1-n)^{\frac{1}{2}}]=0$. Thus a magnetic fall off with $n=\frac{1}{5}$ must be avoided. It is expected that still higher order approximations will reveal other combinations of the frequencies which would lead to unstable orbits. However, the rate at which the amplitude of the oscillation builds up becomes progressively slower for the higher orders (if the initial amplitudes of oscillation are about 1 cm, the number of turns required for the catastrophe to develop is of the order of a^{p-1}, where a is the radius of the orbit in centimeters and p is the order of approximation for which the frequencies interact) so that eventually a commensurability between the frequencies becomes harmless. The phenomenon of instability due to commensurable relationships between frequencies is not confined to the synchrotron but exists as well in the betatron, although there the situation is somewhat simpler since the phase oscillation ω_2 is absent.

7. The synchrotron which has been considered in this paper is one in which the radiofrequency field is frequency modulated. It is, however, obvious that by the time the electrons have obtained 2 Mev or more of kinetic energy the radiofrequency has become nearly constant. From this point onward the equations and their solution would apply equally well to a synchrotron without frequency modulation or to the combination betatron-synchrotron type of accelerator.