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A Five-Dimensional Field Theory

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With the assumption of a five-dimensional space-time the field equations and the interactions of the field and particles are obtained. In this theory, the electrodynamics is in agreement with the classical theory. In the case of the meson theory, the vector, pseudoscalar, and pseudovector theories are obtained.

I. INTRODUCTION

CURRENT field theories are of Maxwellian and scalar types. These theories are genetically unrelated. Here we propose a unified treatment under the assumption of a five-dimensional space-time continuum. Instead of the ordinary four-dimensional space-time we have five, including four space-like (x, y, z, x_0) and one time-like (t) dimensions. If a proper assumption is made on the nature of the charge distributions in the fifth dimension (x_0), the field equations and the interactions with particles can be obtained.

As the momentum and velocity of a particle in the fifth dimension have never been observed, they are assumed to be zero. We also assume the density distribution along the fifth dimension to vary as $\cos(kx_0+\epsilon)$ or $\sin(kx_0+\epsilon)$, equivalent to the saying that the particle in the five-dimensional space is a long line extending in the fifth dimension, and the density varying as $\cos(kx_0+\epsilon)$ along that dimension.

II. THE FIELD EQUATIONS AND THE LAGRANGIAN

We define the five field vector

$$A_\mu=(A_x, A_y, A_z, i\phi, A_0), \qquad (1)$$

the field tensor

$$F_{\mu\nu}=\partial A_\mu/\partial x_\nu-\partial A_\nu/\partial x_\mu, \quad \mu, \nu=0, 1, 2, 3, 4 \qquad (2)$$

the charge density

$$j_\mu=(\rho\dot{x}, \rho\dot{y}, \rho\dot{z}, i\rho, \rho\dot{x}_0), \qquad (3)$$

the magnetic moment tensor

$$M_{\mu\nu}=\begin{pmatrix} 0 & \pi_x & \pi_y & \pi_z & ix \\ -\pi_x & 0 & M_z & -M_y & -iN_x \\ -\pi_y & -M_z & 0 & M_x & -iN_y \\ -\pi_z & M_y & -M_x & 0 & -iN_z \\ -ix & iN_x & iN_y & iN_z & 0 \end{pmatrix}, \qquad (4)$$

and the field equations

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu}=4\pi j_\mu+\frac{\partial M_{\mu\nu}}{\partial x_\nu},$$

or writing explicitly

$$\Delta\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial i^2}-k^2\mathbf{A}=-\frac{4\pi}{c}\mathbf{j}-\mathrm{rot}\,\mathbf{M}+\frac{1}{c}\frac{\partial\mathbf{N}}{\partial t}+\frac{\partial\boldsymbol{\pi}}{\partial x_0},$$
$$\Delta\phi-\frac{1}{c^2}\ddot{\phi}-k^2\phi=-4\pi\rho-\mathrm{div}\,\mathbf{N}-\frac{\partial x}{\partial x_0}, \qquad (5)$$
$$\Delta A_0-\frac{1}{c^2}\ddot{A}_0-k^2A_0=-4\pi\rho\dot{x}_0-\mathrm{div}\,\boldsymbol{\pi}-\frac{1}{c}\frac{\partial x}{\partial t},$$

and the interaction energy between field and charge density

$$\mathcal{E}=\int(M_{\mu\nu}+j_\mu A_\mu)dxdydzdx_0. \qquad (6)$$

Since all the charge quantities vary as $\cos(kx_0+\epsilon)$, each of the variables $(\mathbf{A}, \phi, A_0)$ can be separated into two parts: $(\mathbf{A}_1+\mathbf{A}_2, \phi_1+\phi_2, \mathbf{A}_0+0)$ with

$$\Delta\mathbf{A}_1-\frac{1}{c^2}\frac{\partial^2\mathbf{A}_1}{\partial t^2}-k^2\mathbf{A}_1=-\frac{4\pi}{c}\mathbf{j}-\mathrm{rot}\,\mathbf{M}+\frac{1}{c}\frac{\partial\mathbf{N}}{\partial t},$$
$$\Delta\mathbf{A}_2-\frac{1}{c^2}\frac{\partial^2\mathbf{A}_2}{\partial t^2}-k^2\mathbf{A}_2=-k\boldsymbol{\pi},$$
$$\Delta\phi_1-\frac{1}{c^2}\ddot{\phi}_1-k^2\phi_1=-4\pi\rho-\mathrm{div}\,\mathbf{N}, \qquad (7)$$
$$\Delta\phi_2-\frac{1}{c^2}\ddot{\phi}_2-k^2\phi_2=kx,$$
$$\Delta A_0-\frac{1}{c^2}\ddot{A}_0-k^2A_0=-4\pi\rho\dot{x}_0-\mathrm{div}\,\boldsymbol{\pi}-\frac{1}{c}\frac{\partial x}{\partial t}.$$

Thus all quantities with subscript 1 vary as $\cos(kx_0+\epsilon)$, while the quantities with subscript 2 vary as $\sin(kx_0+\epsilon)$. Since

$$\langle\cos^2(kx_0+\epsilon)\rangle_{\text{Av}}=\langle\sin^2(kx_0+\epsilon)\rangle_{\text{Av}}=\tfrac{1}{2};\quad \langle\cos(kx_0+\epsilon)\sin(kx_0+\epsilon)\rangle_{\text{Av}}=0,$$

Eq. (6) can be written in the form

$$\mathcal{E}=\int\left\{(\mathbf{M}\cdot\mathbf{H}_1)-(\mathbf{N}\cdot\mathbf{E}_1)-\boldsymbol{\pi}\,\text{grad}\,\mathbf{A}_0-\frac{1}{c}xA_0+\left(x\frac{\partial\phi_2}{\partial x_0}-\boldsymbol{\pi}\frac{\partial\mathbf{A}_2}{\partial x_0}\right)-\frac{1}{c}\mathbf{j}\cdot\mathbf{A}_1+\rho\phi_1\right\}dxdydz, \quad (8)$$

except with different choice of the charge units.

On the other hand the field equations in space free of charges are derivable from the following Lagrangian:

$$\mathcal{L}=-\frac{1}{8\pi}\int\left(\frac{\partial A_\mu}{\partial x_\nu}-\frac{\partial A_\nu}{\partial x_\mu}\right)dxdydzdx_0. \quad (9)$$

If we put $Q_\mu=A_\mu$, $P_\mu=\partial\mathcal{L}/\partial Q_\mu$, we get the Hamiltonian function,

$$\mathcal{H}=\sum P_\mu\dot{Q}_\mu-\mathcal{L}=\frac{1}{8\pi}\int\left\{\left[(\text{grad}\,\mathbf{A})^2+\frac{1}{c^2}\frac{\partial\mathbf{A}^2}{\partial t}^2+k^2A^2\right]+\left[(\text{grad}\,A_0)^2+\frac{1}{c^2}A_0{}^2+k^2A_0{}^2\right]-\left[(\text{grad}\,\phi)^2+\frac{1}{c^2}\phi^2+k^2\phi^2\right]\right\}dxdydzdx_0. \quad (10)$$

Here it is to be reminded that $(\phi_1+\phi_2,\ \mathbf{A}_1+\mathbf{A}_2)$ can be separated into independent terms in the Hamiltonian function, since $\langle\cos^2(kx_0+\epsilon)\rangle_{\text{Av}}=\frac{1}{2}$, etc.

III. THE HAMILTONIAN EQUATION CONTAINING FIELD AND PARTICLE

The five-dimensional Hamiltonian function for a particle in the Dirac's form of a particle with spin $\frac{1}{2}$ is

$$c\beta_\mu\left(p_\mu+\frac{e}{c}A_\mu\right)+m_0c^2=0, \quad (11)$$

where β_μ satisfies the relations $\beta_\mu\beta_\nu+\beta_\nu\beta_\mu=2\delta_{\mu\nu}$ and $\beta_0=\beta_1\beta_2\beta_3\beta_4$. Now by assumption that $v_0=0$, $p_0=0$, we see that $\psi^+\beta_4\beta_0\psi=0$ at least for a free particle. Thus to the first approximation

$$j_\mu=\psi^+\beta_4\beta_\mu\psi=\psi^+\beta_4(\beta_1,\beta_2,\beta_3,\beta_4,0)\psi.$$

(*i*) In the case of electrodynamics, we have $k=0$, $j_0=0$, $M_{\mu\nu}=0$, then all the equations of the field and particles in our theory will conform with the ordinary equations.

(*ii*) In the case of the meson theory, we adopt the "single force hypothesis," i.e., $j_\mu\equiv0$. Put

$$M_{\mu\nu}=f\psi^+\beta_4\cdot\tfrac{1}{2}(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu)\psi=f\psi^+\beta_4\begin{pmatrix}0 & \beta_4\beta_3\beta_2 & \beta_4\beta_1\beta_3 & \beta_4\beta_2\beta_1 & -i\beta_1\beta_2\beta_3\\ -\beta_4\beta_3\beta_2 & 0 & \beta_1\beta_2 & \beta_1\beta_3 & i\beta_1\beta_4\\ -\beta_4\beta_1\beta_3 & -\beta_1\beta_2 & 0 & \beta_2\beta_3 & i\beta_2\beta_4\\ -\beta_4\beta_2\beta_1 & -\beta_1\beta_3 & -\beta_2\beta_3 & 0 & i\beta_3\beta_4\\ i\beta_1\beta_2\beta_3 & -i\beta_1\beta_4 & -i\beta_2\beta_4 & -i\beta_3\beta_4 & 0\end{pmatrix}\psi. \quad (12)$$

In accordance with Eqs. (8), (10), and (11), we obtain the complete Hamiltonian function of the particle and the field:

$$\mathcal{H}_c=c\beta_\mu p_\mu+m_0c^2+f\left[\boldsymbol{\sigma}\cdot\mathbf{H}_1+i\beta_4\boldsymbol{\alpha}\cdot\mathbf{E}_1+\beta_4\boldsymbol{\sigma}\cdot\text{grad}\,A_0-\frac{i}{c}\beta_1\beta_2\beta_3A_0+\beta_4k(\boldsymbol{\sigma}\cdot\mathbf{A}_2)-ik\beta_1\beta_2\beta_3\phi_2\right]$$
$$+\frac{1}{8\pi}\int\left\{\left[(\text{grad}\,\mathbf{A})^2+\frac{1}{c^2}\frac{\partial\mathbf{A}^2}{\partial t}+k^2\mathbf{A}^2\right]+\left[(\text{grad}\,A_0)^2+\frac{1}{c^2}A_0{}^2+k^2A_0{}^2\right]-\left[(\text{grad}\,\phi)^2+\frac{1}{c^2}\phi^2+k^2\phi^2\right]\right\}dxdydz.$$

Here we see that $\boldsymbol{\sigma}\cdot\mathbf{H}_1+i\beta_4\boldsymbol{\alpha}\cdot\mathbf{E}_1$ corresponds to the current vector meson coupling and the term $\beta_4\boldsymbol{\sigma}\cdot\text{grad}\,A_0$ is the pseudoscalar coupling, while the last two terms give the pseudovector coupling. Thus all types of coupling necessary come in our scheme with equal importance.

Finally it should be noted that in our theory all quantities must be expressible in the tensor form in consistence with the assumption of the five-dimensional space, or, in other words, all quantities must be invariant under the five-dimensional rotation. As a consequence the coupling constants must be alike for all kinds of coupling.

IV. CONCLUSION

We have obtained the field equations and the interactions of the field and the particles with the assumption of the five-dimensional space-time. The electrodynamics in our theory is in agreement with the classical theory. In the case of the meson theory we obtained the vector, pseudoscalar and pseudovector couplings. But an important and also rather stringent consequence of our theory is that all coupling must appear in equal importance, i.e., the coupling constants must be alike for all kinds of coupling. Moreover, our theory corresponds to the weak coupling of the current theories.*

* Cf. W. Pauli and S. Kusaka, Phys. Rev. **63**, 400 (1943). In this paper they have shown that weak coupling is in better agreement with the experiment.

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Accuracy of the Earth-Flattening Approximation in the Theory of Microwave Propagation*

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A study was made of the maximum ranges and elevations for which the earth-flattening approximation in the theory of microwave propagation is valid. It is found that at a range equal to half the radius of the earth the error introduced by the earth-flattening approximation is only 2 percent, and this independently of the wave-length. The fractional error Δ in the height-gain functions is found to be proportional to the 5/2th power of the elevation, and to the inverse power of the wave-length. Values of Δ for various wave-lengths and elevations are shown in Table I. For wave-lengths of the order of several centimeters the earth-flattening approximation breaks down at elevations greater than a few thousand feet.

1. INTRODUCTION

THE central problem in the theory of microwave propagation is the determination of the electromagnetic field produced by a dipole antenna situated at some elevation above the ground. The electromagnetic field is affected primarily by the polarization of the antenna (vertical or horizontal dipole), the properties of the ground, the variation with elevation of the refractive index of the air, and by the spherical shape of the earth's surface. The last mentioned factor is, of course, very serious for propagation into regions below the horizon. It also introduces great complexity into the mathematical solution of the problem,[1] especially in the presence of a variable refractive index in the atmosphere.

Considerable simplification of the analysis has been achieved in recent years through a device due originally to Schelleng, Burrows, and Ferrell,[2] and later developed by M. H. L. Pryce,[3] whereby

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[1] See Arnold Sommerfeld's article in Frank-Mises, *Differentialgleichungen der Physik* Vol. II, p. 918.

[2] Schelling, Burrows, and Ferrell, Proc. I. R. E. **21**, 427 (1933).

[3] M. H. L. Pryce, unpublished report. See also J. E. Freehafer, Radiation Laboratory Report 447, 1943.