PHYSICAL REVIEW VOLUME 70, NUMBERS 7 AND 8 OCTOBER 1 AND 15, 1946

# Propagation of Radiation in a Medium with Random Inhomogeneities*

PETER G. BERGMANN
*Sonar Analysis Group, 64th Floor, 350 Fifth Avenue, New York 1, New York*
(Received May 13, 1946)

By means of the methods of geometrical optics, approximate formulae are being derived which correlate the statistical properties of the inhomogeneities of the transmitting medium with the fluctuations to be expected in the signal level of radiative energy. Through a further simplification of the formulae obtained, it is possible to predict the dependence of signal fluctuation on range without detailed knowledge of the statistical parameters of the "microstructure" of the transmitting medium.

## INTRODUCTION

WHEN radiative energy is propagated over considerable distances, the transmitting medium is rarely the homogeneous expanse it is assumed to be in elementary theory. The medium is bounded, and it may have an internal structure, such as a density gradient. In addition, the most important non-solid media of propagation, the atmosphere and the ocean, are known to possess a rapidly changeable random structure of comparatively small dimensions, which is caused by local heating, convective currents, and similar factors. In this paper, the modification of the radiative field which is caused by random structure will be derived for the case of small changes and on the assumption of a wave-length so short that the formulae of ray optics are valid. The results obtained may find application in the propagation of either electromagnetic or sound waves of high frequencies in the atmosphere or in a similarly extended medium.

## THE PROBLEM

We shall restrict ourselves at once to the application of ray optical methods. Treatment of the same problem by means of wave optics is planned for the future. The basic equations of ray optics in the stationary case may be written in the form

$$\left.\begin{aligned}(\nabla S)^2&=n^2,\\ \nabla\cdot\left(\frac{I}{n}\nabla S\right)&=0.\end{aligned}\right\}\qquad(1)$$

In these equations, the function $S(\mathbf{r})$ is the optical path length counted from the radiative source to the point $\mathbf{r}$ along the connecting ray path. Each constant value of $S>0$ corresponds to a wave front in the ray optical approximation. $n(\mathbf{r})$ is the local index of refraction, and $I(\mathbf{r})$ is the intensity of the radiative field, measured in units of energy passing per unit time through a unit area cross section perpendicular to the rays. In electromagnetic theory, $I$ is the averaged magnitude of Poynting's vector, while in acoustics, $I$ is the mean square pressure divided by $\rho c$. The first of the Eqs. (1) expresses Huygens' principle that the distance between consecutive wave fronts is inversely proportional to the local index of refraction; while the second equation is the law of conservation of energy. Absorption and scattering are disregarded. For the following discussion, it is convenient to introduce the level $L$, defined as the logarithm (on some base) of $I/n$.[1] We shall measure the level in nepers, that is on the base $e$. In terms of level, Eqs. (1) may be rewritten in the form

$$\begin{aligned}(\nabla S)^2&=n^2,\\ \nabla S\cdot\nabla L+\nabla^2 S&=0.\end{aligned}\qquad(2)$$

It will now be assumed that $n$ is very nearly equal to unity. We shall set

$$n=1+\lambda n',\qquad(3)$$

with $\lambda$ a constant parameter,

$$\lambda\ll 1.\qquad(4)$$

* This work represents in part results of research carried out by the Sonar Analysis Group under contracts between Columbia University and the Office of Scientific Research and Development and between Woods Hole Oceanographic Institution and the Bureau of Ships, Navy Department.

[1] The quantity $I/n$ has no direct physical significance and is chosen merely for convenience. However, because $n$ very nearly equals unity, according to the assumptions made, $I/n$ does not differ significantly from $I$. In most practical cases, $n$ differs from unity by an amount less than $10^{-2}$.

An approximate solution of Eqs. (1) will be obtained in terms of definite integrals.

### APPROXIMATE SOLUTION OF THE BASIC EQUATIONS

If $n'$ vanishes, the solution of Eq. (1) for a point source is

$$\left.\begin{aligned} S_0&=r,\\ L_0&=b(\Omega)-2\ln r,\end{aligned}\right\} \tag{5}$$

where $r$ is the distance from the source and $b$ is a constant or, in the case of a directional source, a function of the angle $\Omega$. That the pair $S_0$, $L_0$, is a solution can be verified directly by substituting the expressions (5) into Eqs. (2) with $n$ equal to unity. We shall call $S_0$, $L_0$ the zero approximation.

The first approximation is obtained if Eq. (2) is expanded into a power series with respect to the parameter $\lambda$. Denoting all first-order variables by primes, we obtain the following conditions for the first approximation:

$$\left.\begin{aligned} \mathbf{r}_0\cdot\nabla S'&=n',\\ \mathbf{r}_0\cdot\nabla L'&=-\nabla^2 S'-\nabla L_0\cdot\nabla S',\end{aligned}\right\} \tag{6}$$

where $\mathbf{r}_0$ is the unit vector pointing directly away from the source.

Fortunately, the two Eqs. (6) need not be solved simultaneously. They can be solved with the further condition that the variables $S'$ and $L'$ are to vanish at the location of the source.

The solution of the first Eq. (6) is provided by the integral

$$S'(\mathbf{r})={}^*\!\!\int_{\rho=0}^{\mathbf{r}} n'(\rho)d\rho. \tag{7}$$

The symbol ${}^*\!\!\int_0^{\mathbf{r}}$ refers to integration along a straight line from the location of the source to the point characterized by the radius vector $\mathbf{r}$. $\rho$ is the variable of integration. For what follows, we shall require the gradient and the Laplacian of $S'$. Figure 1 shows how the path of integration must be transformed if a differentiation of an integral of the type (7) is to be carried out. Considering the unspecified integral

$$J(\mathbf{r})={}^*\!\!\int_0^{\mathbf{r}} A(\rho)d\rho, \tag{8}$$

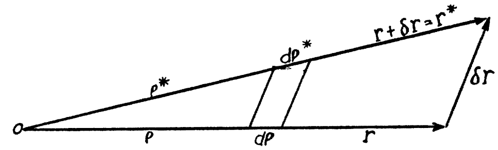


FIG. 1. Variation of the line integral.

we find that the variation of this integral may be written as

$$\begin{aligned}\delta J(\mathbf{r})&={}^*\!\!\int_0^{\mathbf{r}} \delta[A(\rho)d\rho]\\ &={}^*\!\!\int_0^{\mathbf{r}} \delta A(\rho)d\rho+{}^*\!\!\int_0^{\mathbf{r}} A(\rho)\delta d\rho.\end{aligned} \tag{9}$$

The local variation $\delta A(\rho)$ is, according to the figure,

$$\delta A(\rho)=\frac{\rho}{r}\nabla_\rho A(\rho)\cdot\delta\mathbf{r} \tag{10}$$

while the variation of $d\rho$ equals

$$\delta d\rho=\frac{\mathbf{r}_0\cdot\delta\mathbf{r}}{r}d\rho. \tag{11}$$

We find, therefore,

$$\begin{aligned}\delta J(\mathbf{r})&=\frac{1}{r}\delta\mathbf{r}\cdot{}^*\!\!\int_0^{\mathbf{r}}[\rho\nabla_\rho A(\rho)+\mathbf{r}_0 A(\rho)]d\rho\\ &=\frac{1}{r}\delta\mathbf{r}\cdot{}^*\!\!\int_0^{\mathbf{r}}\nabla_\rho[\rho A(\rho)]d\rho.\end{aligned} \tag{12}$$

The gradient of $S'$, Eq. (7), is thus

$$\nabla S'(\mathbf{r})=\frac{1}{r}{}^*\!\!\int_0^{\mathbf{r}}\nabla_\rho(\rho n')d\rho. \tag{13}$$

Iteration of this process of differentiation leads to the result

$$\begin{aligned}\nabla^2 S'(\mathbf{r})&=\nabla\left(\frac{1}{r}\right)\cdot{}^*\!\!\int_0^{\mathbf{r}}\nabla_\rho(\rho n')d\rho\\ &\quad+\frac{1}{r^2}{}^*\!\!\int_0^{\mathbf{r}}\nabla_\rho\cdot[\rho\nabla_\rho(\rho n')]d\rho.\end{aligned} \tag{14}$$

This expression can be further simplified with the help of the identity

$$\mathbf{r}_0 \cdot \overset{*}{\int_0^{\mathbf{r}}} \nabla_\rho Q(\rho) d\rho \equiv Q(\mathbf{r}) - Q(0). \qquad (15)$$

We obtain, after a straightforward computation,

$$\nabla^2 S'(\mathbf{r}) = \frac{2}{r} n'(\mathbf{r}) + \frac{1}{r^2} \overset{*}{\int_0^{\mathbf{r}}} \rho^2 \nabla_\rho{}^2 n'(\rho) d\rho. \qquad (16)$$

We shall now write out the explicit expressions occurring on the right-hand side of the second Eq. (6). We have for $\nabla L_0$ the expression

$$\nabla L_0 = \nabla b - \frac{2}{r} \mathbf{r}_0, \qquad (17)$$

with the further condition

$$\mathbf{r}_0 \cdot \nabla b = 0. \qquad (18)$$

The second Eq. (6) therefore assumes the form

$$\mathbf{r}_0 \cdot \nabla L' = -\frac{1}{r^2} \overset{*}{\int_0^{\mathbf{r}}} \rho^2 \nabla_\rho{}^2 n' d\rho - \frac{1}{r} \nabla b \cdot \overset{*}{\int_0^{\mathbf{r}}} \rho \nabla_\rho n' d\rho \equiv B(\mathbf{r}). \qquad (19)$$

The two terms on the right-hand side of Eq. (19) can be interpreted quite simply. The first term corresponds to the "lens action" of local inhomogeneities, while the second term, which depends on the directivity of the source, represents the change in local intensity caused by the lateral displacement of the "beam." In what follows we shall consider a non-directional source and drop the second term.

The solution of Eq. (19) is the integral

$$\left.\begin{aligned} L' &= \overset{*}{\int_0^{\mathbf{r}}} B(\rho) d\rho \\ &= -\overset{*}{\int_0^{\mathbf{r}}} \left(\frac{1}{\rho} - \frac{1}{r}\right) \rho^2 \nabla_\rho{}^2 n' d\rho, \end{aligned}\right\} \qquad (20)$$

by a simple transformation of the double integrals.

Equations (7) and (20) are solutions of the differential Eq. (6). If conditions at the source are disregarded, these solutions are not unique; it is possible to write down the general solution of Eq. (6). The difference between two solutions of the first Eq. (6) is a function of the angle only. The most general solution is, therefore,

$$\begin{aligned} S_g' &= S_0' + \phi(\Omega), \\ \mathbf{r}_0 \cdot \nabla \phi &= 0. \end{aligned} \qquad (21)$$

$S_0'$ is the particular solution (7), and $\phi(\Omega)$ is an arbitrary function of the solid angle $\Omega$. The addition $\phi(\Omega)$ produces a discontinuous addition to the variable $S$ at the source, and is, thus, inconsistent with the assumption that for vanishing $r$, $S'$ is to vanish. We find that the expressions (7) and (20) represent the only solution which satisfies the conditions at the source.

As for the higher approximations, the expansion described here leads, at each stage, to equations for $S^{(n)}$ and $L^{(n)}$, which have the same type left-hand sides as Eq. (6). The right-hand sides, however, get progressively more involved. At any rate, each successive approximation has a unique solution, which can be expressed in the form of explicit line integrals. These higher approximations will not be considered in the remaining sections of this paper.

## THE SELF-CORRELATION FUNCTION OF THE INDEX OF REFRACTION

In the following sections, we shall derive the mean and the standard deviation of $S$, the optical path length from the source, and of $L$, the level at a fixed point $\mathbf{r}$. First, however, the microstructure must be characterized by certain statistical properties. If we consider a configuration of microstructure patterns, or a time series of distributions of $n'$ in the same medium, then we shall assume that there is a possibility of averaging all quantities which depend on the spatial distribution of $n'$ and its derivatives. The averaged quantities shall be enclosed in angular brackets, $\langle\,\rangle$. Obviously, it is reasonable to define the standard wave velocity so that

$$\langle n'(x, y, z)\rangle = 0. \qquad (22)$$

The significance of this normalization is that the deviation of $n'$ from unity is equally likely to be positive or negative. The second assumption is that the spatial correlation function,

$$\langle n'(x_1, y_1, z_1)n'(x_2, y_2, z_2)\rangle,$$

exists and is a function of the coordinate differences only,

$$\left.\begin{aligned}\langle n'(x_1, y_1, z_1)n'(x_2, y_2, z_2)\rangle \\ \equiv N(x_1-x_2, y_1-y_2, z_1-z_2) \\ \equiv N(x_2-x_1, y_2-y_1, z_2-z_1).\end{aligned}\right\} \quad (23)$$

In other words, it is assumed that the statistical characteristics of the microstructure, while not necessarily isotropic, are homogeneous.

The function $N$ satisfies a number of inequalities. A few shall be listed here. From the inequality

$$[n'(0)\pm n'(\mathbf{r})]^2 \geqslant 0 \quad (24)$$

we have

$$0 \leqslant \langle [n'(0)\pm n'(\mathbf{r})]^2\rangle = 2[N(0)\pm N(\mathbf{r})] \quad (25)$$

or, for all coordinate differences $\mathbf{r}$,

$$\begin{aligned}N(0) \geqslant 0, \\ |N(\mathbf{r})| \leqslant N(0).\end{aligned} \quad (26)$$

An inequality between the $N$-values belonging to two arguments $\mathbf{r}_1$ and $\mathbf{r}_2$ is obtained by means of another positive definite expression. Introduce a parameter $\alpha$ (or $\beta$ or $\gamma$) which serves to number the cases which constitute the "population" for purposes of averaging and then consider the square of the determinant

$$D = \begin{vmatrix} n_\alpha'(0), & n_\alpha'(\mathbf{r}_1), & n_\alpha'(\mathbf{r}_2) \\ n_\beta'(0), & n_\beta'(\mathbf{r}_1), & n_\beta'(\mathbf{r}_2) \\ n_\gamma'(0), & n_\gamma'(\mathbf{r}_1), & n_\gamma'(\mathbf{r}_2) \end{vmatrix}, \quad (27)$$

averaged three times, over $\alpha$, $\beta$, and $\gamma$. This triple mean square is

$$\left.\begin{aligned}\langle D^2\rangle = 6\{N^3(0)+2N(\mathbf{r}_1)N(\mathbf{r}_2)N(\mathbf{r}_1\pm\mathbf{r}_2) \\ -N(0)[N^2(\mathbf{r}_1)+N^2(\mathbf{r}_2)+N^2(\mathbf{r}_1\pm\mathbf{r}_2)]\}.\end{aligned}\right\} \quad (28)$$

The resulting inequality takes the form

$$\left.\begin{aligned}N^3(0)+2N(\mathbf{r}_1)N(\mathbf{r}_2)N(\mathbf{r}_1\pm\mathbf{r}_2) \\ -N(0)[N^2(\mathbf{r}_1)+N^2(\mathbf{r}_2) \\ +N^2(\mathbf{r}_1\pm\mathbf{r}_2)]\geqslant 0.\end{aligned}\right\} \quad (29)$$

This inequality can be transformed if we solve it with respect to $N(\mathbf{r}_1\pm\mathbf{r}_2)$. We find the inequality

$$\begin{aligned}\frac{1}{N(0)}[N(\mathbf{r}_1)N(\mathbf{r}_2)-R] \leqslant N(\mathbf{r}_1\pm\mathbf{r}_2) \\ \leqslant \frac{1}{N(0)}[N(\mathbf{r}_1)N(\mathbf{r}_2)+R],\end{aligned} \quad (30)$$

$$R \equiv \{[N^2(0)-N^2(\mathbf{r}_1)][N^2(0)-N^2(\mathbf{r}_2)]\}^{\frac{1}{2}}.$$

It follows that $N$ will be continuous everywhere if it is continuous at the point 0. For if $\mathbf{r}_2$ is a very small coordinate difference $\boldsymbol{\delta}$, and if

$$N(\boldsymbol{\delta}) = N(0)(1-\epsilon), \quad (31)$$

where $\epsilon$ is again a small quantity, then (30) goes over into

$$\left.\begin{aligned}-\epsilon N(\mathbf{r})-[\epsilon(2-\epsilon)]^{\frac{1}{2}}[N^2(0)-N^2(\mathbf{r})]^{\frac{1}{2}} \\ \leqslant N(\mathbf{r}\pm\boldsymbol{\delta})-N(\mathbf{r}) \leqslant -\epsilon N(\mathbf{r}) \\ +[\epsilon(2+\epsilon)]^{\frac{1}{2}}[N^2(0)-N^2(\mathbf{r})]^{\frac{1}{2}}.\end{aligned}\right\} \quad (32)$$

Both the lower and the upper bound converge toward zero with $\epsilon$. It follows further from the inequality (32) that the rate of change of $N$, defined as

$$\lim_{|\delta|=0}\left\{\frac{1}{|\boldsymbol{\delta}|}[N(\mathbf{r}+\boldsymbol{\delta})-N(\mathbf{r})]\right\}, \quad (33)$$

is everywhere bounded if the following limit exists

$$\lim_{|\delta|=0}\left\{\frac{1}{|\boldsymbol{\delta}|}\left[1-\frac{N(\boldsymbol{\delta})}{N(0)}\right]^{\frac{1}{2}}\right\}, \quad (34)$$

for fixed direction of $\boldsymbol{\delta}$. If that limit vanishes, $N(\mathbf{r})$ is constant everywhere and equal to $N(0)$.

Another set of inequalities can be obtained by considering positive definite expressions of the type

$$Q = \overset{*}{\int_0^{\mathbf{r}}}\phi(\rho)n'(\rho)d\rho\cdot\overset{*}{\int_0^{\mathbf{r}}}\phi^*(\sigma)n'(\sigma)d\sigma \geqslant 0, \quad (35)$$

where the asterisk indicates transition to the conjugate complex. If we choose in particular for the arbitrary function $\phi(\rho)$ the function $e^{ik\rho}$, then we obtain the following

$$\begin{aligned}Q &= \overset{*}{\int_0^{\mathbf{r}}}\int e^{ik(\rho-\sigma)}N(\rho-\sigma)d\rho d\sigma \\ &= 2\overset{*}{\int_0^{\mathbf{r}}}(r-\rho)\cos k\rho\cdot N(\rho)d\rho \geqslant 0\end{aligned} \quad (36)$$

or

$$\int_0^r \left(1-\frac{\rho}{r}\right) \cos k\rho \cdot N(\rho) d\rho \geqslant 0. \tag{37}$$

#### THE MEAN SQUARE DEVIATION OF $S'$

From Eq. (7), we shall now compute an expression for the mean square deviation of the optical path length from the geometrical path length. We have

$$\left.\begin{aligned}\sigma_S{}^2 &= \langle (S_0+\lambda S'+\lambda^2 S'')^2\rangle - \langle S_0+\lambda S'+\lambda^2 S''\rangle^2 \\ &= \lambda^2[\langle S'^2\rangle - \langle S'\rangle^2] + \text{higher terms.}\end{aligned}\right\} \tag{38}$$

In this expression, $\langle S'\rangle$, the mean deviation of the optical from the geometrical path length, vanishes,

$$\langle S'\rangle = \overset{*}{\int_0^r} \langle n'(\rho)\rangle d\rho = 0. \tag{39}$$

For $\langle S'^2\rangle$ we find

$$\langle S'^2\rangle = \overset{*}{\iint_{\rho,\sigma=0}^{r}} N(\rho-\sigma) d\sigma d\rho. \tag{40}$$

This double integral can be transformed as follows:

$$\left.\begin{aligned}\langle S'^2\rangle &= \int_{\sigma=0}^{r} \int_{\tau=-\sigma}^{r-\sigma} N(\tau) d\tau d\sigma \\ &= \int_{\tau=0}^{r} \int_{\sigma=0}^{r-\tau} N(\tau) d\sigma d\tau + \int_{\tau=-r}^{0} \int_{\sigma=-\tau}^{r} N(\tau) d\sigma d\tau,\end{aligned}\right\} \tag{41}$$

and finally

$$\langle S'^2\rangle = 2\overset{*}{\int_0^r} (r-\rho) N(\rho) d\rho. \tag{42}$$

We find, thus, that the mean optical path length equals the actual distance $r$, while the r.m.s. deviation is given in this approximation by the expression

$$\sigma_S = \left[2\overset{*}{\int_0^r} (r-\rho) N(\rho) d\rho\right]^{\frac{1}{2}}. \tag{43}$$

The integral is to be extended over the straight line connecting the source with the point at which the observer is located.

It may be assumed that in most practical cases the function $N(\mathbf{r})$ decreases rapidly for large values of the argument. As a result, expression (42) may be replaced, in fair approximation, by

$$\sigma_S{}^2 \sim 2r \overset{*}{\int_0^\infty} N(\rho) d\rho. \tag{44}$$

Without any knowledge of the details of the function $N$, it can be predicted that, for sufficiently large distances, the r.m.s. fluctuation of the optical path length will increase with the square root of the distance.

#### THE SIGNAL LEVEL FLUCTUATION

From Eqs. (19) and (20), it is possible to obtain an expression for signal level fluctuation. By an argument analogous to that leading to the mean square deviation of the optical path length, we find first that

$$\sigma_L{}^2 = \langle L'^2\rangle. \tag{45}$$

This expression is, in turn, equal to

$$\langle L'^2\rangle = \overset{*}{\iint_{\rho,\sigma=0}^{r}} \left(\frac{1}{\rho}-\frac{1}{r}\right)\left(\frac{1}{\sigma}-\frac{1}{r}\right) \times \rho^2\sigma^2 \langle \nabla_\rho{}^2 n'(\rho) \nabla_\sigma{}^2 n'(\sigma)\rangle d\sigma d\rho. \tag{46}$$

The integrand can be simplified, first of all, by the introduction of the iterated Laplacean of $N(\mathbf{r})$, in accordance with the identity

$$\langle \nabla^2 n'(\rho) \nabla^2 n'(\sigma)\rangle \equiv \nabla^2\nabla^2 N(\rho-\sigma). \tag{47}$$

Equation (46) then assumes the form

$$\langle L'^2\rangle = \overset{*}{\iint_{\rho,\sigma=0}^{r}} \left(\frac{1}{\rho}-\frac{1}{r}\right)\left(\frac{1}{\sigma}-\frac{1}{r}\right) \times \rho^2\sigma^2 \nabla^2\nabla^2 N(\rho-\sigma) d\sigma d\rho. \tag{48}$$

It is now possible to convert the double integral into a single integral by means of transformations similar to those leading to Eq. (42) above. We obtain the expression

$$\langle L'^2\rangle = \frac{1}{15}\frac{1}{r^2} \overset{*}{\int_{\rho=0}^{r}} (r^2+3\rho r+\rho^2) \times (r-\rho)^3 \nabla^2\nabla^2 N(\rho) d\rho. \tag{49}$$

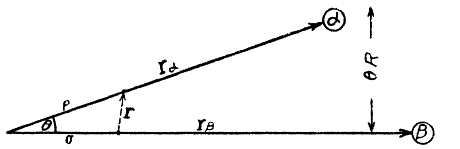

FIG. 2. Two receiving stations.

If $N(\rho)$ is small for large values of the argument, then the expression (49) can be approximated by the much cruder approximation

$$\langle L'^2\rangle \sim \frac{1}{15} r^3 \overset{*}{\int_{\rho=0}^{\infty}} \nabla^2\nabla^2 N(\rho)d\rho. \tag{50}$$

New observations, made recently at the Laboratory of the University of California, Division of War Research, at San Diego, appear to show that the fluctuation of supersonic sound signals over paths in the deep ocean, well removed from both surface and bottom, shows an increase with increasing distance, but the rate of increase did not agree quantitatively with Eq. (50).

### CORRELATION AT TWO RECEIVING STATIONS

In the case of two stations receiving the same signals (Fig. 2), the levels received at either station are given by the expression (20), so that we have

$$\left.\begin{aligned} L_\alpha' &= -\overset{*}{\int_0^{r_\alpha}} \left(\frac{1}{\rho}-\frac{1}{r_\alpha}\right)\rho^2\nabla_\rho^2 n'(\rho)d\rho, \\ L_\beta' &= -\overset{*}{\int_0^{r_\beta}} \left(\frac{1}{\sigma}-\frac{1}{r_\beta}\right)\sigma^2\nabla_\sigma^2 n'(\sigma)d\sigma. \end{aligned}\right\} \tag{51}$$

The average product of $L_\alpha'$ and $L_\beta'$ is given by the expression

$$\langle L_\alpha' L_\beta'\rangle = \overset{*}{\int_{\rho=0}^{r_\alpha}} \overset{*}{\int_{\sigma=0}^{r_\beta}} \left(\frac{1}{\rho}-\frac{1}{r_\alpha}\right)\left(\frac{1}{\sigma}-\frac{1}{r_\beta}\right) \times \rho^2\sigma^2\nabla^2\nabla^2 N(\mathbf{r})d\sigma d\rho, \tag{52}$$

where the two integrals are to be extended over two different paths. $\mathbf{r}$ is a vector which connects the two points $(\sigma)$ and $(\rho)$, as indicated in Fig. 2. This double integral cannot be converted into a single integral, because the vector $\mathbf{r}$ assumes values depending on two parameters. However, it is possible to replace the expression (52) by an approximate expression which, like (50), gives information on the dependence on distance for large distances. First of all, we shall assume that the argument $N(\mathbf{r})$ is significantly different from zero only for small values of the argument $\mathbf{r}$, let us say for

$$|\mathbf{r}| < l, \tag{53}$$

and further, that $l$ itself is small compared with the lateral separation of the stations $\alpha$ and $\beta$, that is,

$$l \ll \tfrac{1}{2}\theta(r_\alpha + r_\beta). \tag{54}$$

It will further be assumed that $\theta$ is a small angle and that

$$\frac{|r_\alpha - r_\beta|}{r_\alpha + r_\beta} \ll 1. \tag{55}$$

The average vector from source to receivers will be called $\mathbf{R}$,

$$\mathbf{R} \equiv \tfrac{1}{2}(\mathbf{r}_\alpha + \mathbf{r}_\beta). \tag{56}$$

In that case, the only portion of the integral which contributes significantly is the obtuse circular section in the $\mathbf{r}$-plane indicated in Fig. 3. In this region, the expression (52) may be replaced, in fair approximation, by

$$\langle L_\alpha' L_\beta'\rangle \sim \int\int \rho\sigma\nabla^2\nabla^2 N(\mathbf{r})d\rho d\sigma. \tag{57}$$

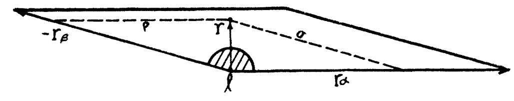

FIG. 3. Area of integration in the $\mathbf{r}$-plane.

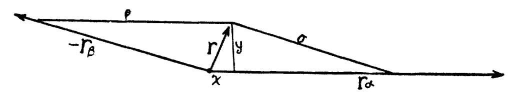

FIG. 4. Coordinate transformation in the $\mathbf{r}$-plane.

This integral, in turn, can be further transformed. If we describe the area of integration by rectangular coordinates, $x$ and $y$, Fig. 4,

$$\left.\begin{aligned} x &= \rho - \sigma\cos\theta, \\ y &= \sigma\sin\theta, \end{aligned}\right\} \tag{58}$$

or

$$\left.\begin{aligned} \rho &= x + y\cot\theta, \\ \sigma &= y\operatorname{cosec}\theta, \end{aligned}\right\} \tag{59}$$

with the Jacobian

$$\frac{\partial\rho}{\partial x}\frac{\partial\sigma}{\partial y} - \frac{\partial\rho}{\partial y}\frac{\partial\sigma}{\partial x} = \operatorname{cosec}\theta, \tag{60}$$

Eq. (53) assumes the form

$$\langle L_\alpha' L_\beta' \rangle \sim \operatorname{cosec}^2 \theta \int\int_{x,y} y(x+y\cot\theta) \times \nabla^2\nabla^2 N(x, y, z) dxdy. \quad (61)$$

It can be shown that the expression which remains under the integral sign is changed but little if the circular sector is expanded into a semicircle. Thus, the expression (57) can be further simplified into

$$\langle L_\alpha' L_\beta' \rangle \sim \frac{1}{2\theta^2} \int\int_{x,y=-\infty}^{\infty} xy\nabla^2\nabla^2 N dxdy + \frac{1}{2\theta^3} \int\int_{x,y=-\infty}^{\infty} y^2\nabla^2\nabla^2 N dxdy. \quad (62)$$

Of these two terms, the second one is large compared with the first, so that the final expression is

$$\langle L_\alpha' L_\beta' \rangle \sim \frac{1}{2\theta^3} \int\int_{x,y=-\infty}^{\infty} y^2\nabla^2\nabla^2 N dxdy, \quad (63)$$

where $y$ is the direction perpendicular to **R** in the plane of source and receivers.

To obtain the usual correlation coefficient, the expression (63) has to be divided through by (50). It is found that the correlation coefficient is independent of the distance between source and receivers and inversely proportional to the cube of the distance between the two receiving stations perpendicular to the line connecting the source with the location of the receivers, $\delta$, as indicated in Fig. 2.

PHYSICAL REVIEW VOLUME 70, NUMBERS 7 AND 8 OCTOBER 1 AND 15, 1946

# A Calculation of the Binding Energies of $H^3$ and $He^4$ with a New Potential

MU-HSIEN WANG
*Department of Physics, National University of Chekiang, Meitan, Kweichow, China*
(Received May 15, 1946)

The binding energies of the nuclei $H^3$ and $He^4$ are calculated by the method of equivalent two-body, using the potential suggested by Wang. The range at which the potential between two nucleons is cut off is the same as that for the case of the deuteron, and the same range for the equivalent two-body is deduced accordingly. The binding energies thus calculated are 7.3 Mev and 15.1 Mev, respectively.

## I. METHOD OF CALCULATION

THE nuclear potential proposed by Wang[1] was previously applied by the author to the calculation of the binding energy of the deuteron and of the neutron-proton scattering.[2] In the present work the binding energies of the nuclei $H^3$ and $He^4$ are computed by using the same potential in the method of the equivalent two-body.

Let $C\phi(r)$ be the potential between any two nucleons. If the Gaussian wave function $N\exp[-\frac{1}{2}\nu(r_{12}^2+r_{13}^2+r_{23}^2)]$ is used, the variation energy for $H^3$ is[3]

$$E(H^3) = \frac{9\nu\hbar^2}{2M} - 3C\frac{4}{\pi^{\frac{1}{2}}}\int_0^\infty \exp(-x^2) \times \phi[x/(3\nu/2)^{\frac{1}{2}}]x^2 dx, \quad (1)$$

where

$$x = (3\nu/2)^{\frac{1}{2}} r_{12} \quad (\text{or } r_{13},\ r_{23}),$$

and $r_{12}$ is the distance between particles 1 and 2. For the nucleus $He^4$, taking $N\exp[-\frac{1}{2}\nu(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)]$ as the wave function, we have

$$E(He^4) = \frac{9\nu\hbar^2}{M} - 6C\frac{4}{\pi^{\frac{1}{2}}}\int_0^\infty \exp(-x^2) \times \phi[x/(2\nu)^{\frac{1}{2}}]x^2 dx, \quad (2)$$

[1] K. C. Wang and H. L. Tsao, Phys. Rev. **66**, 155 (1944); Nature **155**, April 28 (1945).
[2] Mu-Hsien Wang, Phys. Rev. **66**, 103 (1944).
[3] William Rarita and R. D. Present, Phys. Rev. **51**, 788 (1937).