THE PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 70, NOS. 7 AND 8 — OCTOBER 1 AND 15, 1946

Reaction of Radiation on Electron Scattering and Heitler's Theory of Radiation Damping

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(Received June 26, 1946)

The reaction of radiation on the scattering of electrons is treated on the basis of Heitler's theory of radiation damping. Because in this theory no account is taken of virtual processes, the infra-red catastrophe reappears, and the theory gives a total cross section for scattering, which depends critically on the longest wave-length radiation that can be emitted, and which does not agree with the correspondence principle. It seems probable that only by a modification of present theories specific to the domain of high energies and small distances will a satisfactory solution of this simple problem be found.

1. THE INFRA-RED CATASTROPHE IN THE ORDINARY THEORY

AMONG the many problems involving radiation damping there is one which has been the subject of much study and to which no satisfactory answer has yet been obtained. This is the reaction of radiation on the scattering of an electron.

If we consider the scattering of an electron in a fixed potential field, which we may for convenience assume to be weak, then the scattering cross section of the electron is given by

$$d\sigma=\left(\frac{m}{2\pi\hbar^2}V_{kp}\right)^2 d\omega, \tag{1}$$

where **k** and **p** are initial and final momenta,

$$V_{kp}=\int d\tau\, Ve^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{r}\hbar}, \tag{1a}$$

$d\mathbf{p}=p^2dpd\omega$, m is the mass, V the scattering potential. One would expect that the reaction of radiation emitted in the collision might not affect the probability of scattering for low velocity of the electron because the recoil of the emitted radiation should be negligible. But as the velocity of the electron is increased, corrections of order $(v/c)^4$ should appear in the formula for scattering, which are proportional to the square of the electron's charge.

It is clear that these corrections, while presumably small, are in principle subject to direct experimental study. The theory of the interaction of radiation with matter should make it possible to compute them. As is well known, if one applies the accepted principles of quantum electrodynamics to this problem, one does not obtain a reasonable or a finite answer. We should like to discuss the status of this problem from a somewhat new point of view in order that we can see how to apply to it the proposals which have been put forward by Heitler[1] for eliminating the divergence difficulties of the present theory. In fact, it seems not unreasonable to apply all proposed modifications of electrodynamics to this simple problem as a sort of criterion of their adequacy. Judged by this criterion, the proposals put forth by Heitler must be regarded as unsatisfactory.

The difficulties of this problem first appeared in the form of the so-called infra-red catastrophe. In fact, if one computes by perturbation methods the probability that an electron will be scattered with the emission of a quantum of frequency q one obtains the cross section

$$\delta_r d\sigma = \frac{2e^2}{3\pi\hbar c}\frac{(\mathbf{p}-\mathbf{k})^2}{m^2c^2}\frac{dq}{q}\cdot d\sigma, \qquad (2)$$

whose integral will diverge logarithmically for small q. This problem has been analyzed in detail by Bloch and Nordsieck.[2] These authors showed that it is in fact unlikely that a scattering process will take place with the emission of 0, 1, or, in fact, any finite number of quanta. Specifically by a rigorous but non-relativistic solution of the problem for low frequencies, they showed that the increase in probability of scattering with the emission of a large number of quanta is quantitatively compensated by the decrease in the probability of elastic scattering or scattering with the emission of a smaller number of quanta. Thus, they obtained for the total probability of scattering the result (1), as one would indeed expect.

The situation is, however, not as satisfactory as this, because Bloch and Nordsieck neglected the fact that only quanta which are energetically capable of emission, i.e., below the high frequency limit given by the electron's kinetic energy, can in fact be emitted.

Taking this circumstance into account but still calculating non-relativistically, Pauli and Fierz[3] showed that the total cross section for scattering is not unchanged but would in fact vanish because the reaction on the probability of elastic scattering is not adequately compensated by an increased probability for scattering with the emission of high frequency quanta. The divergences at high frequencies which appear in this treatment are again logarithmic.

This subject was further clarified by Dancoff[4] who considered what effect relativistic corrections would have on the high frequency terms of Pauli and Fierz. In doing this he necessarily confined his attention to terms of the second order in the electron's charge. His result was that these terms do not in general converge and that relativistic corrections could not be counted on to alter qualitatively the Pauli-Fierz result. The details of Dancoff's result further suggest that the theory made no sense at all. Thus the precise character of the infinite terms depends on whether the scattering potential is a four-vector or a world scalar and whether the spin of the scattered particle is 0 or $\frac{1}{2}$. It is not possible to believe that in a problem involving only charges moving with low velocity these deductions can have any relation to reality.

If we wish to consider this problem from the point of view proposed by Heitler and, at the same time, wish to be able to take relativistic effects into account, it will be necessary to analyze it in terms of the perturbation theory based on the smallness of the electron's charge. If for a moment we introduce a lowest frequency $\bar{q}$, then it is easy to see the part played by the arguments of Bloch and Nordsieck, Pauli and Fierz, and Dancoff in determining the radiative corrections. This frequency $\bar{q}$ can, in fact, be fantastically low, and this situation may be realized by considering the collision problem as it would be if the electron and the scatterer were both enclosed in a conducting box of enormous dimensions.

When we carry out this calculation we can at first neglect the contribution of processes in which more than one quantum is involved, or which involve a higher order of the electron's charge than the second. We then see that as long as the frequency q is smaller than the upper limit of the spectrum and the velocity of the electron is small compared to the velocity of light, two corrections to the scattering formula

[1] W. Heitler, Proc. Camb. Phil. Soc. **37**, 291 (1941); W. Heitler and H. W. Peng, Proc. Camb. Phil. Soc. **38**, 296 (1942).

[2] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

[3] W. Pauli and M. Fierz, Nuovo Cimento **15**, 167 (1938).

[4] S. M. Dancoff, Phys. Rev. **55**, 959 (1939).

cancel. One of these is the positive correction which comes from the probability of emitting a quantum of frequency q

$$\delta_r d\sigma = \frac{e^2}{4\pi^2 \hbar m^2 c^3} \int_{\bar{q}} \frac{d\mathbf{q}}{q^5} [\mathbf{p}\times\mathbf{q} - \mathbf{k}\times\mathbf{q}]^2 \cdot d\sigma = \frac{2e^2}{3\pi\hbar c} \frac{(\mathbf{p}-\mathbf{k})^2}{m^2c^2} \frac{dq}{q} \cdot d\sigma. \quad (3)$$

The second is a negative correction to the probability of scattering without emitting a quantum of frequency q,

$$\delta_e d\sigma = -\delta_r d\sigma. \quad (4)$$

This second correction may be formally obtained in either of two ways. (a) It may be regarded as caused by the virtual emission and reabsorption of a quantum of frequency q during the scattering process; or (b) it may be regarded as a renormalization of the probability that the incident electron and the scattered electron will in fact have no quanta associated with them. These are, of course, equivalent descriptions. The finite result of Bloch and Nordsieck follows from an expansion of this simple argument to processes involving the multiple emission of quanta. The conclusion which appears above is maintained throughout but, of course, as $\bar{q}$ is made smaller and smaller, the neglect of processes involving more than one quantum finally becomes inadmissible. The argument of Pauli and Fierz is then simply this: The probability of scattering with emission of radiation has no terms for frequency $q > mv^2/2\hbar$. On the other hand, the corrections to the probability of radiationless scattering continue for infinitely high q and lead to a logarithmic divergence. What Dancoff has done is to show that a consistent relativistic calculation for these latter terms does not give generally finite results.

We, therefore, see that the terms involving the virtual emission of quanta as a correction to the probability of radiationless scattering, and which may alternatively be derived as a sort of renormalization of the probability that electrons will be unaccompanied by quanta, have a dual rôle. For low frequencies they are needed to cancel the change in scattering probability caused by the emission of a quantum, a change which, as $\bar{q}$ goes to 0, becomes logarithmically infinite. For high frequencies these terms themselves have nothing left to cancel and give rise to a new logarithmic divergence. Thus, no proposal which is as simple as either including or excluding these terms can give a finite result or a sensible one. This is the essential reason why Heitler's proposals give an unsatisfactory answer when applied to this simple problem.

2. HEITLER'S THEORY: QUALITATIVE CONSIDERATIONS

In the theory of radiation damping developed by Heitler and his collaborators, the transition probability from an initial state, A, to a final state, B, is determined by certain matrix elements U_{AB} which are related to the matrix element H_{AB} of the Hamiltonian by the equation

$$U_{AB} = H_{AB} + i\pi \sum_C \rho_C H_{AC} U_{CB}. \quad (5)$$

The sum goes over all states, C, of the same energy as the initial and final state; ρ_C is the density of states of the type C per unit energy.

The cross section of a process leading from state A to B is given in Heitler's theory by

$$\sigma_{AB} = \frac{2\pi}{\hbar v_A} \rho_B |U_{AB}|^2, \quad (6)$$

where v_A is the velocity of the incident particle, and ρ_B is the density of states of type B per unit energy. Equation (6) is entirely analogous to the usual formula for the cross section except that the matrix element of the Hamiltonian is replaced by that of U.

The last term in Eq. (5) represents the radiation damping. Its function is to keep the transition matrix elements U finite even if the matrix elements of the Hamiltonian or the number of possible final states become very large. Heitler and his collaborators have proved in many papers on the meson theory that all cross sections actually remain finite at high energies while they would increase indefinitely without radiation damping. Also in our case, the result for the cross section is finite under all conditions, but we shall show that the result does not agree with physical expectations as discussed in Section 1.

The matrix elements of the Hamiltonian relevant for our problem are of two types. The first type is the matrix element for simple scattering which is simply the Fourier component of the potential as given in Eq. (1a),

$$(\mathbf{k}|H|\mathbf{p}) = V_{kp}. \tag{7}$$

The other relevant matrix elements correspond to the emission of a quantum $\mathbf{q}$ together with the scattering of the electron from its initial momentum $\mathbf{k}$ to the final momentum $\mathbf{p}$. These matrix elements are

$$(\mathbf{k}|H|\mathbf{pq}) = Cq^{-\frac{1}{2}}(\mathbf{p}-\mathbf{k})\cdot\mathbf{u}\,V_{kp}, \tag{8}$$

where

$$C = e/2\pi\hbar^{\frac{1}{2}}mc^{\frac{1}{2}} \tag{8a}$$

is a constant and $\mathbf{u}$ is a unit vector in the direction of polarization of the quantum. The matrix element (8) is normalized in such a way that the number of states of the quantum can be simply taken equal to the volume in q-space, $d\mathbf{q} = 4\pi q^2dq$. It is easily seen that (8) agrees with Eq. (3) of the first section.

We shall now estimate the order of magnitude of the cross section for scattering with emission of radiation in first approximation; i.e., with the assumption that radiation is small. We shall also estimate the reaction of radiation on the radiationless scattering, represented by the radiation damping term in (5). We shall find that in contrast to the theory outlined in Section 1, these two effects do not cancel, and we shall show that this lack of cancellation makes Heitler's theory unacceptable.

In first approximation we may put $U = H$ for transitions involving radiation. According to (6) and (8), the cross section for transitions with radiation will then be proportional to V^2C^2 (apart from other factors), which is of the order C^2 times the cross section for scattering without radiation.

We now consider the correction which radiation makes to the cross section for radiationless transitions by applying Eq. (5) to such transitions. The radiation reaction is given by the contribution of states C containing a quantum, to the last term of (5). If we make again the approximation $U_{CB} = H_{CB}$ for such states C, their contribution to the last term of (5) will be of the order V^2C^2. Since H_{AB} is of order V, the relative magnitude of the radiation reaction term in (5) is of order VC^2. This is smaller than the relative probability of collisions *with* radiation by a factor of the order V, which can be made arbitrarily small by a suitable choice of the scatterer.

Actually, the effect of radiation reaction on the *cross section* σ_{AB} is even smaller. This is because the damping term in (5) is imaginary in our present approximation while H_{AB} is real. The effect of the damping term on the cross section σ_{kp} is only of the relative order of V^2C^4 which is extremely small because C contains the (small) electron charge as a factor. Actually, closer examination shows that the leading term in the radiation reaction is not given by inserting for U_{CB} its first approximation, H_{CB}. In the next approximation, an imaginary term appears in U_{CB} which is of the relative order V compared with H_{CB}, and this term gives a *real* contribution to the radiation damping term in Eq. (5). The effect of this, both on U_{AB} and on the cross section σ_{AB}, is of the relative order V^2C^2, whereas the cross section for emission of radiation was shown to be of the relative order C^2. Therefore we see that the transitions with emission of radiation are not compensated by a corresponding reduction of the cross section for scattering without radiation.

This failure of compensation is especially serious because the total probability of emission of radiation diverges logarithmically for small frequencies q as was shown after Eq. (2). The total cross section for scattering with and without emission of radiation will, therefore, depend on the lower limit for the frequency, $\bar{q}$, and thereby on the size of the imaginary box in which the radiator is considered to be enclosed. It is clear that such a dependence has no physical meaning. The Heitler theory of radiation damping in its present form therefore does not solve the problem of the infra-red catastrophe. Moreover, the red end of the spectrum does not appear to involve a profound problem which would require going beyond the scope of present quantum mechanics, so that the failure of Heitler's theory must be considered the more serious.

Heitler's theory forbids, in fact, the customary solution of the infra-red problem as outlined in

Section 1. Heitler's prescription is to calculate the matrix element of each phenomenon only in the first approximation in which such a matrix element appears. This prescription serves to eliminate the usual divergences at high frequencies. However, it also eliminates in our case the perturbation by virtual quanta of low frequencies which gave the result (4), and thereby eliminated the infra-red catastrophe. In Heitler's theory this perturbation must be left out and there is, therefore, no adequate compensation for the scattering which takes place with emission of radiation.

3. HEITLER'S THEORY: QUANTITATIVE CALCULATION

For a quantitative discussion of Eq. (5) we shall make the simplifying assumption that the matrix elements of the scattering potential V_{kp} do not depend on the direction of the vectors $\mathbf{k}$ and $\mathbf{p}$. This amounts to assuming an interaction potential of range very short compared to the electron's wave-length. In addition we shall neglect the higher order effects such as the emission of two quanta or the scattering of quanta. We shall, however, make no restrictions on the magnitude of the electronic charge e and thereby of the constant C, nor any restrictions on the magnitude of the matrix element $V_{kp}=V$.

We shall first write down Eq. (5) explicitly for transitions with and without radiation. Using Dirac's notation for the matrix element, we have for transitions without radiation

$$(\mathbf{p}'|U|\mathbf{p})=(\mathbf{p}'|H|\mathbf{p})+i\pi\frac{4\pi mp}{(2\pi\hbar)^3}\int\frac{d\omega''}{4\pi}(\mathbf{p}'|H|\mathbf{p}'')(\mathbf{p}''|U|\mathbf{p})+i\pi\frac{4\pi mp}{(2\pi\hbar)^3}\int\frac{d\omega''}{4\pi}d\mathbf{q}(\mathbf{p}'|H|\mathbf{p}''\mathbf{q})(\mathbf{p}''\mathbf{q}|U|\mathbf{p}).\tag{9}$$

For scattering with emission of radiation, we have the Heitler equation

$$(\mathbf{p}'\mathbf{q}|U|\mathbf{p})=(\mathbf{p}'\mathbf{q}|H|\mathbf{p})+i\pi\frac{4\pi mp}{(2\pi\hbar)^3}\int\frac{d\omega''}{4\pi}\times[(\mathbf{p}'\mathbf{q}|H|\mathbf{p}''\mathbf{q})(\mathbf{p}''\mathbf{q}|U|\mathbf{p})+(\mathbf{p}'\mathbf{q}|H|\mathbf{p}'')(\mathbf{p}''|U|\mathbf{p})].\tag{10}$$

In any matrix element, $\mathbf{q}$ means that a quantum exists in the respective state of the system; $\mathbf{p}$ or $\mathbf{p}'$ or $\mathbf{p}''$ denote the momentum of the electron, p the absolute value of the momentum which is the same for all states, the integral $d\omega''$ is over all directions of $\mathbf{p}''$ and the integral $d\mathbf{q}$ is over magnitude, direction and polarization of the light quantum.

These equations are to be solved with expressions (7) and (8) for the matrix elements of the Hamiltonian. It is reasonable to assume, and it will in fact be shown to be true, that the solution will have the form

$$\begin{aligned}(\mathbf{p}'|U|\mathbf{p})&=A_1+A_2\mathbf{p}\cdot\mathbf{p}'/p^2,\\(\mathbf{p}'\mathbf{q}|U|\mathbf{p})&=q^{-\frac{3}{2}}(B_1\mathbf{p}'\cdot\mathbf{u}+B_2\mathbf{p}\cdot\mathbf{u}).\end{aligned}\tag{11}$$

Inserting (11) into (10) we obtain

$$B_1\mathbf{p}'\cdot\mathbf{u}+B_2\mathbf{p}\cdot\mathbf{u}=VC(\mathbf{p}'-\mathbf{p})\cdot\mathbf{u}+\frac{imp}{2\pi\hbar^3}\int\frac{d\omega''}{4\pi}[V(B_1\mathbf{p}''\cdot\mathbf{u}+B_2\mathbf{p}\cdot\mathbf{u})+VC(\mathbf{p}'-\mathbf{p}'')\cdot\mathbf{u}(A_1+A_2\mathbf{p}''\cdot\mathbf{p}/p^2)].\tag{12}$$

The integration over $d\omega''$ is elementary. The terms containing only one factor involving $\mathbf{p}''$ vanish upon integration, whereas the average of a term of the form $\mathbf{p}''\cdot\mathbf{a}\ \mathbf{p}''\cdot\mathbf{b}$ gives the result $\mathbf{a}\cdot\mathbf{b}/3$ upon averaging. If we also introduce the abbreviation

$$W=(mp/2\pi\hbar^3)V,\tag{13}$$

and then compare terms with $\mathbf{p}\cdot\mathbf{u}$ and terms with $\mathbf{p}'\cdot\mathbf{u}$ on both sides of the equation, we find:

$$\begin{aligned}B_1&=VC+iCWA_1,\\B_2&=-VC+iWB_2-iWCA_2/3.\end{aligned}\tag{14}$$

In a similar way we obtain by inserting (11) into (9) and using the abbreviation (13)

$$A_1+A_2(\mathbf{p}'\cdot\mathbf{p}/p^2)=V+i\int(d\omega''/4\pi)\times\left[W(A_1+A_2\mathbf{p}''\cdot\mathbf{p}/p^2)+\int d\mathbf{q}q^{-3}\times CW(\mathbf{p}''-\mathbf{p}')\cdot\mathbf{u}(B_1\mathbf{p}''\cdot\mathbf{u}+B_2\mathbf{p}\cdot\mathbf{u})\right].\tag{15}$$

Evaluation gives

$$A_1+A_2(\mathbf{p}'\cdot\mathbf{p}/p^2)=V+iWA_1 +iWCp^2\frac{8\pi}{3}\left(B_1-B_2\frac{\mathbf{p}'\cdot\mathbf{p}}{p^2}\right)\ln\frac{q_{\max}}{\bar{q}}. \quad (16)$$

In this equation $q_{\max}$ is the Duane-Hunt high frequency limit, $q_{\max}=mv^2/2h$. On the other hand, $\bar{q}$ is determined by the size of the box in which the system is considered to be enclosed. As this box becomes larger, the logarithm in the last term of Eq. (16) increases indefinitely.

Combining Eq. (14) and (16) and using the abbreviation

$$R=(8\pi/3)C^2p^2\ln(q_{\max}/\bar{q}) =\frac{2}{3\pi}\frac{e^2}{\hbar c}\left(\frac{v}{c}\right)^2\ln\frac{mv^2}{2\hbar\bar{q}}, \quad (17)$$

we find the following solutions for the amplitudes:

$$A_1=V\frac{1+iWR}{1-iW+W^2R},\quad A_2=V\frac{iWR}{1-iW+W^2R/3},\quad B_1=\frac{VC}{1-iW+W^2R},\quad B_2=\frac{-VC}{1-iW+W^2R/3}. \quad (18)$$

The quantity R contains the dependence on the size of the box and becomes infinite for infinite size; however, for any reasonable size, R is quite small.

It will be noted that the amplitudes A_1, etc. all remain finite when R goes to infinity. This shows that in this case as in others the theory of radiation damping gives finite results. It will also be noted that for $R=\infty$ (strong coupling with radiation) the coefficients B_1 and B_2 become vanishingly small, so that in this limit the scattering process paradoxically will take place without the emission of radiation. (This result is not changed by integration over all frequencies q which can be emitted.)

The result for the cross sections becomes quite complicated in the general case. For instance, the result for the cross section without emission of radiation takes the form

$$d\sigma=\lambda\!\!\!^{-2}d\omega W^2 \times\left|\frac{1+iWR}{1-iW+W^2R}+\frac{iWR\cos\alpha}{1-iW+W^2R/3}\right|^2 \quad (19)$$

where $\lambda\!\!\!^{-}=\hbar/p$ is the de Broglie wave-length of the particle, divided by 2π. The further evaluation simplifies in the limit of very large R in which case the result is

$$d\sigma=\lambda\!\!\!^{-2}d\omega(1+3\cos\alpha)^2. \quad (20)$$

The cross section thus becomes of the order $\lambda\!\!\!^{-2}$ which is reasonable. Generally it is clear that $d\sigma$ is a quadratic function of $\cos\alpha$, the angular dependence being due to the interaction with radiation.

We shall now discuss the result in more detail in the limit of small R and thereby substantiate the qualitative considerations of the last section. If we neglect all terms of higher relative order than W^2R^2, the cross section for scattering without radiation (19) reduces to

$$d\sigma=\lambda\!\!\!^{-2}d\omega W^2[1-W^2-4W^2R +W^2R^2(1+\cos\alpha)^2+\cdots] \quad (21)$$

where α is the scattering angle (angle between $\mathbf{p}$ and $\mathbf{k}$). Similarly, if we neglect terms of relative order W^2R, we find from (18)

$$B_1=-B_2=VC(1+iW+\cdots) \quad (22)$$

and the cross section for scattering with emission of radiation of frequency q becomes (cf. (11))

$$\delta_r d\sigma=\lambda\!\!\!^{-2}d\omega\frac{W^2}{1+W^2}C^2\frac{d\mathbf{q}}{q^3}2p^2(1-\cos\alpha)\sin^2\beta, \quad (23)$$

where β is the angle between $\mathbf{u}$ and $\mathbf{p}-\mathbf{p}'$; or, if we integrate over all frequencies and directions for the emitted radiation:

$$\sigma_r=2\lambda\!\!\!^{-2}d\omega RW^2/(1+W^2). \quad (24)$$

Therefore, the ratio of the scattering with radiation to the scattering without radiation is of the order R. On the other hand, the first correction which the radiation reaction makes to the cross section for radiationless scattering is of the relative order W^2R. This correction, therefore, does not compensate the cross section with emission of radiation and the total scattering will therefore depend on R, and thus on the minimum frequency which can be emitted.

4. REMARK ON THE SCATTERING BY A POTENTIAL IN HEITLER'S THEORY

If we disregard radiation and simply consider the scattering by a concentrated potential, the cross section becomes

$$d\sigma = \lambda\!\!\!^{-2} d\omega W^2/(1+W^2), \tag{25}$$

which follows from (19) by neglecting R. The expression (25) is obviously correct for small W in which case it is identical with the Born approximation. For strong potentials (large W) the cross section remains finite and attains the limiting value $4\pi\lambda\!\!\!^{-2}$ which is the correct upper limit if only spherically symmetrical scattering is considered. However, this limit will be attained regardless of the sign of the potential. Actually the result $4\pi\lambda\!\!\!^{-2}$ is reasonable only for an attractive potential (and even in this case there should be fluctuations corresponding to resonance) while in the case of a repulsive potential the cross section can never become greater than $4\pi a^2$ where a is the radius of the region in which the repulsive potential exists. This limit holds even for infinitely strong repulsion and is smaller than $4\pi\lambda\!\!\!^{-2}$, because the assumption that the matrix elements for the scattering V_{kp} are independent of the direction of $\mathbf{k}$ and $\mathbf{p}$, is equivalent to assuming $a \ll \lambda\!\!\!^{-}$.

Moreover, Eq. (25) does not correctly describe the deviations from the Born approximation when this approximation first begins to fail. According to (25) the deviations should be of the relative order W^2; whereas the actual theory gives a deviation of the order $W(\lambda\!\!\!^{-}/a)$ which is much larger.

We do not regard this failure of Heitler's theory as significant because the theory was not intended to describe higher approximations for an arbitrary interaction such as the deviations from the Born approximation for an arbitrary potential. Heitler's theory was actually intended to apply to problems of radiation, and the matrix elements of the Hamiltonian should be taken from the fundamental theory of the interaction of particles with radiation. For this reason we consider the failure of the theory in the problem of the infra-red catastrophe as much more significant than its failure in the scattering problem.

5. CONCLUSIONS

The theory of radiation damping developed by Heitler and his collaborators gives, in all applications that have so far been made, a finite result for all cross sections even in the case of very large interactions. However, the application of the theory to the problem of the reaction of radiation on the scattering of electrons does not give a physically sensible result. The effect of radiation damping on the scattering without radiation is extremely small compared with the cross section for scattering with emission of radiation. The total scattering probability is therefore essentially the sum of the uncorrected scattering with and without radiation. The latter is subject to the infra-red catastrophe and depends on the lower limit $\bar{q}$ of the frequencies which can be emitted, becoming infinite as $\bar{q}$ goes to zero. It is evident that such a dependence can have no physical reality and that the probability of electron scattering should not depend on its ability to emit radio waves of extremely long wave-length.

In the customary theory as outlined in Section 1, the infra-red catastrophe does not occur because the scattering with emission of radiation is compensated by an equal decrease of the scattering without radiation. This decrease is brought about by the consideration of the influence of virtual quanta with long wave-length on the scattering without radiation. This amounts to the inclusion in the theory of matrix elements which are of higher order than is necessary to obtain a non-vanishing transition probability. The inclusion of such matrix elements is forbidden in Heitler's theory. It is seen that these matrix elements play an important role in avoiding the infra-red catastrophe.

It is well known that the inclusion of the perturbation by the virtual quanta leads to divergent results due to the quanta of *high* frequency. This ultraviolet catastrophe is inherent in the customary theory and avoided in Heitler's theory of radiation damping. What we have shown in this paper is that it is not possible to obtain a satisfactory answer either by including the effects of virtual quanta as they are given by present theory or by excluding them entirely. They must be retained for long wave-lengths. They must be eliminated or modified for short wave-lengths.

This implies the existence of a critical wave-length. There are at least three possibilities. There is (1) the wave-length corresponding to the Duane-Hunt limit, (2) that corresponding to the electron's Compton wave-length, or (3) a new length characteristic of a future theory and presumably smaller.

The first of these possibilities, which might for instance amount to some method of systematic elimination of all virtual processes except those involving the creation of particles which in the actual problem are energetically capable of being created, would clearly involve a major and not correspondence theoretic change in the formalism of quantum mechanics. In particular, it would mean giving up the existence of a Hamiltonian and a wave function calculable from it. It does not seem likely that this will prove to be the correct path.

The results of Dancoff show that at least in present relativistic theory the Compton wave-length of the electron does not provide a suitable critical wave-length for the problem in question. It therefore seems most probable to us that a new length will be involved in a correct solution, above which the present quantum theory will have a kind of validity and below which new phenomena will have to be taken into account. It would seem likely that only in this way can the correspondence principle be satisfied. It is not satisfied by Heitler's proposals, and that in the last analysis is why they are unsatisfactory. In any case, we believe that the simple problem here considered may afford a useful test of future theories of radiation.

PHYSICAL REVIEW VOLUME 70, NUMBERS 7 AND 8 OCTOBER 1 AND 15, 1946

Scattering of Fast Neutrons by Boron

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The cross section of boron for the scattering of fast neutrons through angles greater than 30° was measured at neutron energies between 0.2 and 3 Mev.

THE scattering of monoenergetic fast neutrons by boron has been investigated by Kikuchi and Aoki[1,2] for $d-d$ neutrons and by Leipunsky[3] for photo-neutrons of 130-, 220-, and 850-kev energy. In the present experiments the scattering of neutrons of energies from 0.2 to 3 Mev was investigated for the two boron isotopes.

The enriched material available was boron powder containing 53 percent B^{10}. The powder was placed in a cylindrical brass container, $2\frac{15}{16}''$ i.d. The density of the powder was increased by shaking it on a vibrating table. 194 grams of the powder formed a layer $1\frac{1}{2}''$ thick, containing 0.255×10^{24} atoms/cm^2. A sample of normal boron powder was prepared in the same way. The normal sample contained 200 grams of powder and had the same number of atoms per cm^2 as the enriched sample.

The experiments at neutron energies up to 1500 kev were carried out using neutrons obtained by bombarding Li with protons accelerated by the University of Wisconsin's electrostatic generator. For the experiments at 3 Mev the neutrons were produced by bombarding D_2O ice with 200-kev deuterons obtained by means of a Cockcroft-Walton set.

The cross section for the scattering of neutrons through angles greater than 30° was measured.

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