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A New Approach to Kinematic Cosmology—(B)

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Maxwell's equations, Lorentz's equations of motion, and Dirac's equations are investigated on a cosmological background. Maxwell's equations are conform invariant and the equations of motion can easily be made conform invariant. Then, without introducing any new assumptions, Dirac's equations are seen to be conform invariant. In open universes the solutions of Maxwell's and Dirac's equations are the same as in flat Minkowski space. The behavior of these equations is different in closed universes by reasons of topology. In closed universes both Maxwell's and Dirac's equations provide new eigenvalue problems. Dirac's equations exhibit a marked difference between an elliptic and a spherical closed universe.

INTRODUCTION

THIS investigation is the continuation of a paper under the same title.[23] As the first two parts of **A** contained much material of which no use is made here, we have tried to formulate the present paper so that its main contents can be understood without a detailed knowledge of **A**.

In **A** we investigated the possible cosmological backgrounds. Here we shall examine the equations of electrodynamics and Dirac's equations on this background.

In viewing the relation of atomic physics to cosmology, we encounter two widely differing schools of thought. Some investigators believe that the worlds of microphysics and of cosmology are intimately connected. This point of view was perhaps most distinctly represented by the late Sir Arthur Eddington. Dirac, Schrödinger, and Milne seem to share this general attitude. On the other hand, many physicists dismiss this view as formal, speculative, and arbitrary.

Our analysis here seems to point toward a third possibility, which differs from both extreme views, the one claiming intimate connection, the other complete independence between atomic physics and cosmology.

We now give a brief summary of the conclusions reached in this paper.

The possible universes are either *closed* or *open*. The metric form of closed universes is

$$\text{I}: \quad ds^2 = R^2(\tau)\{d\tau^2 - d\rho^2 - \sin^2\rho(d\theta^2 + \sin^2\theta d\varphi^2)\}, \qquad (0.11)$$

and that of open universes is

$$\text{II}: \quad ds^2 = R^2(\tau)\{d\tau^2 - d\rho^2 - \sinh^2\rho(d\theta^2 + \sin^2\theta d\varphi^2)\} \qquad (0.12)$$

or

$$\text{III}: \quad ds^2 = R^2(\tau)\{d\tau^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\}. \qquad (0.13)$$

In the *cosmological coordinate system* (*c.c.s.*), the line elements assume the form

$$ds^2 = \gamma(t, r)ds_0^2, \qquad (0.2)$$

where ds_0^2 is the line element of flat Minkowski space-time, and the functions γ are listed in **A**, Table I.

It was shown in **A** that Maxwell's equations do not involve the function γ, and have, therefore, the same form in all cosmological spaces and, in particular, the same form as in Minkowski space. Under suitable assumptions, the same statement applies to Lorentz's equations of motion and to Dirac's equations. Thus the equations of electrodynamics and Dirac's equations are insensitive to any change of metric; the cosmological background does not reveal itself in the equations. The implication seems unavoidable that the study of Maxwell's, Lorentz's, and Dirac's equations does not point to any connection between microphysics and

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[23] L. Infeld and A. Schild, Phys. Rev. **68**, 250–272 (1945); hereinafter referred to as **A**. The numbering of parts, sections, equations, and footnotes is carried over from this previous work.—We wish to take this opportunity to correct a printing error in **A**; Eq. (A72) should read:

$$\bar{x} \pm (\bar{t}^2 - \bar{\rho}^2)^{\frac{1}{2}} = [\{x \pm (t^2 - \rho^2)^{\frac{1}{2}}\}^{-1} - bp]^{-1}, \quad \bar{\rho}/\bar{t} = \rho/t.$$

cosmological structure. Yet a more detailed analysis shows that such a connection does exist. Anticipating our results, we may say:

In the case of *open* universes, there is *no difference* between Minkowski space and any other cosmology.

In the case of *closed* universes, the situation is different. The reason for this lies not in the differential equations but in the appearance of boundary conditions, which are caused by the fact that the points (τ, ρ), $(\tau, \rho+\pi)$, and $(\tau+\pi, \rho)$ must be identified with a single physical event. The relation of the atomic to the cosmological world is determined, *not* by the *metric, but* by the *topology* of the universe.

We shall now outline briefly the behavior of the equations of electrodynamics and of Dirac's equations in the transition from one cosmological space to another. In particular, we wish to consider under what conditions these equations are *conform invariant*, i.e., retain their form for all universes of one type.

In the case of Maxwell's equations, the conform invariance is immediately demonstrated. No new assumptions enter the argument. The behavior of Lorentz's equations of motion and of Dirac's equations is not so obvious.

The expressions

$$R/r_e, \quad \dot{R}/r_c \tag{0.3}$$

enter Lorentz's equations of motion for an electron and Dirac's equations, respectively. Here R is a function of time defined by one of the line elements (0.11) to (0.13); r_e is the "radius of the electron"

$$r_e = e^2/m \tag{0.4}$$

(m = mass of the electron, $-e$ = charge of the electron, c = velocity of light = 1); r_c is the Compton wave-length

$$r_c = \hbar/m \tag{0.5}$$

($\hbar$ = Planck's constant divided by 2π).

The three ratios

$$R/r_e, \quad R/r_c, \quad r_e/r_c \tag{0.6}$$

are dimensionless pure numbers, the last being the fine structure constant. We now have the following alternative: either r_e and r_c are assumed constant, in which case the first two of the three dimensionless quantities are functions of time; or, all three dimensionless quantities are constants, in which case the electron mass m is a function of time. The second of these assumptions implies the conform invariance of Lorentz's and Dirac's equations.

III. Cosmological Electrodynamics

8. MAXWELL'S EQUATIONS IN COSMOLOGICAL SPACES

In A it was shown that the line elements, suitable for the large scale description of our universe, are of the conformal-Minkowskian form

$$ds^2 = g_{ij}dx^i dx^j = \gamma(t, r)ds_0^2, \tag{8.1}$$

where ds_0^2 is the Minkowskian line element

$$\begin{aligned} ds_0^2 &= {}^0g_{ij}dx^i dx^j \\ &= dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \tag{8.11}$$

Three distinct types of universes are found to be compatible with the "postulate of homogeneity." The corresponding functions γ are of the forms

$$\text{Case I:} \quad \gamma = (1-a)^{-2}f\{t/(1-a)\}, \tag{8.12}$$

$$\text{Case II:} \quad \gamma = \gamma(a), \tag{8.13}$$

$$\text{Case III:} \quad \gamma = \gamma(t), \tag{8.14}$$

where

$$a = t^2 - r^2. \tag{8.15}$$

In universes of type I, the coordinate r of a "fundamental particle," representing a nebula, is a double valued function of the time coordinate t. The simplest way to avoid this difficulty is to identify pairs of points in 4-space with single physical events. The points thus identified are connected by an inversion:

$$(t, r, \theta, \varphi) \rightarrow (-t/a, r/a, \theta, \varphi). \tag{8.16}$$

This transformation is of period two, i.e., it is its own inverse. Also it leaves the cosmological line element (8.1–8.12) invariant in form.

Suppressing the two polar coordinates θ and φ, the tr plane has the *topology of a torus*.

The c.c.s. has the important property that, in it, Maxwell's equations assume the same form for all possible universes, and, in particular, the same form as in flat Minkowski space. We write Maxwell's equations for empty space in the usual form

$$\partial F_{ij}/\partial x^k+\partial F_{jk}/\partial x^i+\partial F_{ki}/\partial x^j=0, \quad (8.21)$$

$$\partial((-g)^{\frac{1}{2}}g^{ik}g^{jl}F_{kl})/\partial x^j=0. \quad (8.22)$$

It follows immediately from (8.1) that

$$(-g)^{\frac{1}{2}}g^{ik}g^{jl}F_{kl}=(-{}^0g)^{\frac{1}{2}}\,{}^0g^{ik}\,{}^0g^{jl}F_{kl}, \quad (8.23)$$

and thus the differential Eqs. (8.21) and (8.22) are the same for all functions γ.

We say that Maxwell's equations are *conform invariant.* By this we mean that they are invariant under a conformal mapping

$$ds'=\lambda ds, \quad (8.24)$$

i.e., a change in the function γ of the cosmological line element (8.1). The result of Eq. (8.23) is expressed by saying that the tensor F_{ij} and the tensor density $(-g)^{\frac{1}{2}}F^{ij}$ are conform invariant. Another important conform invariant quantity is the electromagnetic energy tensor density

$$(-g)^{\frac{1}{2}}E_i{}^j=(-g)^{\frac{1}{2}}(F_{ik}F^{kj}+\tfrac{1}{4}\delta_i{}^jF_{rk}F^{rk}). \quad (8.25)$$

In the presence of charges and currents, Eq. (8.22) must be replaced by

$$\partial((-g)^{\frac{1}{2}}F^{ij})/\partial x^j=(-g)^{\frac{1}{2}}J^i, \quad (8.26)$$

where J^i is the charge-current vector. In order to preserve the conform invariance of Maxwell's equations, $(-g)^{\frac{1}{2}}J^i$ must be conform invariant.

From the above it follows that all electromagnetic fields in cosmological spaces are also possible fields in flat space-time. In the case of universes of types II and III, the converse statement holds: Any electromagnetic field in flat space-time is a suitable solution of Maxwell's equations in such universes. Thus the problem of solving Maxwell's equations in universes of types II and III is dealt with. However, matters are more complicated in universes of type I.

In universes of type I a ***new boundary condition*** must be imposed on the solutions of Maxwell's equations. Since two points connected by the inversion (8.16) are identified as a single event we must demand that the field tensor F_{ij} ***be invariant in form under this inversion.*** Let $F_{ij}'(x)$ be the tensor transform of $F_{ij}(x)$ under the inversion (8.16); then we may write the boundary condition in the compact form

$$F_{ij}'(x)\equiv F_{ij}(x). \quad (8.3)$$

We shall now show how such a suitable F_{ij} may be constructed from any solution ${}^0F_{ij}$ of Maxwell's equations which may not necessarily satisfy the boundary condition (8.3). Since the inversion (8.16) leaves the line element of a universe of type I invariant, it also leaves Maxwell's equations invariant. Thus, ${}^0F_{ij}'$ is a solution of Maxwell's equations, and so is

$$F_{ij}=\tfrac{1}{2}({}^0F_{ij}+{}^0F_{ij}'). \quad (8.4)$$

Since the inversion (8.16) is its own inverse, we deduce that F_{ij} satisfies the condition (8.3) and is therefore a suitable solution.

It is obvious that all possible solutions of Maxwell's equations, which satisfy the condition (8.3), can be written in the form (8.4). Thus all electromagnetic fields in a universe of type I can be constructed from the set of all fields in flat Minkowski space. This general result is, however, of limited use in practice. A simple electromagnetic field ${}^0F_{ij}$ will, by (8.4), generate a field F_{ij} which is, usually, very complicated and difficult to interpret physically.

A simple example of an electromagnetic field in a universe of type I is the usual electrostatic field of a point charge e at the spatial origin $r=0$**:

$$F_{rt}=e/r^2, \quad \text{all other } F_{ij}=0. \quad (8.5)$$

A straightforward calculation shows that the condition (8.3) is satisfied. The usual field of a magnetic pole at the spatial origin also satisfies the boundary condition.

Summarizing, we may say that the description of electromagnetic phenomena in cosmological spaces is completely ***independent of the metric*** of the universe, but is only influenced by the type, or, more precisely, by the *topology* of the universe

9. THE REST COORDINATE SYSTEM

For the detailed discussion of electromagnetic fields in universes of type I, it is convenient to

** Here F_{rt} denotes the component F_{10} belonging to the coordinates $x^1=r$ and $x^0=t$. A similar notation is used throughout whenever there is no danger of confusion.

use a system of coordinates different from that employed in the previous section. Under the coordinate transformation

$$t+r=\tan\tfrac{1}{2}(\tau+\rho),\quad t-r=\tan\tfrac{1}{2}(\tau-\rho). \quad (9.1)$$

the conformal-Minkowskian line element (8.1–8.12) assumes the form

$$ds^2=R^2(\tau)\{d\tau^2-d\rho^2 -\sin^2\rho(d\theta^2+\sin^2\theta d\varphi^2)\}. \quad (9.11)$$

In **A** it was shown that the fundamental particles, representing nebulae, are at rest in this new coordinate system (τ, ρ). We shall therefore refer to it as the *rest coordinate system*, r.c.s. for short. This coordinate system is closely related to that used by Robertson.[24]

A suitable conformal mapping reduces the line element (9.11) to that of an Einstein universe, where $R=R_E=$constant. Thus, in the r.c.s. the solutions of Maxwell's equations are of the same form in all universes of type I, and, in particular, the same as in the Einstein universe. In this respect, the r.c.s. is similar to the c.c.s. Maxwell's equations in an Einstein space are of a form less familiar than in Minkowski space. But on the other hand, the boundary condition to be imposed on the electromagnetic field is considerably simpler in the r.c.s. It is this property which makes the r.c.s. convenient for the study of Maxwell's equations.

It immediately follows from (8.16) and from the transformation Eqs. (9.1), that the points

$$(\tau,\rho,\theta,\varphi)\equiv(\tau,\rho+\pi,\theta,\varphi) \equiv(\tau+\pi,\rho,\theta,\varphi) \quad (9.12)$$

are identified as one single event. This implies two facts: first, the 3-space $\tau=$constant of constant positive curvature is an *elliptic 3-space;* second, the *τ-axis* is *closed.* This last restriction is new and arises quite naturally from the discussion of universes of type I in the c.c.s. We shall see that it has an important effect on the large scale behavior of electromagnetic radiation and possibly on other physical problems.

The $\tau\rho$ plane ($\theta=\varphi=$constant) is represented by a square of side π of which opposite sides are identified. Its topology is that of a torus. Physically, the finite coordinate length π of the τ axis is the time in which light circumnavigates space and returns to the fundamental particle from which it was emitted. The boundary condition to be imposed on the field tensor F_{ij} is that it be *periodic in τ and ρ with period π.*

As an alternative to the "elliptic" universe, which was just discussed, we may consider the "spherical" universe of type I in which the coordinates τ and ρ are of period 2π each. Its behavior is to a great extent similar to that of the elliptic universe. However, it has one serious disadvantage: singularities, of the electromagnetic field, say, always occur in pairs at spatially antipodal points, the two singularities being equal except for a possible difference in sign. It is difficult to see how a physical distinction is to be made between a singularity and its "ghost." For this reason, the elliptic universes are to be preferred.

Solutions of Maxwell's equations, free from singularities, were investigated by Schrödinger[25] in coordinates similar to our rest coordinates, except that spatial coordinates were employed by Schrödinger analogous to cylindrical rather than spherical coordinates. Schrödinger did not introduce the *a priori* condition that the field components be of period π in τ. Therefore, it seems justified to give a fresh discussion of Maxwell's equations in the r.c.s. which, incidentally, includes electromagnetic fields with a singularity at the spatial origin (these are examined in the next section).

From the line element (9.11), we obtain

$$\begin{aligned} &g_{\tau\tau}=R^2,\quad g_{\rho\rho}=-R^2,\quad g_{\theta\theta}=-R^2\sin^2\rho,\quad g_{\varphi\varphi}=-R^2\sin^2\rho\sin^2\theta,\quad (-g)^{\frac{1}{2}}=R^4\sin^2\rho\sin\theta,\\ &g^{\tau\tau}=1/R^2,\quad g^{\rho\rho}=-1/R^2,\quad g^{\theta\theta}=-1/R^2\sin^2\rho,\quad g^{\varphi\varphi}=-1/R^2\sin^2\rho\sin^2\theta. \end{aligned} \quad (9.2)$$

[24] A, Section 4.
[25] E. Schrödinger, "Maxwell's and Dirac's equations in the expanding universe," Proc. Roy. Irish Acad. **46A**, 25–47 (1940). See also B. C. Mukerji, "Über elektromagnetische Wellen im Friedmannschen Raum," Zeits. f. Physik **101**, 270–275 (1936).

The second set of Maxwell's equations (8.22) becomes

$$\begin{aligned}
&\sin\theta(\sin^2\rho F_{\rho\tau})_{,\rho}+(\sin\theta F_{\theta\tau})_{,\theta}+F_{\varphi\tau,\varphi}/\sin\theta=0,\\
&-\sin^2\rho\sin\theta F_{\rho\tau,\tau}+(\sin\theta F_{\rho\theta})_{,\theta}-F_{\varphi\rho,\varphi}/\sin\theta=0,\\
&-\sin\theta F_{\theta\tau,\tau}-\sin\theta F_{\rho\theta,\rho}+F_{\theta\varphi,\varphi}/\sin^2\rho\sin\theta=0,\\
&-F_{\varphi\tau,\tau}/\sin\theta+F_{\varphi\rho,\rho}/\sin\theta-(F_{\theta\varphi}/\sin\theta)_{,\theta}/\sin^2\rho=0,
\end{aligned}\tag{9.21}$$

where the comma denotes partial differentiation; thus $F_{\rho\tau,\tau}=\partial F_{\rho\tau}/\partial\tau$, etc. We introduce the electromagnetic potential vector Φ_i:

$$F_{ij}=\Phi_{i,j}-\Phi_{j,i}.\tag{9.22}$$

Then the first set of Maxwell's equations (8.21) is identically satisfied. Substituting from (9.22) into Eqs. (9.21), we obtain a set of four differential equations for the components Φ_i. These are checked to have the following solution[26] which depends on two arbitrary complex constants.

$$\Phi_\tau=-Ae^{in\tau}\frac{dB_n{}^l}{d\rho}P_l{}^m\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix},\tag{9.31}$$

$$\Phi_\rho=-iAne^{in\tau}B_n{}^lP_l{}^m\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix},\tag{9.32}$$

$$\Phi_\theta=Ame^{in\tau}B_n{}^l\frac{P_l{}^m}{\sin\theta}\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix},\tag{9.33}$$

$$\Phi_\varphi=Ae^{in\tau}B_n{}^l\sin\theta\frac{dP_l{}^m}{d\theta}\begin{Bmatrix}\cos m\varphi\\ -\sin m\varphi\end{Bmatrix}.\tag{9.34}$$

Here, $A=A_{nlm}$ is an arbitrary complex constant, $P_l{}^m=P_l{}^m(\cos\theta)$ are associated Legendre polynomials, and $B_n{}^l=B_n{}^l(\rho)$ is a function of ρ which satisfies the differential equation

$$d^2B_n{}^l/d\rho^2+(n^2-l(l+1)/\sin^2\rho)B_n{}^l=0.\tag{9.4}$$

It is interesting to note that the four components Φ_i can be derived from a single scalar function ψ:

$$\psi=-Ae^{in\tau}B_n{}^lP_l{}^m\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix},$$

$$\Phi_\tau=\psi_{,\rho},\quad \Phi_\rho=\psi_{,\tau},\quad \Phi_\theta=-(m/\sin\theta)\psi,\quad \Phi_\varphi=-(\sin\theta/m)\psi_{,\theta\varphi}.$$

From Eqs. (9.31) to (9.34), we immediately obtain the components of the electromagnetic field tensor F_{ij}:

$$F_{\rho\tau}=Al(l+1)e^{in\tau}\frac{B_n{}^l}{\sin^2\rho}P_l{}^m\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix},\tag{9.51}$$

$$F_{\theta\tau}=Ae^{in\tau}\left(\frac{dB_n{}^l}{d\rho}\frac{dP_l{}^m}{d\theta}+imnB_n{}^l\frac{P_l{}^m}{\sin\theta}\right)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix},\tag{9.52}$$

$$F_{\varphi\tau}=Ae^{in\tau}\left(m\frac{dB_n{}^l}{d\rho}P_l{}^m+inB_n{}^l\sin\theta\frac{dP_l{}^m}{d\theta}\right)\begin{Bmatrix}\cos m\varphi\\ -\sin m\varphi\end{Bmatrix},\tag{9.53}$$

[26] The solutions were actually obtained by solving Maxwell's equations in a c.c.s. and by subsequent transformation to the r.c.s.

$$F_{\theta\varphi}=Al(l+1)e^{in\tau}B_n{}^l\sin\theta P_l{}^m\begin{Bmatrix}\cos m\varphi\\-\sin m\varphi\end{Bmatrix},\tag{9.54}$$

$$F_{\varphi\rho}=Ae^{in\tau}\left(\frac{dB_n{}^l}{d\rho}\sin\theta\frac{dP_l{}^m}{d\theta}+imnB_n{}^lP_l{}^m\right)\begin{Bmatrix}\cos m\varphi\\-\sin m\varphi\end{Bmatrix},\tag{9.55}$$

$$F_{\rho\theta}=-Ae^{in\tau}\left(m\frac{dB_n{}^l}{d\rho}\frac{P_l{}^m}{\sin\theta}+inB_n{}^l\frac{dP_l{}^m}{d\theta}\right)\begin{Bmatrix}\sin m\varphi\\\cos m\varphi\end{Bmatrix}.\tag{9.56}$$

Equations (9.21) are satisfied by virtue of the relation (9.4) and the second-order differential equation for the $P_l{}^m$. The other set of Maxwell's equations (8.21) are satisfied by virtue of (9.22).

As is well known, the uniqueness of the F_{ij} in physical space imposes the conditions that m and l be integers with $|m|\leqslant|l|$. Since l is only involved through $l(l+1)$, we may assume $l\geqslant 0$.

The F_{ij} involve the time coordinate τ through the factor $e^{in\tau}$. The boundary condition that the F_{ij} be of period π in τ limits the constant n to even integral values. We may, without loss of generality, take n positive as Eq. (9.4) involves its square. We therefore have a discrete eigenvalue spectrum for n:

$$n=0,\ 2,\ 4,\ \cdots.\tag{9.6}$$

We shall now discuss solutions which represent free electromagnetic radiation and relegate the investigation of fields with singularities to the next section. For solutions free from singularities the conform invariant, electromagnetic energy-momentum tensor-density $(-g)^{\frac{1}{2}}E_i{}^j$ must be finite everywhere. The components of $(-g)^{\frac{1}{2}}E_i{}^j$ can be computed from (8.25); some of these components involve ρ through the factor $B_n{}^l/\sin\rho$. Thus $B_n{}^l$ must vanish at $\rho=0$ and at $\rho=\pi$ at least as rapidly as $\sin\rho$.

The problem of finding solutions of Eq. (9.4) which are regular throughout the interval $(0,\pi)$ and which vanish at the ends has been solved in detail by Infeld.[27] The "factorization" method used in Infeld's paper immediately yields the $B_n{}^l$ explicitly in terms of elementary functions. It is shown that $n\geqslant 1$, $l\leqslant n-1$, and

$$B_n{}^{n-1}=\sin^n\rho,\tag{9.61}$$

$$B_n{}^0=\sin n\rho.\tag{9.62}$$

The other $B_n{}^l$ are obtained from either of these by use of the following recurrence relations

$$B_n{}^{l-1}=\{l\cot\rho+d/d\rho\}B_n{}^l,\tag{9.63}$$

$$B_n{}^{l+1}=\{(l+1)\cot\rho-d/d\rho\}B_n{}^l.\tag{9.64}$$

It is easily checked that the $B_n{}^l$ are of period π in ρ when n is even.

This completes our survey of electromagnetic fields free from singularities in universes of type I. We add a summary of the values which the parameters n, l, m, may assume:

$$n=2,\ 4,\ 6,\ \cdots,\tag{9.71}$$

$$l=0,\ 1,\ 2,\ \cdots,\ n-1,\tag{9.72}$$

$$m=0,\ \pm1,\ \pm2,\ \cdots,\ \pm l.\tag{9.73}$$

The number of states belonging to a fixed n is

$$2\{1+3+5+\cdots+(2n-1)\}=2n^2.\tag{9.74}$$

The factor 2 appears because Eqs. (9.51) to (9.56) yield two complex solutions of Maxwell's equations for given values of n, l, m, since there is a choice in the terms involving φ.

In all universes of type I the radiation spectrum is discrete. This result depends on the topology but not on the metric of the universe. The frequencies of free radiation are

$$\nu_0,\ 2\nu_0,\ 3\nu_0,\tag{9.8}$$

where

$$\nu_0=1/\pi.\tag{9.81}$$

Physically, this is obvious. Since the τ axis of the universes of type I is closed and of finite length π, the wave-length of a light ray must be a submultiple of π. To put it differently, radiation must complete an integral number of vibrations while circumnavigating space, and must return to the point of emission in its

[27] L. Infeld, "On a new treatment of some eigenvalue problems," Phys. Rev. 59, 737–747 (1941); Section 1.

original phase. The minimum frequency $1/\pi$ corresponds to the wave-length π of light which completes one single vibration on its journey through the universe. This is the lower limit of the infra-red spectrum.

In the case of a "spherical" universe of type I, the condition to be imposed on the electromagnetic field components is that they be of period 2π in τ and ρ. This necessitates only a few changes; we now have

$$n=1, 2, 3, 4, \cdots, \tag{9.91}$$

$$\nu_0=1/2\pi, \tag{9.92}$$

replacing Eqs. (9.71), (9.81), respectively.

10. FIELDS WITH SINGULARITIES

We give here a short investigation of electromagnetic fields with a singularity at the spatial origin. The field tensor F_{ij} is given by Eqs. (9.51) to (9.56), where $B_n{}^l(\rho)$ satisfies the differential equation (9.4), and where n is an even integer in the case of "elliptic" universes of type I, and any integer in the case of "spherical" universes. However, if we wish to investigate singularities, we must drop the condition that $B_n{}^l$ be regular in the interval $(0, \pi)$, or $(0, 2\pi)$, and that it vanish at the ends.

For $l=0$, Eq. (9.4) reduces to

$$d^2B_n{}^0/d\rho^2+n^2B_n{}^0=0.$$

To the solution $B_n{}^0=\sin n\rho$ considered in Section 9, we must now add the second solution

$$\bar{B}_n{}^0=\cos n\rho. \tag{10.1}$$

The factorization method[27] enables us to construct a chain of solutions $\bar{B}_n{}^l$ by use of the recurrence relation

$$\bar{B}_n{}^{l+1}=\{(l+1)\cot\rho-d/d\rho\}\bar{B}_n{}^l. \tag{10.2}$$

Unlike the solutions starting with $\sin n\rho$, the chain of $\bar{B}_n{}^l$'s does not terminate; all integral values of l are admitted.

We now have the solutions

$$B_n{}^0,\ B_n{}^1,\ \cdots,\ B_n{}^{n-1}, \tag{10.3}$$

$$\bar{B}_n{}^0,\ \bar{B}_n{}^1,\ \cdots,\ \bar{B}_n{}^{n-1},\ \bar{B}_n{}^n,\ \cdots. \tag{10.4}$$

In order to complete the investigation we must search for a second set of solutions $B_n{}^{*l}$ when $l\geqslant n$. For this purpose we must first obtain $\bar{B}_n{}^n$ explicitly. When $l=n$, the differential equation (9.4) may be factorized as follows:

$$(n\cot\rho-d/d\rho)(n\cot\rho+d/d\rho)B_n{}^n=0, \tag{10.5}$$

of which one solution satisfies

$$(n\cot\rho+d/d\rho)\bar{B}_n{}^n=0. \tag{10.6}$$

This immediately yields

$$\bar{B}_n{}^n=\sin^{-n}\rho. \tag{10.7}$$

Putting $B_n{}^{*n}=v(\rho)\sin^{-n}\rho$, Eq. (9.4) reduces to

$$d^2v/d\rho^2-2n\cot\rho dv/d\rho=0.$$

Thus

$$v=\int\sin^{2n}\rho d\rho,$$

and, finally,

$$B_n{}^{*n}=\sin^{-n}\rho\int\sin^{2n}\rho d\rho. \tag{10.8}$$

When the integral is evaluated, it is seen that $B_n{}^{*n}$ contains a term of the form $\rho\sin^{-n}\rho$. Thus $B_n{}^{*n}$ violates the condition of periodicity and must be discarded. The same is true of the other solutions $B_n{}^{*l}$, $l>n$, which are obtained from $B_n{}^{*n}$ by repeated application of (10.2).

When $n=0$, the $\bar{B}_0{}^l$ yield the *electrostatic* solutions. Putting $n=l=0$, Eqs. (9.51) to (9.56) reduce to $F_{ij}=0$. However, it is easily verified, directly from Maxwell's equations, that the following two solutions hold:

$$F_{\rho\tau}=e/\sin^2\rho,\quad \text{all other } F_{ij}=0, \tag{10.91}$$

and

$$F_{\theta\varphi}=\mu\sin\theta,\quad \text{all other } F_{ij}=0. \tag{10.92}$$

Both solutions satisfy Maxwell's equations. They are the fields of an electrostatic point charge e and of a magnetic pole μ, respectively, which were mentioned at the end of Section 8. $\bar{B}_0{}^1$, $\bar{B}_0{}^2$, etc., yield the static fields of dipoles and multipoles.

When $n>0$, the $\bar{B}_n{}^l$ yield the electromagnetic fields of oscillating charges, dipoles, and multipoles.

One point must be stressed here for later reference. The charge e of a particle appears, in

(10.91), as a constant of integration. Thus, in particular, Maxwell's equations imply that the charge of an electron is constant and cannot change with time as the universe expands. The electronic charge is a conform invariant.

11. THE EQUATIONS OF MOTION

In the previous sections we proved that Maxwell's equations are conform invariant, i.e., the electromagnetic field can be the same function of coordinates in all universes of the same type, independently of the metric. Electric charges and their motion appear as singularities of the field.

However, an electromagnetic field with singularities is not governed by Maxwell's field equations alone. A new set of relations must be added: *Lorentz's ponderomotive equations of motion.* From the theoretical point of view the ponderomotive equations are unsatisfactory,[28] but for want of a better alternative, we shall now discuss these equations, especially as we are only concerned with the invariance of the formalism.

We have stated that, if a field with singularities exists in a universe, then the same field with the same singularities can exist in any other universe of the same type. This statement was based on the conform invariance of Maxwell's equations. We now see that if it is to be correct, the ponderomotive equations also must be insensitive to the metric of the universe. Only then will all electromagnetic phenomena be truly conform invariant. It is our purpose to show here that this invariance of the equations of motion can be achieved in a simple and natural manner.[29]

The starting point of our considerations is the variational form [30] of the ponderomotive equations.

$$\delta \int (m ds + e\Phi_k dx^k) = 0, \qquad (11.1)$$

where the path of integration is along the real or virtual world line of a particle of charge e and mass m, Φ_k is the potential vector of the external electromagnetic field, and the end points of the path are not varied.

In the previous sections it was shown that e and Φ_k are conform invariant. Thus, if the variational principle (11.1) is to be independent of the metric of the universe, *mds must be conform invariant.*

Under the conformal transformation

$$ds \rightarrow \lambda ds, \qquad (11.21)$$

the behavior of m is given by

$$m \rightarrow \lambda^{-1} m. \qquad (11.2)$$

Also

$$(-g) \rightarrow \lambda^8 (-g). \qquad (11.22)$$

Therefore, we can construct a *conform invariant scalar density*

$$m_0 = (-g)^{1/8} m. \qquad (11.3)$$

Both m and m_0 are in general functions of position in space-time: m is invariant under coordinate transformations, but not under conformal transformations; m_0 is invariant under conformal transformations, but not under coordinate transformations.

This disposes of our problem. The following is merely of the nature of an appendix. The Euler-Lagrange equations of the variational principle (11.1) are easily obtained in tensor form:

$$m\left(\frac{d^2x^k}{ds^2} + \left\{ {k \atop i\ j} \right\} \frac{dx^i}{ds}\frac{dx^j}{ds}\right) + \frac{\partial m}{\partial x^j}\left(\frac{dx^j}{ds}\frac{dx^k}{ds} - g^{jk}\right) = e\frac{dx^j}{ds}F_j{}^k. \qquad (11.4)$$

For the cosmological line element

$$ds^2 = \gamma \eta_{ij} dx^i dx^j = \gamma ds_0^2, \qquad (11.51)$$

the ponderomotive equations become

$$m_0 \frac{d^2x^k}{ds_0^2} + \frac{\partial m_0}{\partial x^j}\left(\frac{dx^j}{ds_0}\frac{dx^k}{ds_0} - \eta^{jk}\right) = e\frac{dx^j}{ds_0}F_{ji}\eta^{ik}. \qquad (11.5)$$

As expected, the function γ does not enter these equations.

Universes of types II and III include as simplest prototypes the flat Minkowski space.

[28] Attempts to obtain equations of motion from field concepts have been made by P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938), M. H. L. Pryce, Proc. Roy. Soc. **A168**, 302 (1938), L. Infeld and P. R. Wallace, Phys. Rev. **57**, 797 (1940).

[29] For similar considerations, compare J. A. Schouten and J. Haantjes, "Über die konforminvariante Gestalt der relativistischen Bewegungsgleichungen," Proc. Akad. Wetensch. Amsterdam **39**, 1059–1065 (1936), Eq. (17).

[30] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice Hall, Inc., New York, 1942), p. 117. Bergmann's electromagnetic potential vector Φ_k differs from ours in sign.

It is natural to assume that m *is constant in a Minkowski universe*, where also $m=m_0$ in the c.c.s. This assumption implies that $m_0=$constant in the c.c.s. for all universes of types II and III. The ponderomotive equations now simplify to

$$m_0 d^2x^k/ds_0{}^2 = e(dx^j/ds_0)F_{ji}\eta^{ik}. \quad (11.6)$$

The prototype of a cosmology of type I is the Einstein universe E, which assumes its simplest form in the r.c.s. In this coordinate system the general line element is

$$ds^2 = R^2\{d\tau^2 - d\rho^2 - \sin^2\rho(d\theta^2+\sin^2\theta d\varphi^2)\} = (R/R_E)^2 ds_E{}^2. \quad (11.71)$$

We introduce

$$m_E = (R/R_E)m, \quad (11.7)$$

which is conform invariant because, under the conformal transformation $ds\rightarrow\lambda ds$, we have $R\rightarrow\lambda R$ and $m\rightarrow(1/\lambda)m$. We now add the assumption that m *is a constant in an Einstein universe*, where also $m=m_E$ in the r.c.s. It follows that $m_E=$constant in the r.c.s. for all universes of type I.

Our latest assumption can be expressed in the dimensionless form given in the Introduction:

$$R/r_e = \text{constant}, \quad (11.72)$$

where r_e is the "radius of the charge e," defined by Eq. (0.4). Later we shall deduce from Dirac's equations that this constant, dimensionless ratio is an integer.

Our derivation of conform invariant equations of motion was based, primarily, on the behavior of the mass m under conformal transformations, as given by Eq. (11.2). Further subsidiary assumptions, stating that m is constant in the universes E, M_2, M_3, were added; these were adopted on grounds of simplicity.

Even without making the "subsidiary assumptions," a good deal of information on the functional dependence of the mass m on position in space-time can be obtained from general principles. The mass m as a function of coordinates must conform to the principles of isotropy, homogeneity,[31] and equivalence.

The spatial isotropy of the universe demands that m be a function of t and r only in the c.c.s., or of τ and ρ only in the r.c.s. The principle of homogeneity states, mathematically, that physical quantities (such as mass) must be invariant in form under the three-parameter group of coordinate transformations which move the world-line of one fundamental particle into that of another. This implies that, in the c.c.s., m must be of the form

$$m = m_c p\{t/(1-t^2+r^2)\}, \quad (11.81)$$

$$m = m_c p(t^2-r^2), \quad (11.82)$$

$$m = m_c p(t), \quad (11.83)$$

in a universe of type I, II, or III, respectively, where m_c is a constant. In the r.c.s., m must be of the form

$$m = m_c \Pi(\tau). \quad (11.84)$$

The function $\Pi(\tau)$ can be arbitrarily assigned in any one universe. However, it is then determined in all other universes of the same type by the conformal property (11.2) of m.

The "principle of equivalence" asserts that the motion of a particle in a purely gravitational field ($eF_{ij}=0$) is independent of the mass of the particle. Alternatively, it states that under the same initial conditions two particles of different masses have identical motions. In the absence of an electromagnetic field, the equations of motion (11.4) reduce to

$$\Pi\left(\frac{d^2x^k}{ds^2}+\left\{\begin{matrix}k\\ i\ j\end{matrix}\right\}\frac{dx^i}{ds}\frac{dx^j}{ds}\right) + \frac{d\Pi}{d\tau}\left(\frac{d\tau}{ds}\frac{dx^k}{ds}-g^{0k}\right)=0 \quad (11.9)$$

in the r.c.s., by virtue of (11.84). The principle of equivalence demands that, in any one universe, the function $\Pi(\tau)$ be the same for all mass particles.

Our "subsidiary assumptions" simply introduce the special choice $\Pi=p=1$ for universes E, M_2, M_3.

One important question remains to be examined. In previous sections, the fundamental particles, which are cosmologically important special cases of free particles, were assumed to move along geodesics. We must now show that the fundamental particles satisfy our generalized equations of motion (11.9); otherwise, our theory would be inconsistent. The simplest way to

[31] A, Section 1.

answer this question is to choose a r.c.s. such that any particular fundamental world-line under examination coincides with the τ axis. Then

$$dx^\alpha/ds=0, \quad g^{0\alpha}=0, \quad (\alpha=1,2,3),$$

and

$$d\tau/ds=1/R(\tau), \quad g^{00}=1/R^2(\tau).$$

Thus the last bracket expression in (11.9) vanishes; for $k=1, 2, 3$, because both terms are zero, and for $k=0$ because it reduces to

$$(d\tau/ds)^2-g^{00}=1/R^2-1/R^2=0.$$

Therefore, the equations of motion (11.9) become

$$\frac{d^2x^k}{ds^2}+\left\{\begin{matrix} k \\ i\ j \end{matrix}\right\}\frac{dx^i}{ds}\frac{dx^j}{ds}=0,$$

which are the equations of a geodesic and are satisfied by the τ axis, by virtue of the spatial isotropy of the universe. It follows from our considerations that the only free particles which move along geodesics are the fundamental particles.

12. PLANCK'S RADIATION LAW IN AN EINSTEIN UNIVERSE

In this section, we derive Planck's radiation law in an Einstein universe, where we choose the unit of length such that $R_E=1$. The general case of universes of type I is of no interest here, as we can hardly consider a state of radiation equilibrium in an expanding universe.

Usually, Planck's law is derived by considering radiation enclosed in a "Jeans' box."[32] Here, any reference to Jeans' box is superfluous because of the finite volume of space.

We are concerned with radiation whose frequency $\nu=n/2\pi$ is a large number, i.e., radiation whose wave-length is small compared to the radius of the universe. For each n there are $2n^2$ complex solutions of Maxwell's equations (Eq. (9.74)). The number of frequencies within an interval Δn is $\frac{1}{2}\Delta n$, since n is limited to even integral values (Eq. (9.71)). Thus, the number N of photons within a narrow frequency interval $\Delta\nu=\Delta n/2\pi$ is given by the Einstein-Bose distribution law:

$$N=\frac{2n^2\cdot\frac{1}{2}\Delta n}{e^{n\beta/2\pi}-1}=\frac{8\pi^3\nu^2\Delta\nu}{e^{\beta\nu}-1}, \tag{12.1}$$

where β is a constant.

The next step is the introduction of the energy of a photon, and then the volume of space, i.e., the metric. The energy ϵ of a photon is given by Einstein's relation

$$\epsilon=h\nu, \tag{12.2}$$

where h is Planck's constant. The total energy of radiation E is:

$$E=Vu=\frac{8\pi h\nu^3\Delta\nu}{e^{\beta\nu}-1}\pi^2, \tag{12.3}$$

where V is the volume π^2 of elliptic space and u the energy density. Dividing by $V=\pi^2$, we finally obtain Planck's radiation law

$$u=\frac{8\pi h\nu^3\Delta\nu}{e^{\beta\nu}-1}. \tag{12.4}$$

In the spherical Einstein universe, the number of frequency levels within an interval Δn is Δn, since n can now assume all integral values. Therefore, N is 2 times the expression given by (12.1). On the other hand, the volume of spherical space is $V=2\pi^2$. Thus, we again arrive at (12.4).

IV—Dirac's Equations

13. INTRODUCTION

The formal work of many authors on the generalization of Dirac's equations to the space-time manifolds of general relativity allows us to dispense here with much tedious calculation. We shall use results obtained by Taub[33] and Schrödinger.[34] These will be combined with results by Schouten and Haantjes.[35]

Schouten and Haantjes consider Dirac equations invariant under conformal transformations.

[32] R. C. Tolman, *The Principles of Statistical Mechanics* (Clarendon Press, Oxford, 1938), Section 93a.

[33] A. H. Taub, "Quantum equations in cosmological spaces," Phys. Rev. **51**, 512–525 (1937).

[34] E. Schrödinger, "Eigenschwingungen des Sphärischen Raumes," Commentationes Pontificiae Academiae Scientiarum **2**, 321–364 (1938). See also reference 25.

[35] Reference 29. The same problem was investigated along different lines by O. Veblen, Proc. Nat. Acad. Sci. **21**, 484–487 (1935), and by P. A. M. Dirac, Ann. Math. **37**, 429–442 (1936).

Since all cosmological spaces are conformal, their results become pertinent to the problem before us. On the other hand, Taub and Schrödinger solve the problem of a free electron in a Robertson coordinate system. They do not formulate the question whether Dirac's equations should depend on the particular form of the function $R(\tau)$ or should, like Maxwell's equations, be insensitive to any choice of R. They could hardly have avoided this question had their coordinate system been, not Robertson's, but the r.c.s. We find only an implicit reference to this problem in Schrödinger's[25] discussion of "alarming phenomena."

For our further argument it is essential to distinguish between frequencies ${}_\tau\nu$ and ${}_s\nu$.[36]

The symbol ${}_\tau\nu$ stands for the frequency of radiation in τ time. The eigenvalues of this frequency were obtained before:

$$ {}_\tau\nu = n/2\pi, \quad n=1, 2, 3, \cdots. \tag{13.1}$$

If only to simplify the language of our argument, we shall ignore the discrete character of this spectrum. For reasonable values of the frequency, i.e., when $n \gg 1$, the spectrum is sufficiently dense for ${}_\tau\nu$ to be treated as a continuous variable.

A source of radiation can emit a photon of any frequency ${}_\tau\nu$. But the radiation, once emitted, keeps its rhythm, measured in τ time, throughout its journey. Actually, this applies in the r.c.s. to radial light signals only; but it is these alone which are of interest to us here. Thus the symbol ${}_\tau\nu$ indicates the *permanence* of radiation in τ time, i.e., the constancy of its frequency measured by the τ clock described in **A**, Section 7.

So far, we have not discussed the sources of radiation. From the point of view of general relativity, these are clock devices indicating permanent frequency ${}_s\nu$ in the proper time s which they measure. An atom emitting radiation, whether represented classically as an oscillator with frequency ${}_s\nu$, or quantum mechanically as a system characterized by differences in energy levels, is such a clock and, in a given universe, ${}_s\nu$ is a set of numbers, with dimensions $[L^{-1}]$. According to relativity theory we have, in the r.c.s.,

$$ {}_s\nu = {}_\tau\nu/R, \tag{13.2}$$

where R has the dimensions $[L]$. This equation expresses the law of red shift.

We shall see later that the atomic and cosmological quantities enter Dirac's equations through a dimensionless combination only. More specifically, let us consider a universe of type I in the r.c.s. Then, for a free electron, or an electron in a central Coulomb field, the only combinations of physical quantities appearing in Dirac's equations are

$$ R/r_c = Rm/\hbar \tag{13.3}$$

and the fine structure constant

$$ r_e/r_c = e^2/\hbar. \tag{13.4}$$

Both are dimensionless! We therefore cannot expect Dirac's equations to yield the characteristic frequencies

$$ {}_s\nu_{(1)}, \ {}_s\nu_{(2)}, \ {}_s\nu_{(3)}, \ \cdots, \ {}_s\nu_{(k)}, \ \cdots. \tag{13.5}$$

What we do expect and do obtain from Dirac's equations is a set of *dimensionless and constant* numbers

$$ \nu_{(1)}, \ \nu_{(2)}, \ \nu_{(3)}, \ \cdots, \ \nu_{(k)}, \ \cdots. \tag{13.6}$$

These must be identified in some simple manner with the observed frequency levels. The simplest identification is

$$ {}_s\nu_{(k)} = A\nu_{(k)}, \tag{13.7}$$

where A has the dimensions $[L^{-1}]$. Since the $\nu_{(k)}$ are found to be constant, and since relativity theory demands that the ${}_s\nu_{(k)}$ be constant, therefore, A must be a constant.

14. THE CONFORM INVARIANT DIRAC EQUATIONS

Schouten and Haantjes[29] have shown how Dirac's wave equations must be generalized to Riemannian space-time, so that they may be conform invariant. In the case of cosmological spaces, where the required mathematical tools are simpler, their conclusions may be reached more directly.

We use the c.c.s. with cartesian spatial coordinates. The cosmological line elements are of the

[36] The subscripts τ and s are written on the left of ν, in order to avoid confusion with tensor indices.

form

$$ds^2=\gamma(t,r)(dt^2-dx^2-dy^2-dz^2)$$
$$=\gamma(t,r)\eta_{ij}dx^i dx^j. \tag{14.1}$$

We introduce the usual two-dimensional spin space,[37] characterized by an arbitrary skew symmetric spin metric $\gamma_{\rho\mu}$, and by a mixed Hermitian quantity $\sigma^{k\dot\lambda\mu}$ which provides the transition from world vectors to spin tensors. In this section (and also in Appendix B), Latin indices refer to space-time coordinates and range over 0, 1, 2, 3, while Greek indices (dotted and undotted) refer to spin coordinates and range over 1, 2.

The world metric imposes a restriction on the $\sigma^{k\lambda\mu}$:

$$g^{kl}=\sigma^{k\dot\lambda\mu}\sigma^{l}{}_{\dot\lambda\mu}=\sigma^{k\dot\lambda\mu}\sigma^{l\dot\rho\sigma}\gamma_{\dot\lambda\dot\rho}\gamma_{\mu\sigma}. \tag{14.2}$$

If we restrict ourselves to linear spin transformations with determinant $+1$, we may connect a particular spin metric with the Riemannian metric of our c.c.s. in the following way:

$$\gamma_{\lambda\mu}=(-g)^{3/8}\epsilon_{\lambda\mu}=\gamma^{3/2}\epsilon_{\lambda\mu},$$
$$\gamma^{\lambda\mu}=(-g)^{-3/8}\epsilon_{\lambda\mu}=\gamma^{-3/2}\epsilon_{\lambda\mu}, \tag{14.21}$$

where

$$\epsilon_{12}=-\epsilon_{21}=1,\quad \epsilon_{11}=\epsilon_{22}=0. \tag{14.22}$$

Let

$${}^0\sigma^{k\dot\lambda\mu}=(-g)^{\frac{1}{4}}\sigma^{k\dot\lambda\mu}=\gamma^2\sigma^{k\dot\lambda\mu}. \tag{14.23}$$

Then, since by (14.1) $g^{kl}=(1/\gamma)\eta^{kl}$, Eq. (14.2) reduces to

$$\eta^{kl}={}^0\sigma^{k\dot\lambda\mu}\,{}^0\sigma^{l\dot\rho\sigma}\epsilon_{\dot\lambda\dot\rho}\epsilon_{\mu\sigma}. \tag{14.3}$$

Thus the ${}^0\sigma^{k\dot\lambda\mu}$ are constants independent of γ and, therefore, the same for all cosmological spaces. They are in fact the Pauli matrices multiplied by $1/\sqrt{2}$.

We begin our discussion of Dirac's equations by considering the law of conservation which the probability current and density 4-vector I^k satisfies

$$\partial\{(-g)^{\frac{1}{2}}I^k\}/\partial x^k=0. \tag{14.4}$$

The world vector I^k induces a Hermitian spin tensor $\chi_{\dot\lambda\mu}$, given by

$$(-g)^{\frac{1}{2}}I^k=(-g)^{\frac{1}{4}}\sigma^{k\dot\lambda\mu}\chi_{\dot\lambda\mu}={}^0\sigma^{k\dot\lambda\mu}\chi_{\dot\lambda\mu}. \tag{14.41}$$

In order to obtain Dirac's equations, we put

$$\chi_{\dot\lambda\mu}=\Psi_{\dot\lambda}\Psi_\mu+X_{\dot\lambda}X_\mu. \tag{14.42}$$

The conservation equation (14.4) now becomes

$${}^0\sigma^{k\dot\lambda\mu}\Psi_{\dot\lambda,k}\Psi_\mu+{}^0\sigma^{k\dot\lambda\mu}\Psi_{\dot\lambda}\Psi_{\mu,k}$$
$$+{}^0\sigma^{k\dot\lambda\mu}X_{\dot\lambda,k}X_\mu+{}^0\sigma^{k\dot\lambda\mu}X_{\dot\lambda}X_{\mu,k}=0, \tag{14.5}$$

where the comma denotes partial differentiation. This equation is satisfied if

$$2^{\frac{1}{2}}\,{}^0\sigma^{k\dot\lambda\mu}(\Psi_{\dot\lambda,k}-ieh^{-1}\Phi_k\Psi_{\dot\lambda})$$
$$=-i\beta(-g)^{\frac{1}{8}}X^\mu=-i\beta_0\epsilon^{\mu\rho}X_\rho,$$
$$2^{\frac{1}{2}}\,{}^0\sigma^{k\dot\lambda\mu}(X_{\dot\lambda,k}-ieh^{-1}\Phi_kX_{\dot\lambda})$$
$$=-i\beta(-g)^{\frac{1}{8}}\Psi^\mu=-i\beta_0\epsilon^{\mu\rho}\Psi_\rho. \tag{14.6}$$

These are Dirac's equations for an electron. In it, $-e$ is the electronic charge, Φ_k the potential 4-vector of the external electromagnetic field, and β is an invariant, given by

$$\beta=m/h. \tag{14.71}$$

By (14.21) and (14.6), we have

$$\beta_0=(-g)^{1/8}\beta=(-g)^{1/8}m/h=m_0/h. \tag{14.7}$$

It is important to note that β can be an arbitrary function of the space-time coordinates without violating the conservation equations.

Equations (14.6) are invariant under any linear spin transformation with constant components and determinant $+1$. Unless we restrict spin transformations in this way, the partial derivatives in Eqs. (14.6) must be replaced by covariant derivatives, which would add unnecessary complication.

In Section 11, we postulated that, under the conformal transformation $ds\rightarrow\lambda ds$,

$$m\rightarrow(1/\lambda)m. \tag{14.8}$$

This was necessary to ensure the conform invariance of Lorentz's ponderomotive equations of motion. This same assumption now ensures the *conform invariance of Dirac's equations:* By (11.3), (14.7), and (14.8), β_0 is a conform invariant density, i.e., β_0 is independent of the function

[37] L. Infeld and B. L. v. d. Waerden, "Die Wellengleichung des Elektrons in der allgemeinen Relativitätstheorie," Sitz. Ber. preuss. Akad. Wiss. 380–402 (1933). See also O. Veblen and J. Von Neumann, "Geometry of complex domains," Princeton University Mimeographed Notes (1935–36).

γ but may depend on the type of the universe; thus γ does not enter the Eqs. (14.6).

It is easily seen that the probability current 4-vector density (14.41) is conform invariant. This is in accord with the usual identification of the probability current 4-vector with the macroscopic electric current 4-vector whose density was shown in Section 8 to be conform invariant.

We now add the subsidiary assumptions adopted in Section 11, that m is constant in the simple universes E, M_2, M_3. This immediately implies in the case of universes of type II and III that Dirac's equation and its solutions are exactly identical with those in flat Minkowski space.

In the case of universes of type I, the new boundary condition must be satisfied, although not necessarily by the wave functions themselves. However, as in the case of Maxwell's equations, Dirac's equations in cosmological spaces are best solved in the r.c.s. In this coordinate system, Eq. (14.8) and the assumption of the constancy of m in the Einstein universe, can be combined in the dimensionless form

$$mR/\hbar = R/r_c = \text{constant}, \tag{14.9}$$

where r_c is given by Eq. (0.5).

15. THE FREE ELECTRON

We consider here universes of type I in the r.c.s. Schrödinger has discussed Dirac's equations in these cosmological spaces in great detail,[38] using Robertson's coordinate system. Since the transition to the r.c.s. is very simple, being a pure time transformation, we can take over Schrödinger's results with a few necessary changes. In the appendix, we give a short outline of the derivation of the wave equations by use of two-dimensional spinors.

For the line element

$$ds^2 = R^2(\tau)[d\tau^2 - d\rho^2 - \sin^2\rho(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{15.1}$$

Dirac's equations for a free electron are

$$\sin\rho[i\omega_{1,\tau} - i\omega_{3,\rho} + \beta_E\omega_1] = -\omega_{4,\theta} + (i/\sin\theta)\omega_{4,\varphi}, \tag{15.11}$$

$$\sin\rho[i\omega_{2,\tau} - i\omega_{4,\rho} + \beta_E\omega_2] = -\omega_{3,\theta} - (i/\sin\theta)\omega_{3,\varphi}, \tag{15.12}$$

$$\sin\rho[i\omega_{3,\tau} - i\omega_{1,\rho} - \beta_E\omega_3] = \omega_{2,\theta} - (i/\sin\theta)\omega_{2,\varphi}, \tag{15.13}$$

$$\sin\rho[i\omega_{4,\tau} - i\omega_{2,\rho} - \beta_E\omega_4] = \omega_{1,\theta} + (i/\sin\theta)\omega_{1,\varphi}, \tag{15.14}$$

where the comma denotes partial differentiation, and

$$\beta_E = R\beta = mR/\hbar. \tag{15.15}$$

This is the same quantity as in (14.9) and thus, by our assumptions, β_E is *the same constant* for all universes of type I in the r.c.s.

The components of the probability current 4-vector density are explicitly given by

$$(-g)^{\frac{1}{2}}I^\tau = (\bar{\omega}_1\omega_1 + \bar{\omega}_2\omega_2 + \bar{\omega}_3\omega_3 + \bar{\omega}_4\omega_4), \tag{15.16}$$

$$(-g)^{\frac{1}{2}}I^\rho = (-\bar{\omega}_1\omega_3 - \bar{\omega}_2\omega_4 - \bar{\omega}_3\omega_1 - \bar{\omega}_4\omega_2), \tag{15.17}$$

$$(-g)^{\frac{1}{2}}I^\theta = (i/\sin\rho)(-\bar{\omega}_1\omega_4 - \bar{\omega}_2\omega_3 + \bar{\omega}_3\omega_2 + \bar{\omega}_4\omega_1), \tag{15.18}$$

$$(-g)^{\frac{1}{2}}I^\varphi = (1/\sin\rho\sin\theta)(-\bar{\omega}_1\omega_4 + \bar{\omega}_2\omega_3 + \bar{\omega}_3\omega_2 - \bar{\omega}_4\omega_1), \tag{15.19}$$

where the horizontal bar denotes the complex conjugate.

Our purpose is to solve Eqs. (15.11) to (15.14) for the four components ω_1, ω_2, ω_3, ω_4 of the wave function under the usual conditions and under the "new boundary conditions" (see Section 9). Schrödinger has shown [39] that the wave function must be either single valued in all four components and for all states, or else always double valued, in which case the two branches of the wave function must differ by a minus sign only. Thus, in the case of an "elliptic" universe, our new boundary conditions require that the wave components be periodic in τ and ρ with period π, or else always change their sign when π is added to τ or to ρ. In either case the components of $(-g)^{\frac{1}{2}}I^i$, given by Eqs. (15.16) to (15.19), are single valued. In a "spherical" universe the same statement applies with π replaced by 2π.

We now meet a rather surprising phenomenon: The solution of Dirac's equations is radically different in elliptic and in spherical universes. This is the first time that we encounter conclusions which are essentially and observationally distinct for the two possible topologies of universes of type I. This difference seems not to have been noticed previously. The solution of

[38] Reference 34, Eqs. (5.15), (6.4); and reference 25, Eq. (3.6).

[39] E. Schrödinger, "Die Mehrdeutigkeit der Wellenfunktion," Ann. d. Physik [5] 32, 49–55 (1938).

Dirac's equations given by Schrödinger is correct for spherical universes only. In elliptic universes the wave function reduces to a particularly simple form.

Schrödinger[40] obtained the solution of Dirac's equations (15.11) to (15.14) in the separated form

$$\omega_1 = Ane^{-i(\nu\tau - m\varphi)} f_k^m(\theta) f_n^k(\rho), \quad (15.21)$$

$$\omega_2 = Ane^{-i(\nu\tau - m\varphi)} g_k^m(\theta) f_n^k(\rho), \quad (15.22)$$

$$\omega_3 = A(\beta_E + \nu) e^{-i(\nu\tau - m\varphi)} f_k^m(\theta) g_n^k(\rho), \quad (15.23)$$

$$\omega_4 = A(\beta_E + \nu) e^{-i(\nu\tau - m\varphi)} g_k^m(\theta) g_n^k(\rho), \quad (15.24)$$

where the first two of the separation constants ν, n, k, m are subject to the relation

$$\nu^2 = n^2 + \beta_E^2. \quad (15.25)$$

The functions $f_n^k(\rho)$, $g_n^k(\rho)$ are defined by the simultaneous differential equations

$$f_n^{k\prime} - (k/\sin\rho) f_n^k = -ing_n^k, \quad (15.31)$$

$$g_n^{k\prime} + (k/\sin\rho) g_n^k = -inf_n^k, \quad (15.32)$$

$$(f_n^{k\prime} = df_n^k/d\rho, \text{ etc.}).$$

As implied by the notation, $f_k^m(\theta)$ and $g_k^m(\theta)$ satisfy an analogous set of equations obtained from (15.31), (15.32) by writing k, m, θ, for n, k, ρ, respectively.

Eliminating g_n^k from Eqs. (15.31), (15.32), we obtain the second-order equation

$$f_n^{k\prime\prime} - [(k^2 - k\cos\rho)/\sin^2\rho - n^2] f_n^k = 0, \quad (15.33)$$

which can immediately be "factorized"[27] thus:

$$\begin{aligned}[(k+\tfrac{1}{2})\cot\rho - 1/2\sin\rho + d/d\rho] \\ \times[(k+\tfrac{1}{2})\cot\rho - 1/2\sin\rho - d/d\rho] f_n^k \\ = [n^2 - (k+\tfrac{1}{2})^2] f_n^k,\end{aligned} \quad (15.34)$$

$$\begin{aligned}[(k-\tfrac{1}{2})\cot\rho - 1/2\sin\rho - d/d\rho] \\ \times[(k-\tfrac{1}{2})\cot\rho - 1/2\sin\rho + d/d\rho] f_n^k \\ = [n^2 - (k-\tfrac{1}{2})^2] f_n^k.\end{aligned} \quad (15.35)$$

We may assume $k \geqslant 0$; if $k < 0$, we have

$$(f_n^k, g_n^k) = (g_n^{-k}, f_n^{-k}),$$

by Eqs. (15.31), (15.32). We also assume $n \geqslant 0$; if $n < 0$, we have

$$(f_n^k, g_n^k) = (f_{-n}^k, -g_{-n}^k).$$

[40] Reference 34, Eqs. (8.18) and (7.17).

If $k = n - \frac{1}{2}$, we obtain the first-order equation

$$[n\cot\rho - 1/2\sin\rho - d/d\rho] f_n^{n-\frac{1}{2}} = 0, \quad (15.36)$$

which yields the solution

$$\begin{aligned} f_n^{n-\frac{1}{2}} &= A\sin^n\rho\cot^{\frac{1}{2}}\tfrac{1}{2}\rho \\ &= 2^{\frac{1}{2}} A\sin^{n-\frac{1}{2}}\rho\cos\tfrac{1}{2}\rho. \end{aligned} \quad (15.37)$$

Other functions f_n^k are given by the recurrence relation

$$\begin{aligned} f_n^{k-1} = [(k-\tfrac{1}{2})\cot\rho \\ - 1/2\sin\rho + d/d\rho] f_n^k, \end{aligned} \quad (15.38)$$

and the g_n^k are obtained from the f_n^k by Eq. (15.31).

The function $f_n^{n-\frac{1}{2}}$ given by (15.37), and the f_n^k, g_n^k derived from it, lead to a single valued current vector density if either n is always an integer, or else, $n = s + \frac{1}{2}$ is always half an odd integer. Then the $(-g)^{\frac{1}{2}} I^i$ are of period 2π in ρ, so that the solution obtained above is valid in a *spherical* universe only. Schrödinger[41] adopts the case where n is half-integral. It then follows from similar consideration of the functions $f_k^m(\theta)$, $g_k^m(\theta)$, that the constants n, k, m are limited to the values

$$n = \pm 3/2, \ \pm 5/2, \ \pm 7/2, \ \cdots = s + \tfrac{1}{2}, \quad (15.41)$$

$$k = \pm 1, \ \pm 2, \ \pm 3, \ \cdots; \ |k| < |n|, \quad (15.42)$$

$$m = \pm 1/2, \ \pm 3/2, \ \pm 5/2, \ \cdots; \quad |m| < |k|. \quad (15.43)$$

Our boundary condition for spherical universes, that the wave function be of period 2π in τ, imposes the new restriction that ν be integral. Thus, not every integral s is admissible; s and ν must be integral solutions of (Eq. 15.25):

$$\nu^2 = (s + \tfrac{1}{2})^2 + \beta_E^2. \quad (15.5)$$

This is a remarkable requirement. It implies that the actual eigenvalues ν and s depend on the number theoretical properties of the dimensionless physical constant β_E which is roughly of the order of 10^{40}.

If we assume the existence of the lowest possible frequency ν_0, when $s = 0$, then β_E must be of the form

$$\beta_E^2 = \nu_0^2 - 1/4, \quad (15.51)$$

[41] Reference 34, Eq. (8.19), where n'', j, m correspond to our n, k, m.

ν_0 being an integer. Eq. (15.5) reduces to

$$\nu^2 = s(s+1) + \nu_0^2. \tag{15.52}$$

A simple solution of this diophantine equation is given by

$$\nu = a(4b^2+1) - b, \tag{15.53}$$

$$\nu_0 = a(4b^2-1) - b, \tag{15.54}$$

$$s = 4ab - 1, \tag{15.55}$$

where a and b are positive or negative integers.

We now turn to the examination of *elliptic* universes of type I. The $f_n{}^k$, obtained from (15.37) and from the recurrence relation (15.38), are now unsuitable as they do not have a period π. However, a satisfactory solution is obtained by putting $k=0$. It immediately follows that $m=0$.[42] Eqs. (15.31) and (15.32) simplify to

$$f_n' + ing_n = 0, \tag{15.61}$$

$$g_n' + inf_n = 0. \tag{15.62}$$

Hence

$$f_n = g_n = e^{-in\rho}. \tag{15.63}$$

The wave components now assume the simple form

$$\omega_1 = \omega_2 = Ane^{-i(\nu\tau + n\rho)}, \tag{15.71}$$

$$\omega_3 = \omega_4 = A(\beta_E + \nu)e^{-i(\nu\tau + n\rho)}. \tag{15.72}$$

The boundary conditions are satisfied if ν and n are even integers. It may be noted that the angular components of the probability current 4-vector density, $(-g)^{\frac{1}{2}}I^\theta$ and $(-g)^{\frac{1}{2}}I^\varphi$, are both zero. This is important as both these components involve the factor $1/\sin\rho$ which does not have period π in ρ.

The even integers ν and n are subject to the relation (15.25):

$$\nu^2 = n^2 + \beta_E^2.$$

If the lowest possible frequency ν_0, for $n=0$, is to exist, then β_E *must be integral:*

$$\beta_E = \nu_0. \tag{15.8}$$

We now have the well-known Pythagorean problem

$$\nu^2 = n^2 + \nu_0^2 \tag{15.81}$$

which must be solved in even integers.

The dimensionless integer β_E is the same constant which was denoted by R/r_c in the introduction, (0.6). We may add the assumption, adopted by many physicists and in particular by Eddington, that the reciprocal of the fine structure constant

$$r_c/r_e = 137 \tag{15.9}$$

exactly. It then follows that all three dimensionless constants

$$R/r_e,\ R/r_c,\ r_c/r_e, \tag{15.91}$$

are integers.

APPENDIX B

We give here a short sketch of the derivation[43] of Dirac's equations (15.11) to (15.14) for a free electron in a closed universe, by use of the two-dimensional spinor formalism.[37] Since Dirac's equations are conform invariant, it is sufficient to consider the Einstein universe, with $R_E=1$, and we therefore start with the line element

$$ds^2 = d\tau^2 - d\rho^2 - \sin^2\rho(d\theta^2 + \sin^2\theta d\varphi^2). \tag{B1}$$

We adopt the simple spin metric

$$\gamma_{\rho\mu} = \epsilon_{\rho\mu}. \tag{B2}$$

We then obtain for the $\sigma^{k\dot{\lambda}\mu}$ defined by Eq. (14.2), the expressions

$$\sigma^{0\dot{\lambda}\mu} = 2^{-\frac{1}{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma^{1\dot{\lambda}\mu} = 2^{-\frac{1}{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma^{2\dot{\lambda}\mu} = (2^{\frac{1}{2}}\sin\rho)^{-1}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$$
$$\sigma^{3\dot{\lambda}\mu} = (2^{\frac{1}{2}}\sin\rho\sin\theta)^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{B3}$$

[42] Reference 34, p. 352 (3), where j corresponds to our k.

[43] The authors wish to thank Dr. B. A. Griffith for supplying the calculations sketched in this section.

The affine spin connection is characterized by expressions $\Gamma_{\mu k}{}^{\nu}$. These are defined by

$$\sigma^{k\dot{\lambda}\mu}{}_{,s}+\left\{\begin{matrix} k \\ r\ s \end{matrix}\right\}\sigma^{r\dot{\lambda}\mu}+\Gamma_{\dot{\rho}s}{}^{\dot{\lambda}}\sigma^{k\dot{\rho}\mu}+\Gamma_{\sigma s}{}^{\mu}\sigma^{k\dot{\lambda}\sigma}=0, \tag{B41}$$

where $\left\{\begin{matrix} k \\ r\ s \end{matrix}\right\}$ are the Christoffel symbols of the metric (B1). For a free electron, i.e., in the absence of an electromagnetic field, we also have

$$\Gamma_{\alpha s}{}^{\alpha}=\Gamma_{\dot{\alpha}s}{}^{\dot{\alpha}}. \tag{B42}$$

The relations (B41), (B42) suffice to determine the $\Gamma_{\mu k}{}^{\nu}$ uniquely. The only non-vanishing components are found to be

$$\begin{aligned}\Gamma_{12}{}^{1}&=-\Gamma_{22}{}^{2}=\tfrac{1}{2}i\cos\rho,\\ \Gamma_{23}{}^{1}&=-\Gamma_{\dot{1}3}{}^{\dot{2}}=\tfrac{1}{2}\cos\rho\sin\theta+\tfrac{1}{2}i\cos\theta.\end{aligned} \tag{B43}$$

Dirac's equations are

$$2^{\frac{1}{2}}\sigma^{k}{}_{\dot{\lambda}\mu}(\Psi^{\dot{\lambda}}{}_{,k}+\Gamma_{\dot{\rho}k}{}^{\dot{\lambda}}\Psi^{\dot{\rho}})=i\beta_E X_{\mu}, \tag{B51}$$

$$2^{\frac{1}{2}}\sigma^{k\dot{\mu}\lambda}(X_{\lambda,k}-\Gamma_{\lambda k}{}^{\rho}X_{\rho})=i\beta_E\Psi^{\dot{\mu}}, \tag{B52}$$

where β_E is the constant given by (15.15). Explicitly, these equations assume the form

$$\sin\rho[i\Psi^{\dot{1}}{}_{,\tau}-i\Psi^{\dot{2}}{}_{,\rho}-i\cot\rho\Psi^{\dot{2}}+\beta_E X_1]=\Psi^{\dot{2}}{}_{,\theta}+\tfrac{1}{2}\cot\theta\Psi^{\dot{2}}+(i/\sin\theta)\Psi^{\dot{1}}{}_{,\varphi}, \tag{B61}$$

$$\sin\rho[i\Psi^{\dot{2}}{}_{,\tau}-i\Psi^{\dot{1}}{}_{,\rho}-i\cot\rho\Psi^{\dot{1}}+\beta_E X_2]=-\Psi^{\dot{1}}{}_{,\theta}-\tfrac{1}{2}\cot\theta\Psi^{\dot{1}}-(i/\sin\theta)\Psi^{\dot{2}}{}_{,\varphi}, \tag{B62}$$

$$\sin\rho[iX_{1,\tau}+iX_{2,\rho}+i\cot\rho\, X_2+\beta_E\Psi^{\dot{1}}]=-X_{2,\theta}-\tfrac{1}{2}\cot\theta X_2-(i/\sin\theta)X_{1,\varphi}, \tag{B63}$$

$$\sin\rho[iX_{2,\tau}+iX_{1,\rho}+i\cot\rho X_1+\beta_E\Psi^{\dot{2}}]=X_{1,\theta}+\tfrac{1}{2}\cot\theta X_1+(i/\sin\theta)X_{2,\varphi}. \tag{B64}$$

These equations immediately reduce to Schrödinger's form, i.e., Eqs. (15.11) to (15.14), under the 4-dimensional transformation

$$\omega_1=\sin\rho\sin^{\frac{1}{2}}\theta[\Psi^{\dot{1}}+\Psi^{\dot{2}}+X_1+X_2], \tag{B71}$$

$$\omega_2=\sin\rho\sin^{\frac{1}{2}}\theta[-\Psi^{\dot{1}}+\Psi^{\dot{2}}-X_1+X_2], \tag{B72}$$

$$\omega_3=\sin\rho\sin^{\frac{1}{2}}\theta[\Psi^{\dot{1}}+\Psi^{\dot{2}}-X_1-X_2], \tag{B73}$$

$$\omega_2=\sin\rho\sin^{\frac{1}{2}}\theta[\Psi^{\dot{1}}-\Psi^{\dot{2}}-X_1+X_2]. \tag{B74}$$

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On Higher Order Transitions

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In the June 1–15, 1946 number of *The Physical Review*, E. F. Lype discusses thermodynamic equilibria of higher order from the point of view of an expansion of the thermodynamic potentials in Taylor's series, and obtains results which, when applied to transitions of the second order, differ from the well-known results of Ehrenfest by a factor of two. Application of the results to the experiments of Keesom on helium and of Clusius and Perlick on methane is held to validate the method of thermodynamic potential and Taylor's series.

The difference between the results of Ehrenfest and of Lype is not to be attributed to any failure of mathematical rigor on the part of Ehrenfest, but to two different conceptions of the nature of the physical phenomena, which are essentially incompatible with each other. To bring out the difference it will be sufficient to restrict ourselves for the present to transitions