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The Magnetic Moments of H^3 and He^3*

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A measurement of the magnetic moments of the nuclei H^3 and He^3 would yield information concerning the deviations from L-S coupling in these nuclei. It is shown that the sum of the moments of the two nuclei can be directly related to the amount of admixture of the ^2P, ^4P, and ^4D eigenfunctions with the ^2S function. Thus the measurement of both moments would lead to direct information concerning the contributions of these functions to the ground state of the two nuclei. The individual moments depend to some extent on the detailed properties of the wave functions, but if only the ^2S and ^4D functions contribute appreciably to the ground state, and if particularly simple forms of these functions are assumed, the moment of each nucleus is shown to be expressible in terms of the amount of admixture of the two functions. If then the amount of ^4D function is taken to be 4 percent on the basis of an estimate by Gerjuoy and Schwinger, the moments of H^3 and He^3 are found to be 2.71 and -1.86 nuclear magnetons, respectively.

1. INTRODUCTION

THE existence of the quadrupole moment of the deuteron indicates that the neutron-proton interaction must involve an interaction of tensor type in order that the ground state of the deuteron contain the required mixture of S and D eigenfunctions.[1] Evidence for a similar deviation from L-S coupling in heavier nuclei is found in the marked decrease of the magnetic moments of the odd-odd nuclei with increasing atomic weight.[2] Some of the consequences of assuming that the tensor interaction acts between the particles in H^3 to the same degree as in the deuteron have been obtained by Gerjuoy and Schwinger.[3] They carried out a variation calculation with results indicating that in the ground state the probability for the system to be in a $^4D_{\frac{1}{2}}$ state is about 4 percent, while the system is to be otherwise found in the $^2S_{\frac{1}{2}}$ state. In general it might be expected that $^2P_{\frac{1}{2}}$ and $^4P_{\frac{1}{2}}$ functions would also be mixed in but their contribution was assumed to be small since they would appear in second order only if the tensor interaction were treated as a perturbation.

Unfortunately, the computation led to a binding energy that was only a fraction of the observed energy, so that the above conclusions concerning the character of the wave function cannot be considered to be very reliable. A somewhat more hopeful method of obtaining information concerning the relative contributions of states of different orbital and spin angular momenta to the ground state of a nucleus is to determine its magnetic moment, since this moment depends in a very sensitive way on just such contributions. In general, it would also be useful to determine the quadrupole and higher moments for this purpose but these moments vanish for nuclei, such as H^3 and He^3, which have a total angular momentum of $\frac{1}{2}$.

An exact calculation of the moments of the nuclei cannot at present be carried out because the results contain integrals which depend on the detailed properties of the wave functions which are, of course, not known. However, in the approximation that the neutron-neutron and proton-proton interactions are identical, the sum of the moments of H^3 and He^3 is independent of the detailed behavior of the wave functions. This will be shown in Section 2. As a consequence of this result, measurements of the moments of both nuclei could be combined to give very useful information concerning the relative contributions of the various possible terms to the ground states of the nuclei.

The moments of the individual nuclei can be expressed in simple terms if the wave function is

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[1] W. Rarita and J. Schwinger, Phys. Rev. 59, 436 and 556 (1941).
[2] S. Millman and P. Kusch, Phys. Rev. 60, 91 (1941).
[3] E. Gerjuoy and J. Schwinger, Phys. Rev. 61, 138 (1942).

assumed to contain only ^2S and ^4D components, and if certain simplifying assumptions, roughly equivalent to those made by Gerjuoy and Schwinger,[3] are made concerning the functions representing these states. The moments are calculated under these conditions in Section 3.

2. THE SUM OF THE MOMENTS OF H^3 AND He^3

In order to bring out the symmetry of the two nuclei with respect to the interchange of neutrons and protons it is desirable to introduce a notation which has this symmetry. Therefore, we label the variables referring to the neutrons in H^3 and to the protons in He^3 with the indices 1, 2, and we label the variables referring to the other particle in each of the nuclei with the index 3. Then the magnetic moments of the nuclei are given in units of the nuclear magneton by

$$\mu_{\mathrm{H}}=\langle L_3{}^z+2\mu_n(S_1{}^z+S_2{}^z)+2\mu_p S_3{}^z\rangle_{\mathrm{Av}}, \quad (1)$$

and

$$\mu_{\mathrm{He}}=\langle L_1{}^z+L_2{}^z+2\mu_p(S_1{}^z+S_2{}^z)+2\mu_n S_3{}^z\rangle_{\mathrm{Av}}, \quad (2)$$

where $L_i{}^z$ is the z-component of the orbital angular moment of the ith particle, $S_i{}^z$ is the z-component of the spin of that particle, and μ_n, μ_p are the magnetic moments of neutron and proton. The average is to be taken over the wave function of that ground state of the nucleus for which the z-component of the total angular momentum has its maximum value, $\frac{1}{2}$.

If the difference between neutron-neutron and proton-proton forces is neglected, the wave functions of the two nuclei are identical so the averages indicated in both Eq. (1) and in Eq. (2) involve the same wave function. Then the sum, $\mu=\mu_{\mathrm{H}}+\mu_{\mathrm{He}}$, of the moments of the two nuclei is

$$\mu=\langle L^z+2(\mu_n+\mu_p)S^z\rangle_{\mathrm{Av}}, \quad (3)$$

where $L^z=L_1{}^z+L_2{}^z+L_3{}^z$ and $S^z=S_1{}^z+S_2{}^z+S_3{}^z$ are the z-components of the total orbital and spin angular momenta, respectively. The average here is to be taken over the common wave function of the two nuclei.

Since $L^z+S^z=J^z$, the z-component of the total angular momentum, S^z can be eliminated from Eq. (3) to give

$$\mu=\mu_n+\mu_p-2(\mu_n+\mu_p-\tfrac{1}{2})\langle L^z\rangle_{\mathrm{Av}}, \quad (4)$$

in a state for which the projection of the total angular momentum on the z axis has its maximum value, $\frac{1}{2}$. The average in Eq. (4) can be simplified by means of the usual vector addition rule

$$\langle L^z\rangle_{\mathrm{Av}}=J^z\langle J(J+1)+\mathbf{L}^2-\mathbf{S}^2\rangle_{\mathrm{Av}}/2J(J+1). \quad (5)$$

Thus

$$\mu=\mu_n+\mu_p-2(\mu_n+\mu_p-\tfrac{1}{2})\langle\tfrac{3}{4}+\mathbf{L}^2-\mathbf{S}^2\rangle_{\mathrm{Av}}/3. \quad (6)$$

Now the ground state of each of the nuclei is made up of a combination of ^2S, ^2P, ^4P, and ^4D eigenfunctions and for each of these functions, $\mathbf{L}^2=L(L+1)$ and $\mathbf{S}^2=S(S+1)$. The average value of $\mathbf{L}^2-\mathbf{S}^2$ is therefore obtained by multiplying $L(L+1)-S(S+1)$ by the amount of admixture of the corresponding state, and adding. If, for simplicity, the amount of admixture is designated by the term symbol for the state, Eq. (6) becomes

$$\mu=\mu_n+\mu_p-2(\mu_n+\mu_p-\tfrac{1}{2})(3\,{}^4D-{}^4P+2\,{}^2P)/3. \quad (7)$$

Equation (7) would be most useful for obtaining an estimate of the relative contributions to the ground state of either nucleus of the states other than the ^2S state from measurements of the two magnetic moments. In the absence of such measurements, we give the value of the sum of the moments which is obtained from the results of Gerjuoy and Schwinger.[3] They found $^4D=0.04$, $^2P={}^4P=0$. With these values, the sum of the moments is 0.849, if the moments of the neutron and proton are taken to be[2] -1.911 and 2.790. This is to be compared to the value $\mu_n+\mu_p=0.879$ which would be expected if L-S coupling held rigorously in these nuclei.

A result similar to Eq. (7) applies to the sum of the moments of any pair of nuclei which can be obtained from each other by the interchange of protons and neutrons.

3. THE MAGNETIC MOMENTS OF H^3 AND He^3

In order to calculate the moments of H^3 and He^3, it is necessary to make some simplifying assumptions concerning the wave function of the ground state. Following Gerjuoy and Schwinger,[2] it is assumed that this function contains only ^2S and ^4D eigenfunctions and that these eigenfunctions are of a particularly simple form. The ^2S function will be taken to be that function which is antisymmetric for interchange of the spins of the two like particles. No detailed statement concerning the spatial dependence will be required. There are four possible types of ^4D eigenfunction, of which a particularly simple one was chosen by Gerjuoy and Schwinger. The spin dependent part of this function was symmetrical in the variables $\boldsymbol{\rho}$ and $\mathbf{r}$ where

$$\begin{aligned}\boldsymbol{\rho}&=\mathbf{r}_1-\mathbf{r}_2,\\ \mathbf{r}&=\mathbf{r}_3-(\mathbf{r}_1+\mathbf{r}_2)/2.\end{aligned}\tag{8}$$

For the calculation of the moments it is only this symmetry property that is needed. As long as the spin dependent factor in the ^4D function is either symmetric or antisymmetric in $\mathbf{r}$ and $\boldsymbol{\rho}$, it will be found that the moment is independent of the detailed spatial form of the function. Thus the functions used here are more general than those used by Gerjuoy and Schwinger, but they are not perfectly general since there is no *a priori* reason against the appearance of a combination of the symmetric and antisymmetric types of function. Nevertheless, it is felt that the simple results which will be obtained on the assumption that only one or the other type of function occurs may be useful as an indication of the magnitude of the influence of the ^4D state on the magnetic moment.

The averages indicated in Eqs. (1) and (2) contain no cross terms between the ^2S and ^4D eigenfunctions since these functions are orthogonal separately in space and spin. Therefore Eq. (1) becomes

$$\begin{aligned}\mu_{\mathrm{H}}={}&{}^2S\langle L_3{}^z+2\mu_n(S_1{}^z+S_2{}^z)+2\mu_pS_3{}^z\rangle_S\\ &+{}^4D\langle L_3{}^z+2\mu_n(S_1{}^z+S_2{}^z)+2\mu_pS_3{}^z\rangle_D,\end{aligned}\tag{9}$$

where the coefficients ^2S and ^4D are, as before, the squares of the amplitudes of these functions and $\langle\,\rangle_S$ and $\langle\,\rangle_D$ indicate averages over the ^2S and ^4D wave functions, respectively. Considering first the ^2S term, the orbital angular momentum makes no contribution since the S state has spherical symmetry. Of the spin terms, only the spin of the proton contributes since the assumed ^2S function is antisymmetric in the neutron spins, as stated in the beginning of the section. The value of the first term in Eq. (9) is therefore $^2S\mu_p$.

Since the three spins are parallel in the ^4D state, the expectation values of the three spins are equal. Thus

$$\langle S_1{}^z\rangle_D=\langle S_2{}^z\rangle_D=\langle S_3{}^z\rangle_D=\langle S^z\rangle_D/3.$$

By means of the usual vector addition rule it is found that $\langle S^z\rangle_D=-\frac{1}{2}$, so Eq. (9) becomes

$$\mu_{\mathrm{H}}={}^2S\mu_p-{}^4D(2\mu_n+\mu_p)/3+{}^4D\langle L_3{}^z\rangle_D.\tag{10}$$

In order to evaluate the last term, it is necessary to make use of the particular symmetry properties of the ^4D function which have been assumed. In terms of the variables $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)/3$, the orbital angular momentum operators are $\mathbf{L}_1=[(\mathbf{r}_1-\mathbf{R})\times\nabla_1 i/j]$, etc. Expressed in terms of the variables $\mathbf{r}$ and $\boldsymbol{\rho}$ given in Eq. (8), these operators satisfy the following relations:

$$\mathbf{L}_3=2(\mathbf{r}\times\nabla_r i/j)/3\tag{11}$$

$$\mathbf{L}=\mathbf{L}_1+\mathbf{L}_2+\mathbf{L}_3=(\mathbf{r}\times\nabla_r i/j)+(\boldsymbol{\rho}\times\nabla_\rho i/j).\tag{12}$$

Because of the assumed symmetry of the ^4D function in $\mathbf{r}$ and $\boldsymbol{\rho}$, the average values of the two terms in Eq. (12) are equal.[4] Therefore $\langle\mathbf{r}\times\nabla_r i/j\rangle_D=\frac{1}{2}\langle\mathbf{L}\rangle_D$. It follows from Eq. (11) that

$$\langle L_3{}^z\rangle_D=\tfrac{1}{3}\langle L^z\rangle_D=\tfrac{1}{3},\tag{13}$$

according to Eq. (5). Inserting this result in Eq. (10) and making use of the normalization condition $^2S+{}^4D=1$, we find

$$\mu_{\mathrm{H}}=\mu_p-2\,{}^4D(\mu_n+2\mu_p-\tfrac{1}{2})/3.\tag{14}$$

The magnetic moment of He³ can be obtained directly by subtracting μ_{H} from Eq. (7) with $^4P={}^2P=0$. It is

$$\mu_{\mathrm{He}}=\mu_n-2\,{}^4D(2\mu_n+\mu_p-1)/3.\tag{15}$$

For L-S coupling, the magnetic moments of H³ and He³ would be equal to the moments of the proton and neutron, respectively. Any observed deviation from these values can be used as a measure of the amount of mixing of the ^4D state, as shown by Eqs. (14) and (15). It is to be remembered in applying these formulae that they are valid only if the contribution of P states to the ground state is negligible and, even then, only if the S and D functions have the particularly simple forms which have been assumed.

If we accept the estimate $^4D=0.04$, the magnetic moments of the nuclei are found to be $\mu_{\mathrm{H}}=2.71$; $\mu_{\mathrm{He}}=-1.86$.

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[4] It is to be noted that symmetry is required only for the spin dependent factor in the function. The spherically symmetrical, space dependent factor makes no contribution to the orbital angular momentum.