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On the Self-Energy of the Electron

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Some evidence is given that the self-energy of an electron in the hole theory is finite, but coincides with mc^2 only if e^2/hc satisfies a particular equation.

INTRODUCTION AND DISCUSSION OF THE RESULTS

ONE of the very few tentative ways of explaining the finiteness of the self-energy of the electron without hypothesis *ad hoc* was made by Weisskopf,[1] who considered an electron in the "vacuum" of the hole theory and reached the conclusion that its self-energy diverges only logarithmically in every approximation; i.e., if the self-energy W is expanded in a series of approximations

$$W=\sum_n W^{(n)}, \tag{1}$$

corresponding to an expansion in powers of the parameter e^2/hc, $W^{(n)}$ does not diverge more strongly than

$$W^{(n)}\sim mc^2(e^2/hc)^n[\log(h/mca)]^n, \tag{2}$$

where a is a "critical length" giving the "dimension" of the electron.

Although Weisskopf did not obtain a finite self-energy, a great step was made, since the radius of the electron appears in (2) only logarithmically, and the electron mass appears for the first time as a coefficient in the self-energy.

Weisskopf considered his result as an evidence that the critical lenght a is of the order of $(h/mc)\exp(-hc/e^2)$, but it must be pointed out that if the series (1) is an alternating one, it is possible, and also very probable, that W remains finite also if a is infinitely small.

If we consider the physical meaning of the different approximations, it appears also very probable that the series (1) is an alternating one. Weisskopf has shown that the reduction of the order of the divergence in the hole theory is caused by a polarization of the "vacuum." In the first approximation this polarization is caused only by the repulsion between identical particles in a completely degenerate Fermi gas. The second approximation gives the effect of the electrostatic repulsion, and therefore this approximation gives a further increase of the polarization and a reduction of the self-energy, i.e., a negative term. Since in the second approximation the electrostatic repulsion is calculated for electrons which are nearer than in reality to the polarizing one, the third approximation will be positive, and so on.

It is therefore possible that the series (1) converges for every value of a to a limit

$$W=mc^2f[(h/mca),(e^2/hc)], \tag{3}$$

and that f is finite also for vanishing a:

$$\lim_{a=0}f[(h/mca),(e^2/hc)]=g(e^2/hc). \tag{4}$$

The requirement that the whole mass of the electron should be electromagnetic would then give a condition for the numerical value of e^2/hc, which should be a root of the equation

$$g(e^2/hc)=1. \tag{5}$$

In order to control these conjectures we shall calculate the second approximation of the electro-

[1] V. F.Weisskopf, Zeits. f. Physik 89, 27 and 90, 817 (1934); Phys. Rev. 56, 72 (1939).

static self-energy in the hole theory and shall find that it diverges according to (2) and is in effect negative.

We have not yet calculated the "second" approximation of the electrodynamic self-energy, which is in reality the fourth approximation of the perturbation produced by the interaction of the electrons with the transverse part of the electromagnetic field; but we have no reasons to believe that the result will be a different one. Anyway, the fact that the second approximation of the electrostatic self-energy is negative is sufficient reason for not rejecting the hope that the series (1) will converge also for vanishing a.

In conclusion we may say that it is possible that the divergence of the self-energy in the hole theory may be only apparent, and may be caused by failure of the expansion in powers of e^2/hc. In this case the exact solution of the problem which must be obtained without expanding powers of e^2/hc will give a finite self-energy; but this solution will be consistent with the assumption of the electromagnetic origin of the mass only if the numerical value of the fine-structure constant satisfies a condition of the type (5).

THE SECOND APPROXIMATION OF THE ELECTROSTATIC SELF-ENERGY

The electrostatic self-energy, according to Weisskopf, is given by

$$2E_S = \sum_{qrst} a_q^+ a_r a_s^+ a_t A(qrst) - 2\sum_{qr} a_q^+ a_r \sum_{s^-} A(qrs^-s^-) + \sum_{s^-t^-} A(s^-s^-t^-t^-), \tag{6}$$

where

$$a_q = a^i(\mathbf{q}), \quad (i = 1, 2, 3, 4) \tag{7}$$

are the well-known operators of the second quantization, and

$$A(qrst) = A(q^i r^j s^k t^l) = \frac{e^2 h^2}{\pi V} \frac{\{u^{i*}(\mathbf{q}) u^j(\mathbf{r})\}\{u^{k*}(\mathbf{s}) u^l(\mathbf{t})\}}{(\mathbf{q}-\mathbf{r})^2}, \quad (\mathbf{q}+\mathbf{s} = \mathbf{r}+\mathbf{t}). \tag{8}$$

Since only the first term of (6) has non-diagonal elements, we obtain for the diagonal elements of the electrostatic self-energy in second approximation

$$4W_S^{(2)} = \sum_{qrstuvwx} a_q^+ a_u a_r^+ a_v a_s^+ a_w a_t^+ a_x A(qurv) A(swtx)(E_q + E_r - E_u - E_v)^{-1}, \tag{9}$$

where (u, v, w, x) are permutations of (q, r, s, t) with

$$u \neq q, \quad v \neq r, \quad w \neq s, \quad x \neq t \tag{10a}$$

and

$$(u, v, w, x) \neq (r, q, t, s). \tag{10b}$$

Using the well-known relations between the a_q, we obtain after some changes of the indices

$$W_S^{(2)} = A + B \tag{11}$$

with

$$A = \tfrac{1}{2} \sum_{qrst} N_q N_r (1-N_s)(1-N_t) A(qsrt)[A(sqtr) - A(srtq)](E_q + E_r - E_s - E_t)^{-1} \tag{11a}$$

and

$$B = \tfrac{1}{4} \sum_{qrst} N_q (1-2N_r)(1-N_s)(1-2N_t) A(qrrs) A(sttq)(E_q - E_s)^{-1}. \tag{11b}$$

The second approximation of the self-energy of an electron at rest is

$$\tilde{W}^{(2)} = W^{(2)}(\text{Vac.}+1) - W^{(2)}(\text{Vac.}), \tag{12}$$

and its electrostatic part may be obtained by deriving $W_S^{(2)}$ with respect to N_0+ and putting afterwards $N_p+ = 0$ and $N_p- = 1$. We obtain in this way

$$c\tilde{A} = \sum_{r^-s^+t^+} A(0^+s^+r^-t^+)[A(s^+0^+t^+r^-) - A(s^+r^-t^+0^+)](mc - R - S - T)^{-1}$$

$$- \sum_{q^-r^-s^+} A(q^-s^+r^-0^+)[A(s^+q^-0^+r^-) - A(s^+r^-0^+q^-)](-Q - R - S - mc)^{-1}, \tag{13}$$

where

$$Q = (m^2c^2 + q^2)^{\frac{1}{2}}, \quad R = (m^2c^2 + r^2)^{\frac{1}{2}}, \quad S = (m^2c^2 + s^2)^{\frac{1}{2}}, \quad T = (m^2c^2 + t^2)^{\frac{1}{2}}; \tag{14}$$

and changing the indices in the first sum we have

$$c\tilde{A} = -\sum_{q^+r^+} A(0^+r^+s^-q^+)[A(r^+0^+q^+s^-) - A(r^+s^-q^+0^+)](Q+R+S-mc)^{-1}$$
$$+\sum_{q^-r^-} A(q^-s^+r^-0^+)[A(s^+q^-0^+r^-) - A(s^+r^-0^+q^-)](Q+R+S+mc)^{-1} \quad (13')$$

with

$$\mathbf{s} = \mathbf{q} + \mathbf{r}. \quad (15)$$

The sum over the spin directions may be made in the usual way:[2]

$$\frac{V^2c\pi^2}{e^4h^4}\tilde{A} = \sum_{qr}\{-[32QRSr^4(Q+R+S-mc)]^{-1}Sp[(1+\beta)(R+\boldsymbol{\alpha}\cdot\mathbf{r}+\beta mc)]$$
$$\times Sp[(S-\boldsymbol{\alpha}\cdot\mathbf{s}-\beta mc)(Q+\boldsymbol{\alpha}\cdot\mathbf{q}+\beta mc)]+[32QRSq^2r^2(Q+R+S-mc)]^{-1}$$
$$\times Sp[(1+\beta)(R+\boldsymbol{\alpha}\cdot\mathbf{r}+\beta mc)(S-\boldsymbol{\alpha}\cdot\mathbf{s}-\beta mc)(Q+\boldsymbol{\alpha}\cdot\mathbf{q}+\beta mc)]$$
$$+[32QRSr^4(Q+R+S+mc)]^{-1}Sp[(1+\beta)(R-\boldsymbol{\alpha}\cdot\mathbf{r}-\beta mc)]$$
$$\times Sp[(S+\boldsymbol{\alpha}\cdot\mathbf{s}+\beta mc)(Q-\boldsymbol{\alpha}\cdot\mathbf{q}-\beta mc)]-[32QRSq^2r^2(Q+R+S+mc)]^{-1}$$
$$\times Sp[(1+\beta)(R-\boldsymbol{\alpha}\cdot\mathbf{r}-\beta mc)(S+\boldsymbol{\alpha}\cdot\mathbf{s}+\beta mc)(Q-\boldsymbol{\alpha}\cdot\mathbf{q}-\beta mc)]\}$$
$$= -mc\sum_{qr}\left\{\frac{(Q+2R+S)[r^2-(S-Q)^2]}{2QRSr^4[(Q+R+S)^2-m^2c^2]}+\frac{QR(Q+R-S)-m^2c^2S}{2QRSq^2r^2[(Q+R+S)^2-m^2c^2]}\right\}; \quad (16)$$

the sum over q and r may be substituted by an integral:

$$\tilde{A} = -mc^2\left(\frac{e^2}{hc}\right)^2\int\left\{\frac{q^2(Q+2R+S)[r^2-(S-Q)^2]}{2QRSr^2[(Q+R+S)^2-m^2c^2]}+\frac{QR(Q+R-S)-m^2c^2S}{2QRS[(Q+R+S)^2-m^2c^2]}\right\}\frac{d\Omega_q d\Omega_r dqdr}{\pi^2} \quad (17)$$

Since only

$$S = [m^2c^2 + (\mathbf{q}+\mathbf{r})^2]^{\frac{1}{2}} \quad (18)$$

depends on the relative directions of $\mathbf{q}$ and $\mathbf{r}$, we may use the transformation

$$\int f\frac{d\Omega_q d\Omega_r dqdr}{\pi^2} = 8\int f\, d\cos(\mathbf{qr})\, dqdr = 8\int_0^p\int_0^p\frac{dqdr}{qr}\int_{S_1}^{S_2} fS\, dS, \quad (19)$$

with

$$S_1 = [m^2c^2+(q-r)^2]^{\frac{1}{2}}, \quad S_2 = [m^2c^2+(q+r)^2]^{\frac{1}{2}}; \quad (20)$$

on the other hand, since we are interested only in the asymptotic behavior of A for $p \gg mc$, and f is finite for q and r finite, we may neglect mc with respect to q and r, and write

$$\tilde{A} \cong -mc^2\left(\frac{e^2}{hc}\right)^2\int_{mc}^{P}\int_{mc}^{P} g_A(Q,R)\frac{dQdR}{QR}, \quad (21)$$

where

$$g_A(Q,R) = 4\int_{|Q-R|}^{Q+R}\frac{Q(Q+2R+S)[R^2-(S-Q)^2]+R^3(Q+R-S)}{R^3(Q+R+S)^2}dS \quad (22)$$

has the values

$$g_A(Q,R) = 4\frac{4Q^3-2Q^2R+R^3}{QR^2} - 4\frac{4Q^3-2QR^2+R^3}{R^3}\log\frac{Q+R}{Q} \quad \text{for } R \leqslant Q \quad (22a)$$

and

$$g_A(Q,R) = 4\frac{4Q^3-2Q^2R+QR^2}{R^3} - 4\frac{4Q^3-2QR^2+R^3}{R^3}\log\frac{Q+R}{R} \quad \text{for } R \geqslant Q. \quad (22b)$$

[2] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, New York, 1936), p. 150.

Owing to the fact that $g_A(Q, R)$ is a homogeneous function of degree zero, it is easy to see that for $P \gg mc$

$$\tilde{A} \cong -mc^2(e^2/hc)^2 \tfrac{1}{2}[g_A(Q, 0)+g_A(0, R)] \log^2 (P/mc); \tag{23}$$

and then we obtain from (22a) and (22b) that

$$\tilde{A} \cong -(4/3)mc^2(e^2/hc)^2 \log^2 (P/mc). \tag{24}$$

In the same way we have for B:

$$c\tilde{B} = \sum_{qst} N_q(1-N_s)(1-2N_t)A(q0^+0^+s)A(sttq)/2Q \qquad (\mathbf{q}=\mathbf{s})$$
$$= \sum_{qt} A(q^-0^+0^+q^+)[A(q^+t^+t^+q^-)-A(q^+t^-t^-q^-)]/2Q; \tag{25}$$

$$\frac{V^2c\pi^2}{e^4h^4}\tilde{B} = \sum_{qt} [32Q^3Tq^2(t-q)^2]^{-1} Sp[(1+\beta)(Q+\boldsymbol{\alpha}\cdot\mathbf{q}+\beta mc)(\boldsymbol{\alpha}\cdot\mathbf{t}+\beta mc)(Q-\boldsymbol{\alpha}\cdot\mathbf{q}-\beta mc)]$$
$$= mc \sum_{qt}[4Q^3Tq^2(t-q)^2]^{-1}(q^2-\mathbf{t}\cdot\mathbf{q}); \tag{26}$$

or, putting

$$\mathbf{r} = \mathbf{t}-\mathbf{q}, \tag{27}$$

$$\frac{V^2\pi^2}{e^4h^4}\tilde{B} = m \sum_{qr}(q^2+R^2-T^2)/(8Q^3Tq^2r^2). \tag{26'}$$

rom here we obtain

$$\tilde{B} = mc^2\left(\frac{e^2}{hc}\right)^2 \int \frac{q^2+R^2-T^2}{8Q^3T}\frac{d\Omega_q d\Omega_r dq dr}{\pi^2} \cong mc^2\left(\frac{e^2}{hc}\right)^2 \int_{mc}^{P}\int_{mc}^{P} g_B(Q, R)\frac{dQdR}{QR}, \tag{28}$$

re

$$g_B(Q, R) = \int_{|Q-R|}^{Q+R} \frac{Q^2+R^2-T^2}{Q^3}dT \tag{29}$$

the values

$$g_B(Q, R) = 4R^3/3Q^3 \quad \text{for } R \leqslant Q \tag{29a}$$

$$g_B(Q, R) = 4/3 \quad \text{for } R \geqslant Q; \tag{29b}$$

then

$$\tilde{B} \cong (2/3)mc^2(e^2/hc)^2 \log^2 (P/mc). \tag{30}$$

ntroducing the critical length $a = h/p$, we conclude from (11), (24), and (30) that

$$\tilde{W}_S{}^{(2)} \cong -(2/3)mc^2(e^2/hc)^2 \log^2(h/mca). \tag{31}$$