PHYSICAL REVIEW VOLUME 70, NUMBERS 5 AND 6 SEPTEMBER 1 AND 15, 1946

Equations of State for Radially Symmetric Distributions of Matter

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(Received June 3, 1946)

It is well known that the line element corresponding to the homogeneous universe can be derived from the physical assumptions that the material with which we are dealing is radially symmetric and spatially isotropic. In order to obtain a more general cosmology the following physical assumptions are made: 1. The matter in the body is radially symmetric. 2. The matter comprises a perfect fluid at rest. And 3. The matter obeys an equation of state of the form $p=p(\rho)$ where p is the pressure ρ is the density. It is shown that the above assumptions lead to three possible cosmologies. One of these is of course the homogeneous universe and the other two are new.

INTRODUCTION

WHEN considering possible models for the universe one of the most important, hitherto discovered, is the homogeneous universe. This model can be derived from the physical assumptions[1] that the material with which we are dealing is radially symmetric and spatially isotropic. On the basis of these assumptions it can be shown that the line element[2] for this model is given by

$$ds^2=dt^2-\frac{e^{g(t)}}{(1+r^2/4R_0^2)^2}\times(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2), \quad (1)$$

where $g(t)$ is an arbitrary function of t. Further the pressure[3] p and density ρ are given by

$$8\pi p=-\frac{e^{-g(t)}}{R_0^2}-\frac{d^2g}{dt^2}-\tfrac{3}{4}\dot{g}^2+\Lambda, \quad (2)$$

$$8\pi\rho=\frac{3e^{-g(t)}}{R_0^2}+\tfrac{3}{4}\dot{g}^2-\Lambda, \quad (3)$$

where the dot denotes differentiation with respect to t and Λ is the cosmological constant. Since both p and ρ are functions of t alone, it is obvious that an equation of state of the form $p=p(\rho)$ always exists for the homogeneous universe.

In attempting to obtain a more general model than the homogeneous universe Professor R. C. Tolman suggested to me, some time ago, the following method of attack. The line element is taken to have the form[4]

$$ds^2=e^{\nu}dt^2-e^{\mu}(dr^2+r^2\sin^2\theta d\phi^2+r^2d\theta^2), \quad (4)$$

where $\mu=\mu(r,t)$, $\nu=\nu(r,t)$, and the material body with which we are dealing is assumed to have the following physical properties.

1. The matter in the body is radially symmetric.
2. The matter comprises a perfect fluid at rest in the coordinate system of (4).
3. The matter obeys an equation of state of the form $p=p(\rho)$.

It is quite obvious that the homogeneous universe satisfies these assumptions. We shall show that there are two additional models which will satisfy all of the above assumptions.

2. GENERAL FORM OF THE GRAVITATIONAL POTENTIALS

If T_j^i denotes the components of the energy-momentum tensor then our second assumption implies[5] $T_1^1=T_2^2=T_3^3=-p$, $T_4^4=\rho$ and $T_j^i=0$, $i\neq j$. Using the expressions for T_j^i as given by Tolman[6] we find

$$8\pi p=e^{-\mu}\left(\frac{\mu'^2}{4}+\frac{\mu'\nu'}{2}+\frac{\mu'+\nu'}{r}\right)-e^{-\nu}\left(\frac{d^2\mu}{dt^2}+\tfrac{3}{4}\dot{\mu}^2-\frac{\dot{\mu}\dot{\nu}}{2}\right)+\Lambda, \quad (2.1)$$

[1] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, Cambridge, England, 1934), pp. 362–370.
[2] Reference 1, p. 369.
[3] Reference 1, p. 377.

[4] The same form of line element was recently used by Einstein and Strauss in discussing the influence of the expansion of space on the gravitation fields surrounding stars. (Rev. Mod. Phys. 17, 120–124 (1945).)
[5] Reference 1, p. 243.
[6] Reference 1, p. 252.

$$8\pi p=e^{-\mu}\left(\frac{\mu''+\nu''}{2}+\frac{\nu'^2}{4}+\frac{\mu+\nu'}{2r}\right)$$

$$-e^{-\nu}\left(\frac{d^2\mu}{dt^2}+\tfrac{3}{4}\dot{\mu}^2-\frac{\dot{\mu}\dot{\nu}}{2}\right)+\Lambda, \quad (2.2)$$

$$8\pi\rho=-e^{-\mu}\left(\mu''+\frac{\mu'^2}{4}+\frac{2\mu'}{r}\right)+\tfrac{3}{4}e^{-\nu}\dot{\mu}^2-\Lambda, \quad (2.3)$$

$$2\dot{\mu}'-\dot{\mu}\nu'=0. \quad (2.4)$$

In the above expressions the prime notation indicates partial differentiation with respect to "r" and the dot notation indicates partial differentiation with respect to "t."

Before proceeding it might be pointed out that many writers on relativistic cosmology prefer to take the value of the cosmological constant Λ to be zero. We retain this constant in our equations since its value does not affect the simplicity or complexity of our calculations in any way.

When $\dot{\mu}=0$ it is not difficult to show that a pressure-density relationship can exist only if the line element can be transformed to a static form in which both μ, ν are both independent of t. If the gravitational potentials μ, ν depend only on r a pressure-density relationship always exists and the problem is usually carried forward by investigating special pressure-density relationships of physical interest. Since this case has already been discussed to some extent in other papers we shall not treat it further here and shall assume throughout that $\dot{\mu}\neq 0$.

With the above restriction Eq. (2.4) can be integrated to give

$$\nu=2\log\dot{\mu}+\phi(t), \quad (2.5)$$

where ϕ is an arbitrary function of "t." Making use of (2.5) Eqs. (2.1)–(2.4) take the form

$$8\pi p=A\frac{\partial}{\partial t}\left[e^{\mu/2}\left(\frac{\mu'^2}{2}+\frac{2\mu'}{r}\right)-B\right], \quad (2.6)$$

$$8\pi p=A\frac{\partial}{\partial t}\left[e^{\mu/2}\left(\mu''+\frac{\mu'}{r}\right)-B\right], \quad (2.7)$$

$$8\pi\rho=-e^{-\mu}\left(\mu''+\frac{\mu'^2}{4}+\frac{2\mu'}{r}\right)+\tfrac{3}{4}e^{-\varphi}-\Lambda, \quad (2.8)$$

where

$$A=e^{-3\mu/2}/\dot{\mu}, \quad B=e^{3\mu/2}\left(\frac{e^{-\varphi}}{2}-\frac{2\Lambda}{3}\right).$$

The equality of (2.6) and (2.7) immediately yields an equation which can be integrated with respect to t. This leads to the equation

$$e^{\mu/2}\left(\mu''-\frac{\mu'^2}{2}-\frac{\mu'}{r}\right)=\psi(r),$$

where $\psi(r)$ is an arbitrary function of r.

Since the divergence $T_{j;i}{}^{i}$ of the energy-momentum tensor must vanish we can obtain the following two relations,

$$p'=-[(p+\rho)\dot{\mu}']/\dot{\mu}, \quad (2.10)$$

$$2\dot{\rho}=-3(p+\rho)\dot{\mu}. \quad (2.11)$$

Eliminating $\dot{\mu}$ gives us the single relation

$$(p+\rho)\dot{\rho}'=\dot{\rho}\rho'. \quad (2.12)$$

By inspection we can see that $\dot{\rho}=0$ or $\rho'=0$ are two solutions of (2.12). If $\dot{\rho}=0$ then (2.11) implies $p+\rho=0$ since we have assumed $\dot{\mu}\neq 0$. The positive character of the pressure and density means that $p+\rho=0$ cannot be satisfied unless $p=\rho=0$. This could hold, however, only in empty space and an equation of state for this case would be meaningless. The other possibility $\rho'=0$ implies $p'=0$ since we have assumed an equation of state of the form $p=p(\rho)$. For this case Eq. (2.10) gives us $p+\rho=0$ or $\dot{\mu}'=0$. We discard $p+\rho=0$ as before and deal with $\dot{\mu}'=0$. When this is so (2.4) would require $\nu'=0$. Further investigation shows that this case leads to the homogeneous universe. In order then to obtain a more general model than the homogeneous universe we now know that neither pressure nor the density can be independent of t or r.

In order to carry through the integration of (2.12) let us define the function $q(\rho)$ by means of

$$dq/d\rho=1/(p+\rho). \quad (2.13)$$

This is legitimate because $p=p(\rho)$. Then (2.12) takes the form

$$\frac{\partial}{\partial r}\log\dot{\rho}=\frac{\partial}{\partial r}q(\rho). \quad (2.14)$$

Thus

$$\log\dot{\rho}=q(\rho)+\text{arbitrary function of } t. \quad (2.15)$$

Equation (2.15) can be integrated by a similar device used to integrate (2.12) and we find that ρ must have the form

$$\rho=\rho(v), \quad (2.16)$$

where $v=h(t)+k(r)$ and h, k are arbitrary functions of their arguments. Moreover a change of time scale $\bar{t}=h(t)$ leaves the form of the line element invariant so there is no loss in generality in taking $h(t)=t$. Thus

$$v=t+k(r). \tag{2.17}$$

From (2.11) we have

$$\dot{\mu}=-2\dot{\rho}/3(p+\rho)=-\frac{2}{3}\frac{dq}{d\rho}\dot{\rho}. \tag{2.18}$$

This implies

$$\mu=-\tfrac{2}{3}q(\rho)+f(r), \tag{2.19}$$

where $f(r)$ is an arbitrary function of r. Thus, because of (2.16), we have that μ must have the form

$$\mu=F(v)+f(r). \tag{2.20}$$

Moreover the form of the second gravitational potential ν is given by (2.5) to be

$$\nu=2\log(dF/dv)+\phi(t). \tag{2.21}$$

We have by the above analysis obtained the general functional form which the gravitational potentials μ, ν must have. In order to find explicit expressions for μ, ν we must still find explicit expressions for $F(v)$, $k(r)$, $f(r)$, and $\phi(t)$. These will be obtained in the next section.

3. DETERMINATION OF THE GRAVITATIONAL POTENTIALS

From the preceding section we have seen that the gravitational potential μ must satisfy an equation of the type

$$e^{\mu/2}\left(\mu''-\frac{\mu'^2}{2}-\frac{\mu'}{r}\right)=\psi(r), \tag{3.1}$$

and further μ must have the form

$$\mu=F(v)+f(r). \tag{3.2}$$

When $\psi(r)=0$, the solution of (3.1) is easily obtained to be $\mu=-2\log(Ar^2+B)$ where A, B are arbitrary functions of "t." For this case the density is given by (2.8) to be $8\pi\rho=12AB+\frac{3}{4}e^{-\varphi(t)}-\Lambda$. Thus the density is a function of t alone. By a result obtained in the previous section we see that this can only be so, under the assumptions we have made, when the solution can be reduced to the solution corresponding to the homogeneous universe.

Returning to Eq. (3.1) we shall make a change of variable by means of $x=r^2/2$. The equation then becomes

$$e^{\mu/2}\left(\frac{\partial^2\mu}{\partial x^2}-\frac{1}{2}\left(\frac{\partial\mu}{\partial x}\right)^2\right)=\psi[(2x)^{\frac{1}{2}}]/2x=h(x). \tag{3.3}$$

Since ψ is still an unknown function we have replaced $\psi[(2x)^{\frac{1}{2}}]/2x$ by $h(x)$. Similarly the form of u is given by

$$\mu=F(v)+z(x), \tag{3.4}$$

where

$$v=t+y(x), \tag{3.5}$$

and $z(x)=f[(2x)^{\frac{1}{2}}]$, and $y(x)=k[(2x)^{\frac{1}{2}}]$. In order to simplify our notation we shall, from now on, denote total differentiation by a letter subscript. Thus $F_v=dF/dv$, $y_x=dy/dx$ and so on. Substituting (3.4) into (3.3) we obtain

$$e^{(F+z)/2}[(F_{vv}-\tfrac{1}{2}F_v^2)y_x^2+F_v(y_{xx}-y_xz_x)+z_{xx}-\tfrac{1}{2}z_x^2]=h(x). \tag{3.6}$$

Since x and v are independent variables, we can satisfy (3.6) only if

$$y_{xx}-y_xz_x=ay_x^2, \tag{3.7}$$

$$z_{xx}-\tfrac{1}{2}z_x^2=by_x^2, \tag{3.8}$$

$$e^{F/2}(F_{vv}-\tfrac{1}{2}F_v^2+aF_v+b)=c, \tag{3.9}$$

$$h(x)=ce^{z/2}y_x^2, \tag{3.10}$$

where a, b, c are constants. From (3.7)

$$z_x=y_{xx}/y_x-ay_x. \tag{3.11}$$

Substituting (3.11) into (3.9) we have

$$2y_xy_{xxx}-3y_{xx}^2=(2b+a^2)y_x^4. \tag{3.12}$$

This last equation can be integrated to give

$$y_{xx}^2=(2b+a^2)y_x^4+c_1y_x^3, \tag{3.13}$$

where c_1 is a constant of integration.

For the remainder of the discussion it is best to divide up the analysis into several cases according as c_1, a, b, $2b+a^2$ do or do not vanish. We shall illustrate the analysis by two cases, one of which leads to a new solution, and the other is used to show that for certain values of c_1, a, b no solution exists which will admit an equation of state.

Case 1

$c_1=b=a=0$. Thus $y_{xx}=0$ and $y=c_2x+c_3$ where c_2, c_3 are constants of integration. Since $v=t+y$ the constant of integration c_3 can be taken to be zero as it can be absorbed into the time scale. From (3.11) $z_x=0$. We can take $z=0$ as a little consideration will show that the constant of integration that arises in this case can be absorbed into the r scale. Thus $\mu=F(v)$ where $v=t+c_2r^2/2$. Under the restrictions imposed in this case Eq. (3.9) becomes

$$e^{F/2}(F_{vv}-\tfrac{1}{2}F_v^2)=c. \quad (3.14)$$

The substitution $F=\log(-c/12w)^2$ reduces this equation to

$$w_{vv}=6w^2, \quad (3.15)$$

which can be integrated to give

$$w_v^2=4w^3-c_3. \quad (3.16)$$

The solution of (3.16) is $w=\wp(v)$ where $\wp(v)$ is the Weierstrass elliptic function with invariants 0, c_3. Thus $F=\log(-c/12\wp(v))^2$. From the expressions we have derived we find our gravitational potentials are given by

$$\mu=\log(-c/12\wp(v))^2,$$

$$\nu=\log(4\wp_v^2/\wp^2)+\varphi(t), \quad (3.17)$$

$$v=t+c_2r^2/2. \quad (3.18)$$

In our gravitational potentials only one unknown function $\phi(t)$ remains. This can be determined by the condition that the density ρ must reduce to a function of v alone. From (2.8), (3.16), (3.17), and (3.18) the density is given by the equation

$$8\pi\rho=6\alpha(\wp\wp_v+c_3v)+\tfrac{3}{4}e^{-\varphi(t)}-6\alpha c_2c_3t-\Lambda, \quad (3.19)$$

where $\alpha=(12/c)^2$. Since v and t are independent variables, ρ can be a function of v alone only if

$$e^{-\varphi}=8(\alpha c_2c_3t+\beta), \quad (3.20)$$

where $\beta=(\Lambda+K)/6$ and K is a constant. When this is so the pressure and density are given by

$$8\pi\rho=6\alpha c_2(\wp\wp_v+c_3v)+K, \quad (3.21)$$

$$8\pi p=-\alpha[\wp\wp_v+6c_3v-5c_3(\wp/\wp_v)]-K. \quad (3.22)$$

We thus see that an equation of state of the form $p=p(\rho)$ exists for this case and that (3.21), (3.22) provide a parametric representation of that relationship.

Case 2

Returning to Eq. (3.13) we take $b=0$, $a=1$, $c_1=2$. The equation then becomes

$$y_{xx}^2=y_x^4+2y_x^3. \quad (3.23)$$

This can be integrated to give

$$y_x=2/[(x+\alpha)^2-1], \quad (3.24)$$

where α is constant of integration. From (3.24) we find

$$y=\log|(x+\alpha-1)/(x+\alpha+1)| + \text{constant of integration}. \quad (3.25)$$

Since we are using this case only for purposes of illustration we shall take both α and the constant of integration to be zero. Thus

$$y=\log|(x-1)/(x+1)|. \quad (3.26)$$

Similarly z can be determined by (3.11) to be

$$z=2\log|x-1|. \quad (3.27)$$

In the expression for z we have again dropped a constant of integration. For this case then

$$v=t+\log|(x-1)/(x+1)|, \quad (3.28)$$

$$\mu=F(v)-2\log|x-1|. \quad (3.29)$$

Returning to Eq. (2.8) we note that the transformation $x=r^2/2$ puts this in the form

$$8\pi\rho=-e^{-\mu}\left(2x\mu_{xx}+\frac{x}{2}\mu_x^2+3\mu_x\right)+\tfrac{3}{4}e^{-\varphi(t)}-\Lambda. \quad (3.30)$$

Substituting (3.29), Eq. (3.30) becomes

$$8\pi\rho=\frac{-e^{-F}}{(x+1)^2}[x(8F_{vv}+2F_v^2-4F_v+12)-6(x^2+1)(F_v-1)]+\tfrac{3}{4}e^{-\varphi}-\Lambda. \quad (3.31)$$

Expressing x in terms of v and t by means of (3.28) we find that (3.31) can be put into the form

$$8\pi\rho=A(v)+B(v)e^{-2t}+\psi(t), \quad (3.32)$$

where

$$A(v) = -e^{-F}(2F_{vv}+\tfrac{1}{2}F_v^2-4F_v+6), \quad (3.33)$$

$$B(v) = e^{-(F-2v)}(2F_{vv}+\tfrac{1}{2}F_v^2+2F_v), \quad (3.34)$$

$$\psi(t) = \tfrac{3}{4}e^{-\phi(t)} - \Lambda. \quad (3.35)$$

Since v, t are independent variables (3.32) implies that ρ is a function of v alone only if

$$B(v) = \text{constant} = 2k, \quad \psi(t) = -2ke^{-2t} + \text{constant}.$$

Using (3.34) we see that $F(v)$ must satisfy the differential equation

$$F_{vv}+\tfrac{1}{4}F_v^2+F_v = ke^{(F-2v)}. \quad (3.36)$$

We already know however that F must satisfy (3.9). Putting $a=1$, $b=0$ this equation can be written

$$F_{vv}-\tfrac{1}{2}F_v^2+F_v = ce^{-(F/2)}. \quad (3.37)$$

Subtracting (3.37) from (3.36) we find

$$\tfrac{3}{4}F_v^2 = ke^{(F-2v)} - ce^{-(F/2)}. \quad (3.38)$$

Differentiating (3.38) with respect to v and substituting for F_{vv} from (3.37) we find

$$F_v(\tfrac{3}{4}F_v^2 - \tfrac{3}{2}F_v + \tfrac{3}{2}ce^{-F/2}) = ke^{(F-2v)}(F_v-2) + \tfrac{1}{2}ce^{-(F/2)}F_v. \quad (3.39)$$

Eliminating F_v^2 by (3.38) Eq. (3.39) takes the form

$$\tfrac{3}{4}F_v^2 = ke^{(F-2v)}. \quad (3.40)$$

From (3.38) and (3.40) we see that $c=0$. However (3.10) implies $h(x)=0$, and in turn (3.3) gives us $\psi(r)=0$. This case has already been dealt with and we have seen that $\psi(r)=0$ leads us to a solution which corresponds to the homogeneous universe. Thus the second case which we have discussed does not lead to a new solution.

It was pointed out that in order to obtain a complete analysis of the problem under consideration it was necessary to investigate several special cases according as c_1, a, b, $2b+a^2$, do or do not vanish. The author has carried through a complete investigation of every case possible and only two new solutions for the gravitational potentials exist, one already obtained and the other will be given in the conclusion of this paper.

CONCLUSION

Starting from a line element of the form

$$ds^2 = e^{\nu}dt^2 - e^{\mu}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2),$$

we have been able to show that the physical assumptions given in the introduction implies that the gravitational potentials can be reduced to one of the following three forms.

1. $e^{\nu}=1, \quad e^{\mu}=e^{g(t)}/(1+r^2/4R^2)^2$ (homogeneous universe).
2. $e^{\nu}=\wp_v^2/2(\alpha c_2c_3t+\beta)\wp^2(v), \quad e^{\mu}=c^2/144\wp^2(v)$ where $v=t+c_2r^2/2$.
3. $e^{\nu}=\wp_v^2/2(\alpha c_2c_3t+\beta)\wp^2(v), \quad e^{\mu}=c^2/144r^4\wp^2(v)$ where $v=t+c_2/2r^2$.

The line element corresponding to the third solution can easily be obtained from that of the second by the transformation $\bar{r}=1/r$. It is not difficult to show that the second and third solutions lead to the same equation of state. As a cosmological model the third solution is not likely to be of interest because of the singularity at $r=0$. In order to determine whether the second is of interest one should investigate the physical properties, such as behavior of particles, light rays, etc., of a model whose gravitational potentials are those given by (2). The author hopes to carry this investigation out at a later time.

Even if these solutions do not provide an interesting cosmology they are at least two new solutions of the field equation that are valid inside matter. As such they may be of interest in problems of the type recently discussed by Einstein and Strauss.[4] More than this the analysis which we have carried through points the way to determine several more new solutions of this type. By solving Eqs. (3.7), (3.8), (3.9) we are lead to several more solutions of the field equations which will be valid inside matter. We know however that these new solutions will not admit an equation of state of the form $p=p(\rho)$.

The author would like to thank Professor Tolman for suggesting this problem, and to say that Eq. (2.9) of the present paper was obtained from him in a private conversation.