PHYSICAL REVIEW VOLUME 70, NUMBERS 5 AND 6 SEPTEMBER 1 AND 15, 1946

Ionization Currents in Non-Uniform Electric Fields

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(Received January 17, 1945)

It is demonstrated that calculation of the ionization current in a gaseous discharge by means of the classical Townsend equation $i=i_0e^{\int\alpha dx}$ is likely to lead to large errors when the field distribution is not uniform. For a field approximately inversely proportional to distance from the cathode, the error was greater than 25 percent when the field intensity changed by more than 2.5 percent per mean free path of electrons just able to ionize. For fields of the type found at the cathode end of a glow discharge, therefore, the Townsend equation is seldom if ever applicable. A differential-difference equation for the electron current as a function of the electron energy and distance from the cathode was derived, and, by the use of semi-empirical functions where adequate data are not available, the ionization currents were calculated by step-by-step numerical methods for a restricted range of pressure and applied voltage. The results agree with measured currents within the range where the assumed functions apply. The method is much more laborious than integration of the Townsend equation, but it yields more information since the actual electron-energy distribution at each point of the discharge is found.

INTRODUCTION

SINCE its formulation by J. S. Townsend in 1900, theTownsend equation $i=i_0e^{\alpha d}$ for the current in a gaseous discharge has been used almost exclusively for the calculation of such currents. When the electric field in the discharge is not uniform the equation is written in the form $i=i_0e^{\int\alpha dx}$ in which x is the distance measured from the cathode in the field direction, and for each small increment of distance the value of the "Townsend coefficient" α corresponding to the field strength in that increment is taken from experimental data such as that of Kruithoff and Penning,[1] for neon and argon, of Bowls[2] for nitrogen, and of Hale[3] for hydrogen. The calculated currents have been and are being used for many purposes, notably in attempts to prove or disprove various theories of the mechanism responsible for sparkover, in attempts to calculate sparking potentials, in attempts to calculate the voltages at which corona onset occurs at points and around wires, and in attempts to calculate the field distribution and the current density in glow discharges.[4]

In most discharges, including some cases of all the types of discharges just mentioned, the field intensity is likely to be far from uniform. Even when the electrodes are parallel planes the field is usually badly distorted by space charge. Since the probability of ionization by collision depends rather upon the energy of the colliding electrons than upon the field intensity where the collision takes place, the use of the Townsend equation is justified only when the distribution of energies of the electrons, in each part of the discharge, is the same as would exist in a uniform field of the same intensity.

The investigation which is reported here was undertaken, first, to determine the magnitude of the error incurred by using a Townsend coefficient measured in a uniform field to calculate the ionization in a field which is not uniform; second, to establish a limit to the variation of the field per electron mean free path, above which limit the Townsend equation $i=i_0e^{\int\alpha dx}$ cannot be expected to give even approximately correct results; and, third, to outline a method for calculating the current in more variable fields which is correct in principle and which can be applied to certain cases with results verified by experiment.

APPARATUS

In order to test experimentally the calculation of ionization currents with Townsend's equation, it was necessary to construct an ionization chamber in which the field intensity was not uniform but was calculable, and in which the

[1] A. A. Kruithoff and F. M. Penning, Physica **3**, 515 (1936); **4**, 430 (1937).
[2] W. E. Bowls, Phys. Rev. **53**, 293 (1938).
[3] D. H. Hale, Phys. Rev. **54**, 241 (1938); **56**, 815 (1939).
[4] L. B. Loeb and J. M. Meek, *The Mechanism of the Electric Spark* (Stanford University Press, California, 1941), includes an extensive bibliography; W. Rogowski, Zeits. f. Physik **100**, 1 (1936); W. Rogowski, Zeits. f. Physik **82**, 473 (1933); A. von Engel and M. Steenbeck, *Elektrische Gasentladungen* (Verlagsbuchhandlung, Julius Springer, Berlin, 1934); and M. J. Druyvesteyn and F. M. Penning, Rev. Mod. Phys. **12**, 87 (1940).

primary current emitted from the cathode could be found. The field between concentric cylinders meets the first requirement, and, if the smaller cylinder is made the cathode, the field distribution somewhat resembles that known to exist in a glow discharge between plane electrodes. A cold cathode illuminated with ultraviolet light was chosen as a source of primary current to avoid density gradients which might occur around a hot filament and also to make the primary current very small and yet controllable. Small currents were necessary to avoid distortion of the field by space charge; the primary currents used were about 10^{-12} ampere.

The apparatus used is shown in Fig. 1. The anode of the ionization chamber was of copper, $3\frac{1}{2}''$ inside diameter and 8″ long, with an elongated opening on one side for the illumination of the cathode. The opening was covered with fine copper gauze and was placed opposite two quartz windows sealed to the glass envelope with graded seals. Two cathodes were used, the one in the lower half of the tube having a diameter of $\frac{1}{8}''$ and the upper one $\frac{7}{16}''$. Both were of nickel, insulated from each other and from the anode with quartz. In order to reduce field distortion near the ends of the chamber, two guard rings of $1\frac{1}{2}''$ diameter were mounted at the ends.

The two cathodes, the guard rings, and the anode were connected to points on a voltage, divider so constructed that each metal part of the tube was supplied with the potential appropriate to its position in the field. Thus the field between the larger cathode and the anode was exactly the same as would have existed in the same space if the larger cathode had been removed, leaving the smaller one extending throughout the tube. By this means electrons could be released at two different points in the non-uniform field and the resulting ionization compared. Use of the larger cathode meant a decrease in the maximum field intensity of 71.4 percent, a decrease in the total voltage acting on the discharge of 37.6 percent, but a decrease in length of the ionizing region of only 8.95 percent, compared with the smaller cathode values.

The ionization currents were measured with a quadrant electrometer connected to indicate the voltage drop across a J. G. White ceramic resistor, and the currents were occasionally checked by measuring the rate-of-charge of a standard capacitor. No difficulty was experienced in these experiments which could be ascribed to polarization of the high resistors used.

The gas used was hydrogen, purified by passing slowly over copper shot and powdered copper at 450°C to a Pyrex reservoir. It was admitted to the ionization chamber through a very fine capillary and a long spiral immersed in liquid air. Mercury contamination was avoided by placing this liquid-air trap between the chamber and the McLeod gages used for measuring pressure, and an additional liquid-air trap ahead of the mercury diffusion pump.

Each time air was admitted to the chamber the tube was baked at 450°C for 16 hours while the pressure was kept below 10^{-6} mm by constant pumping. The tube was then filled with about half an atmosphere of clean hydrogen and baked for 8 hours more, pumped out and refilled with about 10 cm of clean hydrogen and subjected to a glow discharge for at least an hour. Although the size of the metal parts precluded outgassing by induction heating, most of the gases near the surfaces probably were removed or replaced by hydrogen. The chamber was pumped out each night and each time a glow discharge occurred.

EXPERIMENTAL PROCEDURE

With the ionization chamber filled with clean hydrogen at a known pressure and with one of the cathodes illuminated with a steady ultra-

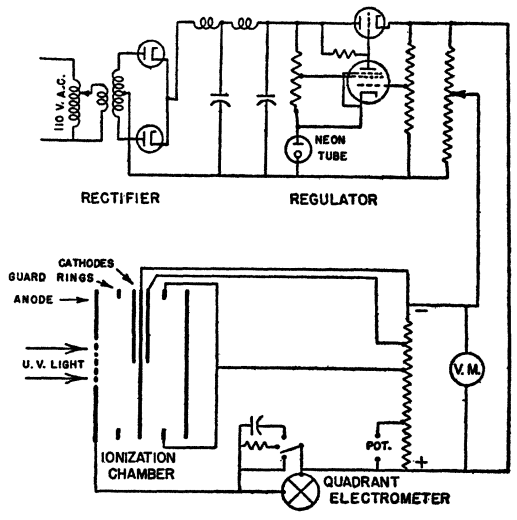

FIG. 1. Apparatus and electric circuit.

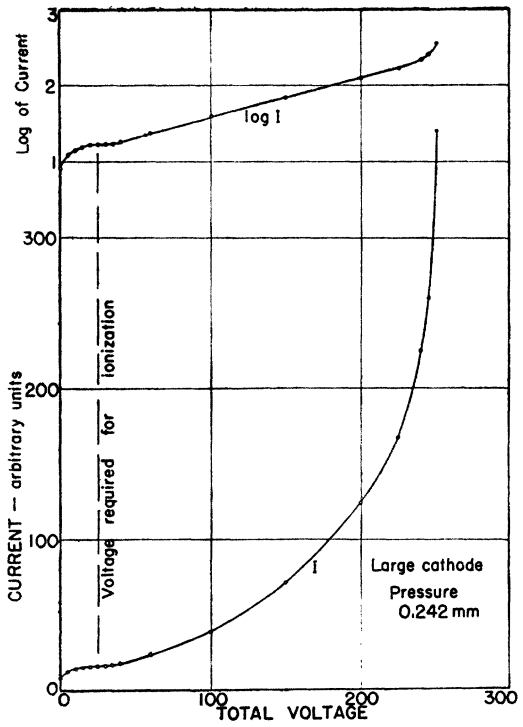

FIG. 2. Typical current-voltage graph.

violet light, the current to the anode was measured for a range of voltages from zero to the value for which secondary processes at the cathode began to cause additional emission. A curve typical of the results obtained is shown in Fig. 2.

It will be noted that considerable current is obtained with no voltage applied, because of the initial velocity of the emitted electrons. The free anode was found to charge up to approximately 0.4 volt negative, indicating a maximum electron energy of about this value. The 2536-angstrom line of the mercury arc corresponds to 4.87 volts, which would indicate that the work function of the nickel cathode in hydrogen was about 4.5, compared to the value of 5.03 quoted for outgassed nickel. At pressures below 0.01 mm it was found that the emission changed erratically with time and possibly with applied voltage, probably because of ion bombardment. At the lowest obtainable pressure the emission was only a few percent of its value between 0.01 and 10 millimeters. Such behavior of a nickel-hydrogen surface was observed previously by Hale.[5]

Since ionization by collision does not occur in hydrogen for electron energies of less than 15.4 volts, it would at first appear simple to collect and measure the total emitted current at any voltage less than this. The problem is complicated, however, by back diffusion, especially at the higher pressures. In fact, back diffusion sets a rather low limit upon the pressure which can be used; it was found impossible to ascertain surely the primary current for pressures higher than about 4 mm for the large cathode and about 10 mm for the smaller one. At 4-mm pressure the mean free path of 15-volt electrons is about 0.4 mm and, apparently, sufficient collisions occur near the cathode to reflect an appreciable proportion of the electrons back into it. Diffusion out through the ends of the cylinder also is possible, but less troublesome because of the greater distance. More than 92 percent of the solid angle seen from the illuminated part of the cathode is covered by the anode cylinder; obviously most of the current which gets away from the cathode region will be collected, even in weak fields.

From curves like Fig. 2 the primary current was obtained throughout the range of pressures for which it could be identified, and the ratio of total current to primary current was computed. At each pressure at least two curves were taken for each cathode, with illumination intensities differing by a factor of two or more. Agreement of the computed ratios was taken to mean that space-charge distortion of the field was negligible. A rough check on the primary current could be obtained by comparing results for the two cathodes, but differences between the two quartz windows and possibly also differences in the two cathode surfaces made it impossible always to get the same emission after moving the arc.

The upper curve of Fig. 2 is a graph of the logarithm of the current plotted against voltage, and it shows two interesting features. The sharp break at the upper end comes at a voltage only slightly lower than that necessary to start a glow discharge, and is attributed to the beginning of secondary emission at the cathode due to positive ion bombardment. The voltage at which the rise occurs is independent of the magnitude of the current and also is approximately the same no matter which cathode is illuminated. The latter fact is ascribed to diffusion of positive ions to the small cathode, causing the additional emission

[5] D. H. Hale, Phys. Rev. 56, 1199 (1939).

from it even when the original illumination was directed at the large cathode. A positive ion accidentally deflected to the small cathode would strike it with a great deal more energy than it would have had at the large one; it would therefore be more likely to cause secondary emission. Ionization in the gas by positive ions would cause a bend of the type shown in Fig. 2, but ionization by positive hydrogen ions of the energies possible here has never been observed and is therefore unlikely.

Secondary emission due to photoelectric action at the cathode by photons arising in the discharge is possible, and if present could account for the bend in Fig. 2. However, in this chamber few photons from the discharge would strike the cathode unless they were radiated after collisions very close to the cathode surface. If such photons were responsible for the bend there would be no reason for expecting it to occur at the same voltage for both cathodes, since the electron energy in the first free path, and therefore presumably the most probable character of the first collision, is greatly different.

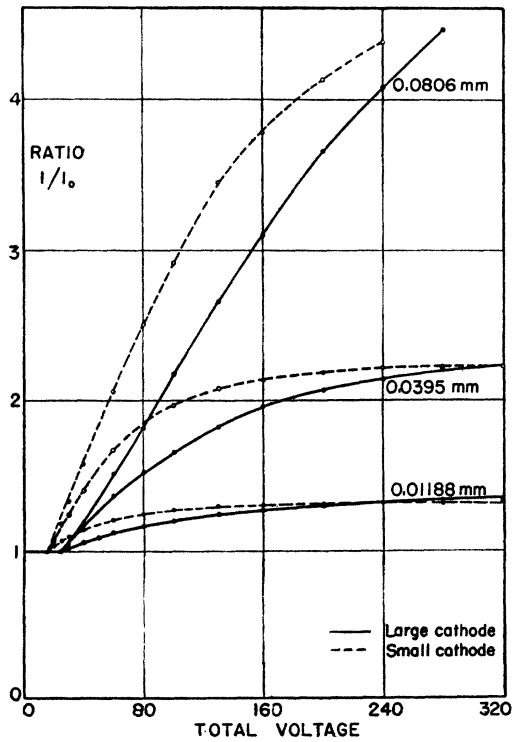

FIG. 3. Ionization ratio—lower pressures.

The linearity of the middle part of the log i curve is not easy to explain; apparently it is a fortuitous result of the particular geometry of the tube. At lower pressures the curve bends downward, and at higher pressures upward. If Townsend's equation applied accurately, the log i curve of Fig. 2 would be straight if and only if the Townsend coefficient α were directly proportional to the field intensity. Actually it increases more rapidly at low and more slowly at high intensities, but since both low and high field regions occur in this chamber a compensation effect might be proposed. It will be shown, however, that Townsend's equation is not applicable under the conditions of Fig. 2, and the theoretical explanation of the strikingly linear curve obtained near this particular pressure would require a knowledge of how the electron energy distribution in the chamber changes with voltage.

EXPERIMENTAL RESULTS

Figures 3 and 4 are typical of the results obtained. On the graphs the ratio of total current to primary current is plotted against the total voltage from the smaller cathode to the anode; the curves for the large cathode therefore show the ionization resulting when electrons are re-

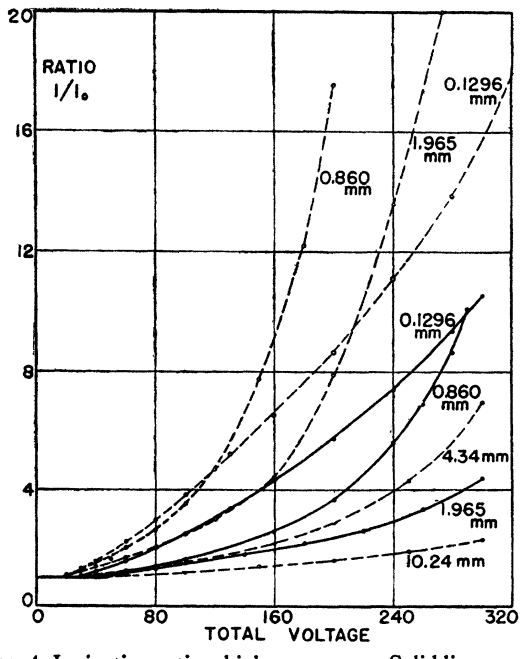

FIG. 4. Ionization ratio—higher pressures. Solid lines represent large cathode; dashed lines represent small cathode.

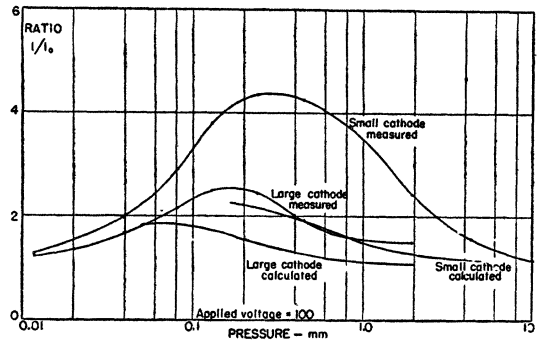

FIG. 5. Measured ionization compared with ionization calculated from Townsend equation. Applied voltage =100 v.

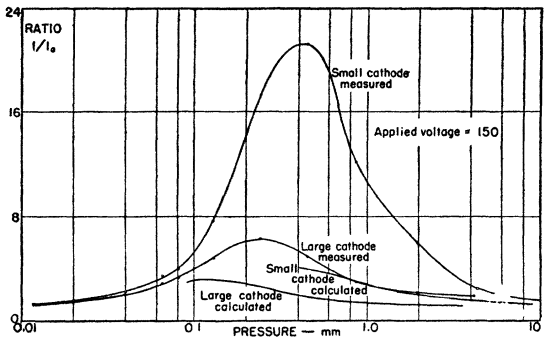

FIG. 6. Measured ionization compared with ionization calculated from Townsend equation. Applied voltage =150 v.

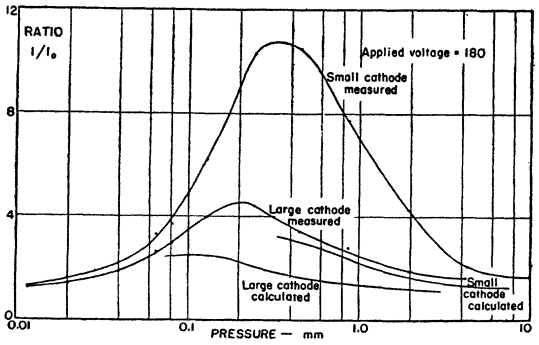

FIG. 7. Measured ionization compared with ionization calculated from Townsend equation. Applied voltage =180 v.

leased at a point in the field outside of the region where the highest field exists. Ionization by collision occurs only when the electrons have 15.4 volts of energy; when electrons start from the large cathode the total voltage must be about 25 volts before ionization begins. For this reason the curves for the smaller cathode always rise first.

Figure 3 shows that at the lower pressures and higher voltages practically as much ionization occurs when the electrons start outside of the high field region as when they fall through it. This is because the electrons, in either case, gain enough energy within a short distance from the cathode to reach their maximum ionizing efficiency, and the total ionization then depends mainly upon the number of collisions the electrons make before reaching the anode. Since the distance is less than 10 percent greater for the small cathode, the extra ionization should not exceed this amount. In fact, for the higher voltages, electrons coming from the small cathode may actually produce less ionization, because the ionizing efficiency falls for very high electron energies.

In Fig. 3 it will be noted that the plots have a downward curvature. From this it is inferred that the ionization is caused principally by primary electrons, cumulative ionization being prevented by the fact that secondary electrons are formed so far out in the weak-field region that they seldom gain enough energy and make enough collisions to ionize before they reach the anode. Harnwell[6] has shown that when high energy electrons ionize, one of the resulting pair of electrons usually has most of the excess energy, the other starting with very little. Therefore the secondary electrons must gain energy from the field before they are able to ionize. At higher pressures, as shown in Fig. 4, cumulative ionization occurs and the curves of current against voltage bend upward. At these pressures ionization occurs nearer to the cathode and the secondary electrons resulting have a much larger probability of forming new ion pairs.

From Fig. 4 it will also be noted that at the higher pressures electrons coming from the small cathode are responsible for much more ionization than those starting farther out in the field. This does not mean that the additional ionization occurs in the small region adjacent to the small cathode, but rather that the electrons starting from the small cathode have more energy and, provided they make enough collisions to expend their energy, they can form more new ion pairs. In Fig. 4 it will be noted that at very high pressures the ionization falls markedly. This is be-

[6] G. P. Harnwell, Phys. Rev. **34**, 661 (1929). See also J. T. Tate and R. R. Palmer, Phys. Rev. **40**, 731 (1932).

cause at these pressures so many collisions occur that much of the electron energy is lost in excitation before ionizing begins. The result is that the total ionization is much less than at lower pressures.

Figures 5–7 are cross plots from the experimental data, and show the ratio of total current to primary current as a function of pressure for certain constant voltages. On these graphs are also shown the results of calculating the ionization by the use of Townsend's $i=i_0e^{\int \alpha dx}$. These calculations were made by a process of numerical integration, dividing the field into a number of thin cylindrical slabs, taking the Townsend coefficient for each slab from the work of Hale, and applying Simpson's rule to find the integral. By taking very thin slabs in the part of the field which varies rapidly, the accuracy of the computation is made equal to the accuracy of the data when about ten slabs are used.

Comparison of the calculated and measured ratios on the graphs will show that the agreement is not very good for any of the pressures in the range of this experiment, and becomes very poor for pressures below 1 mm. Calculations were limited by the range of Hale's data on the Townsend coefficient to pressures above about 0.08 mm for the large cathode and about 0.4 mm for the smaller. It will be noted that at the lowest pressure for which the calculation can be made, on Figs. 6 and 7, the calculated and measured curves for the large cathode again approach agreement. It should be emphasized that this agreement is entirely coincidental; it occurs when the measured curve falls because of the few collisions made by the fast electrons before reaching the anode, whereas the calculated curve levels off when Townsend's coefficient reaches its maximum value as a function of X/p. Here it must be remembered that Townsend's coefficient can be measured only if the field is uniform and long enough to permit the electrons to reach a terminal distribution of energies characteristic of the field intensity used. Hence the approximate agreement shown by the curves for the large cathode at about 0.05- or 0.10-mm pressure on Figs. 5–7 is entirely fortuitous.

At the higher pressures, approaching 10 mm in this ionization chamber, the number of collisions is so great that the distribution of electron energies at any point in the field begins to approximate that which would be found in the corresponding uniform field. Therefore agreement with the calculations should steadily improve as the results are carried to higher pressures. Limitations of this experiment prevented reaching pressures high enough to get good agreement, but it can be stated definitely that good agreement with Townsend-equation results should not be expected for conditions which resemble those existing in this chamber for pressures less than about 10 millimeters of mercury. At this pressure 16-volt electrons in hydrogen make about 250 collisions per centimeter path, while the field intensity in this chamber changes at a maximum rate of about 630 percent per centimeter or about 2.5 percent per mean free path of electrons just able to ionize. Hence this can be used as a rough upper limit to the field-intensity variation permissible when Townsend's equation is to be used. For this chamber, at this limit, the ratio of total current to primary current was about 25 percent higher than Townsend's equation would indicate.

The conclusion is therefore inescapable that when the field intensity decreases rapidly relatively to the length of electronic mean free paths all calculations using $e^{\int \alpha dx}$ are in error, the ionization being much greater than calculated. Thus the theories of the glow discharge presented by von Engel and Steenbeck[7] and by Rogowski[8] among others, in which the field distribution, current variation with voltage, length of the dark space, and ionization in the dark space, are computed in this way are in principle entirely wrong. This explains the fact that Druyvesteyn's experiments[9] failed to show the types of variations expected, except for limited regions in certain gases. It also explains the apparent failure of Meek's criterion in sparkover theory observed by Weissler and confirmed by Fisher.[10] This criterion predicates the formation of a pre-spark streamer upon a certain density of ionization at a particular point in the discharge path. Attempts to test the criterion experimentally by calculating the ionization and measuring the sparking po-

[7] A. von Engel and M. Steenbeck, *Elektrische Gasentladungen* (Verlagsbuchhandlung, Julius Springer, Berlin, 1934), Vol. II, pp. 68–80.
[8] W. Rogowski, Zeits. f. Physik **82**, 473 (1933).
[9] M. J. Druyvesteyn, Physica **5**, 875 (1938).
[10] L. H. Fisher and G. L. Weissler, Phys. Rev. **66**, 95 (1944).

tential in known non-uniform fields will fail if the methods used for calculating the ionization are faulty.

DIRECT CALCULATION OF IONIZATION

The experimental work which has been presented shows that the calculation of ionization by the use of Townsend's equation cannot be relied upon when the field intensity varies rapidly. The reason is that the Townsend coefficient α depends upon the distribution of velocities characteristic of the field intensity in the uniform field where α is measured, and is inapplicable whenever the electrons do not have that distribution of velocities. Therefore the calculation of the ionization in a rapidly-varying field must be directed primarily at discovering what velocity distribution the electrons do have; when this is known the resulting ionization can be computed. The problem can be attacked as follows:

Consider a discharge path of the type used in this work, consisting of the field between concentric cylinders. Let ω be the electron energy, X the field intensity, r the distance from the center, and $i(\omega, r)$ the net number of electrons, having energies between ω and $\omega+d\omega$, which pass a hypothetical cylinder of radius r per second. Call class A the class of all electrons in a cylindrical slab between r and $r+dr$ which have energies between ω and $\omega+d\omega$ and which may therefore enter into the current $i(\omega, r+dr)$.

Electrons enter class A in four ways: (1) By coming from smaller r. The number of these per second is $i(\omega, r)d\omega$. (2) By acceleration in the slab from lower ω. The number of these is $i(\omega, r)eXdr$. (3) By ionizing, where the original energy was between $\omega+\omega_i$ and $\omega+\omega_i+d\omega$. Here it is assumed that only the ionizing energy ω_i is lost by the striking electron, and that the new electron starts with zero energy. (4) By exciting gas molecules, where the original energy was between $\omega+\omega_e$ and $\omega+\omega_e+d\omega$. If more than one kind of excitation is involved, ω_e will have several distinct values.

Electrons also leave class A in four ways: (1) By migration to larger r. The net number of these is $i(\omega, r+dr)d\omega$. (2) By acceleration in the slab to higher energy. The number accelerated out of class A is $i(\omega+d\omega, r)eXdr$. (3) By ionizing, falling to the energy $\omega-\omega_i$. When ionization occurs, a new electron appears, assumed to have energy zero. (4) By exciting, falling to the energy $\omega-\omega_e$.

In a steady discharge the gain to class A per second must equal the loss, making the net loss equal to zero. The net loss to migration is

$$[i(\omega, r+dr)-i(\omega, r)]d\omega=(\partial i/\partial r)d\omega dr.$$

The net loss to acceleration is

$$eX[i(\omega+d\omega, r)-i(\omega, r)]dr=eX(\partial i/\partial\omega)d\omega dr.$$

To calculate the net losses due to ionization and to excitation, let P_i be the probability of ionizing; that is, the number of ion pairs formed per unit path length per electron. P_i will be a function only of the energy of the electron. Similarly let P_e be the probability of excitation; if more than one type of excitation must be considered there will be several distinct values of P_e. Let $g(\omega, r)$ be the average value of the cosine of the angle between the direction of each electron path and the field direction; that is, $g(\omega, r)$ is the average ratio of the component of the path in the field direction to the total path, for electrons of class A. Then the average distance traveled by electrons in the slab is approximately dr/g, and the number of class A electrons which ionize is $(P_i/g)id\omega dr$, so that the net loss to class A by ionization is

$$\left[\frac{P_i(\omega)}{g(\omega)}i(\omega)-\frac{P_i(\omega+\omega_i)}{g(\omega+\omega_i)}i(\omega+\omega_i)\right]d\omega dr,$$

where only the functional variation with ω is written explicitly, but the functions with the exception of P may also vary with r. Similarly excitation causes a net loss to class A of

$$\left[\frac{P_e(\omega)}{g(\omega)}i(\omega)-\frac{P_e(\omega+\omega_e)}{g(\omega+\omega_e)}i(\omega+\omega_e)\right]d\omega dr.$$

Finally, by equating the net loss from all causes to zero, the steady-state equation is found:

$$\frac{\partial i}{\partial r}+eX\frac{\partial i}{\partial\omega}+\frac{P_i(\omega)}{g(\omega)}i(\omega)-\frac{P_i(\omega+\omega_i)}{g(\omega+\omega_i)}i(\omega+\omega_i)+\frac{P_e(\omega)}{g(\omega)}i(\omega)-\frac{P_e(\omega+\omega_e)}{g(\omega+\omega_e)}i(\omega+\omega_e)=0. \quad (4)$$

This equation is subject to the following boundary conditions: (1) The number of electrons re-

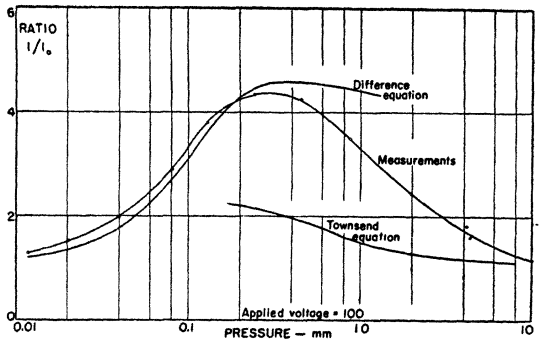

FIG. 8. Results of numerical integration of the difference equation.

leased from the cathode per second is i_0. These have a distribution of low energies which in most cases can be neglected. (2) In any cylindrical slab a number of new electrons, equal to the total number of ionizing collisions, start with zero velocity.

Equation (4) is not open to exact solution for several reasons. In the first place, it is a combination differential-difference equation, and equations of this type have only recently been successfully studied[11] chiefly by the method of Laplacian transformation which becomes complicated when the coefficients are not constant. Secondly, not much information is available on the physical functions which appear as coefficients. The probability of ionization P_i can be taken from the curves of Compton and Van Voorhis[12] or of Tate and Smith;[13] the two sets agree for low energy electrons. The probability of excitation has been measured in a few cases for certain gases; for hydrogen it is almost wholly unknown. As Brode[14] has pointed out, the probability of excitation of a particular state is not always the same as the probability that a corresponding radiation will be emitted, and data from optical experiments are not, therefore, applicable even if they were available. The function g, representing the average ratio of path in the field direction to total path, is very difficult to calculate even in uniform fields; it depends upon the angles through which colliding electrons are scattered and especially upon how much the electron paths after scattering are bent by the field. For this reason it is a function of field distribution as well as electron energy. For low pressures and high fields the ratio is high, particularly for high energy electrons which are preferentially scattered through angles near zero and 180°. For low fields the ratio will approach zero, since under pure diffusion conditions as many electrons move backward as forward.

Within the limitations of knowledge of the coefficients, however, Eq. (4) can be handled by a step-by-step method, in which the range of each of the independent variables is divided into small intervals and the density of electrons in the intervals calculated successively. Under the conditions of high field intensity and low electron density electrons do not gain energy from collisions; furthermore, in a divergent field most of the electrons found at any instant in a given part of the field will have come from points nearer the cathode rather than from collisions nearer the anode followed by backward scattering. Therefore if the electron density is calculated in steps ranging from the cathode outward and from the highest energy downward, the calculations for any interval will be affected only by the results of calculations which have already been made.

Calculations of the ratio of total current to primary current were made by a step-by-step solution of the difference equation for an applied voltage of 100 volts and a range of pressures from 0.01 to 1.28 mm. In these calculations the following assumptions were made: (1) The probability of ionization was taken from the curves Tate and Smith.[13] (2) For high energy electrons, one-half the energy losses were assumed to be caused by ionization; for energies below 30 volts the ratio was changed to one-fourth. The remaining losses were ascribed to excitation and subsequent radiation. This assumption is based upon the observation that high energy electron beams form approximately one new ion pair (requiring about 16 electron volts) for each 30 to 35 volts of energy lost by the beam. Since excitation begins at lower energies than does ionization, low energy electrons will lose proportionally more energy to excitation. (3) The function g, representing the average value of the cosine of the angle between an electron path and

[11] J. Neufeld, Proc. Camb. Phil. Soc. **30**, 389 (1934). See also A. E. Heins, Am. J. Math. **63**, 435 (1941).
[12] K. T. Compton and C. C. Van Voorhis, Phys. Rev. **27**, 724 (1926).
[13] J. T. Tate and P. T. Smith, Phys. Rev. **39**, 270 (1932).
[14] R. B. Brode, Rev. Mod. Phys. **5**, 257 (1933).

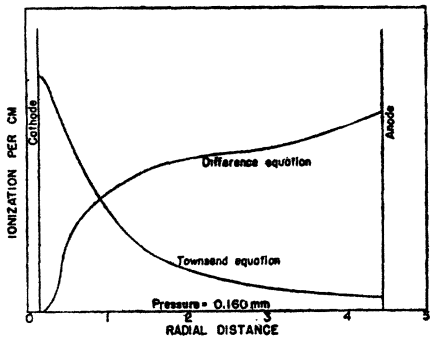

FIG. 9. Location of the ionization, as indicated by calculation.

the field direction, was assumed to be equal to $\cos \frac{1}{2}\pi(1-\omega/\omega_r)$ in which ω is the actual energy of the electrons and ω_r is the energy they would have acquired in falling freely from the cathode. This expression is empirical and admittedly open to question; it gives the right kind of variation in that it does not differ much from unity for high energy electrons which have made few collisions, but it is not very satisfactory at low energies and this may introduce large errors at high pressures.

In choosing the size of the intervals for a step-by-step calculation a compromise must be made to avoid excessive labor. In this work the energy intervals were taken equal to the ionizing energy and the distance intervals equal to the distance through which an electron must fall to gain the ionizing energy. The actual length of the distance intervals is therefore smaller in the high field region.

Figure 8 shows the results obtained by this numerical procedure. At high pressures the indicated currents are too high, in contrast to the currents computed from Townsend's equation which, for fields diverging from the cathode, gives results which are too low. At low pressures the agreement with experiment is as good as can be expected from the crude assumptions necessary.

In Fig. 9 is shown the location of the ionization, as indicated by the two types of calculation. It will be observed that the difference equation ascribes most of the ionization to the low field region near the anode (corresponding to the region of the negative glow in a glow discharge) while the Townsend equation places the ionization chiefly in the high field region near the cathode (corresponding to the cathode dark space in the glow discharge). Thus even if the total ionization were found to be the same by the two methods, there is an essential difference in the result. The existance of the negative glow in a low pressure discharge is itself a strong indication of the correctness of the difference-equation result, although not absolutely conclusive because electrons able to cause radiation of visible light need not have enough energy to ionize. A conclusive test should be possible by the use of auxiliary anodes which could be moved in to intercept the electrons at different distances from the cathode, the potentials of the various anodes being so fixed that the field is not disturbed.

In Fig. 10 are plotted the energy distributions of the electrons for several different pressures, as determined from the difference equation. Information on the energy distribution cannot be obtained from the Townsend equation, since the Townsend coefficient, which expresses the integrated effect of all the electron energies present, cannot be unscrambled. This additional information partly offsets the greatly increased labor required to calculate the current from the difference equation. It also opens the results of

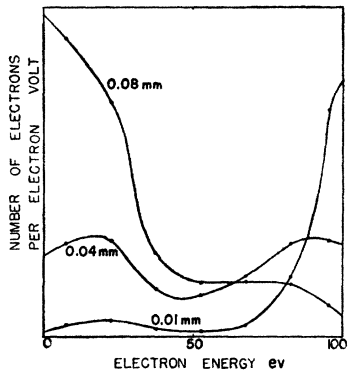

FIG. 10. Energy distribution of electrons reaching the anode, calculated by means of the difference equation.

the calculations to experimental test by another method, since at the low pressures and high fields used here it should be possible to measure directly the energy distribution by retarding-potential methods.

The author owes his thanks to Captain Leonard B. Loeb for giving much time to discussion of this work while he was on leave from his University service for active duty in the United States Navy.