fracting areas to a dark grain. The tilting was done in the plane containing the diffracted beam, i.e., the axis of tilt perpendicular to the line joining the white and dark areas for a given crystal. As far as analysis of individual diffracting areas is concerned, two cases were observed: in some, the pattern remains in position and does not change in intensity for a few consecutive positions of the foil and then disappears abruptly; others change shape and disappear gradually. The first may be interpreted as due to advanced lattice distortions in which the continuity of the lattice may have been spoiled. Their persistence over an appreciable angle is due to the imperfectly parallel x-ray beam. The other type of spots indicates a gradual continuous change of orientation of the lattice.

Naturally it was attempted to obtain thin foils of strain free copper. For that purpose a well-annealed disk of copper was electrolytically reduced to the proper thickness and polished on both sides. The resulting microradiographs are shown in Fig. 6 in two positions of the foil. There is practically no deformation present as far as comparison with the other microradiographs is concerned. A few grains show a faint criss-cross pattern indicating a slight deformation.

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Non-Linear Propagation of Underwater Shock Waves

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Comparative measurements at large and small distances have been made on the pressure wave from a small underwater explosion. It is shown that: (1) there is a small departure from the $1/r$ law for the peak pressure, (2) there is no detectable change in shape or "spreading" of the pressure wave with propagation, and (3) the shock front is discontinuous within the limitations of the apparatus. A theory is developed which is in quantitative agreement with (1) and (3) but in disagreement with (2). The theory predicts a small spreading of the profile of the wave which is large enough to have been detected. The theory shows that a spherically diverging wave of any amplitude always becomes discontinuous eventually. In practice the distance over which it must propagate before becoming discontinuous may be enormous.

INTRODUCTION

It has been known for a long time that elastic waves of finite amplitude propagate with change of shape, and in air the effect is large. In water the medium is linear (for positive pressures) over a much greater range of pressure than is air, and the effect is correspondingly smaller.

In this paper experimental data will be given on the propagation of underwater shock waves, and compared with a theory developed for the expected change of shape of a wave of finite amplitude in water. An estimate will also be made of the distance a spherically diverging wave must propagate before becoming discontinuous.

A submarine explosion (and also a subterranean explosion) emits a pressure wave with a steep (essentially discontinuous) shock front, followed by a gradually diminishing tail. The dependance of the pressure at any given point on time is approximated by $p=p_{\max}\exp(-tc_0)/\sigma(t>0)$, $p=0(t<0)$, t being measured from the time of arrival of the shock front. This disturbance propagates with the velocity of sound c_0, except very close to the explosion. σ is the "space constant," or distance from the shock front over which the wave falls to an eth of its maximum.

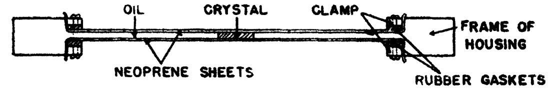

FIG. 1. Cross section of circular housing for oil-immersed tourmaline crystal pressure gauges.

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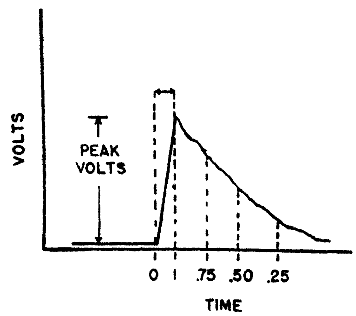

FIG. 2. Schematic oscillogram of pressure wave from explosion, showing measurements. Time of rise to peak, 0–1.

EXPERIMENTAL PROGRAM

In the experiments described below the explosions were produced by No. 6 Dupont blasting cap, with powder fuse. These are thin-walled copper cylinders containing a charge of approximately 0.3 gram of mercury fulminate and tetryl. The charge has a diameter of 0.56 cm and length 1.4 cm. Oscillograms of the pressure waves from this charge at distances from 1 ft. (30.5 cm) to 31.3 ft. (956 cm) were obtained by use of tourmaline pressure gauges, a broad-band television amplifier, and cathode-ray oscillograph. The experiments were performed in the Potomac river, at a depth of $4\frac{1}{2}$ ft. Except for the gauges, the apparatus was essentially that described in reference 1.

The gauges were of two types. (1) Single slabs of tourmaline with metal foil electrodes coated with various kinds of plastic, usually an air-dried synthetic rubber, as in reference 1. (2) Slabs and foil electrodes for which the dielectric was oil, held between two thin sheets of neoprene (thickness 0.0035 in.) as shown in Fig. 1. In this second type, the dielectric was reduced to a minimum and the results were somewhat more consistant than with the first type. This second type was intended to represent over the interval of observation ($\sim$30 μsec.) a crystal suspended in an infinite medium, with a minimum of mechanical "loading" of the crystal, other than its electrodes. The tourmaline crystals were 0.020 in. thick in all cases. The absolute value of the pressure in the explosion wave was determined from gauges calibrated in continuous waves. This calibration was in fair agreement with the theoretical calibration as computed from the piezoelectric constant of tourmaline.

Since the non-linear effects in water are quite small, it is necessary to compare oscillograms of

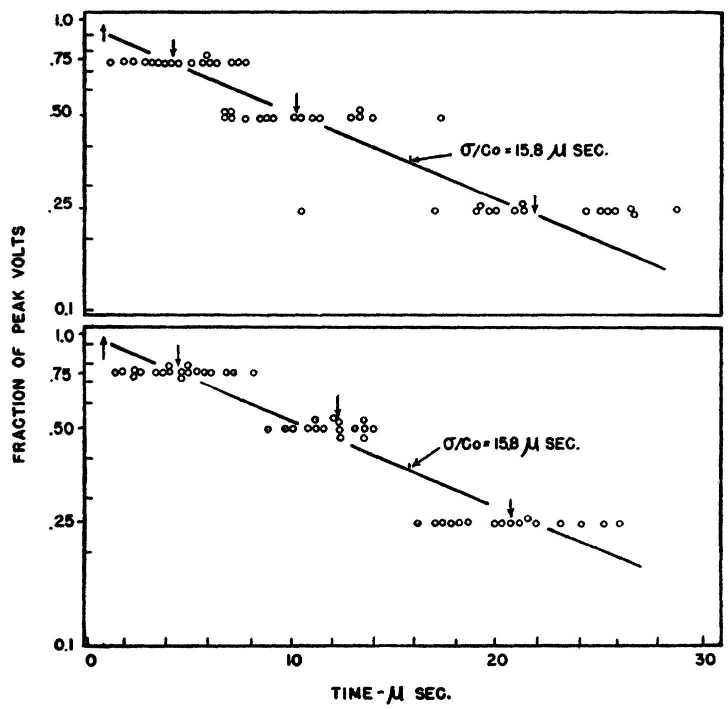

FIG. 3. Measurements of oscillograms from oil-immersed crystals. Upper curve, distance to explosion 1 ft. Lower curve 31.3 ft. Arrows indicate means.

[1] M. F. M. Osborne and S. D. Hart, "Transmission, reflection and guiding of an exponential pulse by a steel plate in water, II. Experiment," J. Acous. Soc. Am. 15, 170 (1946).

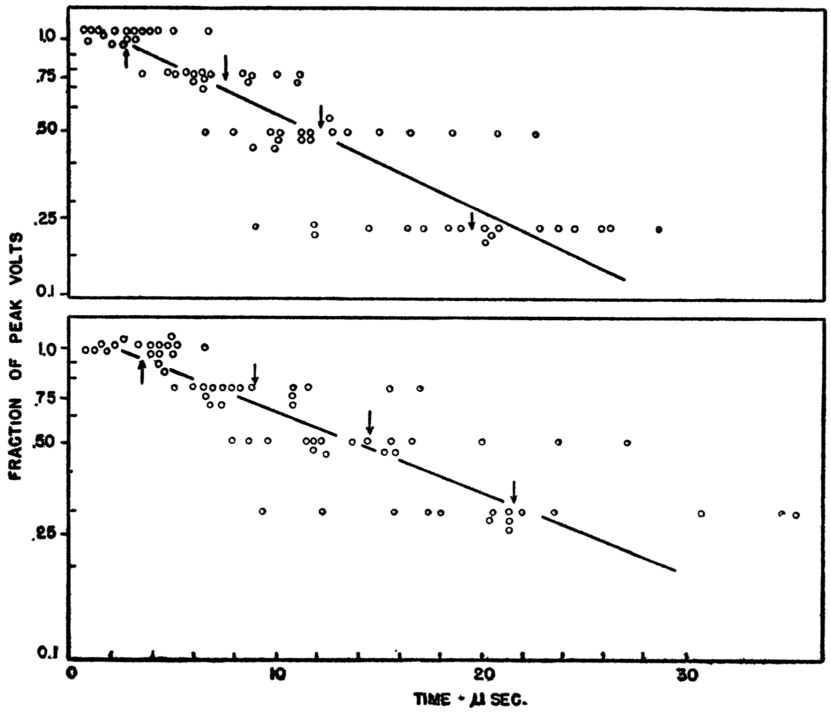

FIG. 4. Measurements of oscillograms from plastic-coated crystals. Upper curve, distance to explosion 1 ft. Lower curve, 31.3 ft. Arrows indicate means.

the explosion wave over as large a range of distance from the explosion as possible, compatible with the durability and linearity of the gauge, and the sensitivity of the recording apparatus. In most cases observations at only the two extremes of distances stated above were made since the change in shape of the explosion wave with distance is very small. In making observation, the crystal face was always oriented with the normal to its face in the direction of propagation, i.e., "face-on." The measured quantities are illustrated in Fig. 2.

EXPERIMENTAL RESULTS

The results of the measurements indicated in Fig. 2, comparing the explosion wave at 1 ft. and 31.3 ft., are shown in Figs. 3 and 4, and summarized in Tables I and II. In the case of the peak voltage measurements, the ratio $(\text{p.v.}\times\text{dist.})_{1'}/(\text{p.v.}\times\text{dist.})_{31.3'}$ is given, since for perfectly linear propagation ($1/r$ law for pressure) this ratio would be unity, and hence departures from unity indicates non-linearity. Each of the points in Figs. 3 and 4 represent the mean of 3–6 observations with a single, different gauge. The errors are standard errors, computed with each point as a single observation.

It will be observed that the dispersion of data from the plastic-coated crystals is greater than for the oil-immersed crystals. No significant

TABLE I. Theoretical and observed increase of time scale from 1′ to 31.3′.

1 Interval	2 η	3 Theoretical increase of time interval	4 Theoretical increase according to Kirkwood and Bethe	5 Observed oil-immersed crystals	6 Observed plastic-coated crystals
		$s_0'(r)(1-\eta)/c_0$ μsec.	$\left[\left(\frac{\ln r/a_0}{\ln r_0/a_0}\right)^{\frac{1}{4}}-1\right]\theta_0 \ln(1/\eta)$ μsec. $\theta_0=\sigma/c_0$	μsec.	μsec.
0–1	1	0	0	$+0.12\pm0.17$	$+0.8\pm0.5$
0–$\frac{3}{4}$	$\frac{3}{4}$	$+1.3$	$+1.4$	-0.3 ± 0.5	$+1.8\pm1.0$
0–$\frac{1}{2}$	$\frac{1}{2}$	$+2.6$	$+3.5$	$+2.1\pm1.0$	$+2.1\pm1.7$
0–$\frac{1}{4}$	$\frac{1}{4}$	$+3.9$	$+7.0$	-0.3 ± 1.4	$+2.1\pm2.3$

change in time scale with distance is indicated by the latter data; the former show a very small increase both in the time scale and time of rise measurements at 31.3 ft. over that at 1 ft.

Figure 5 indicates, as might be expected, for a discontinuous shock front, that the time of rise is primarily an instrumental effect. Since the crystal slabs were of irregular shape, the square root of the area was taken as a typical dimension. The time of rise increases with the square root of the area, probably because of imperfect orientation of the gauges "face-on." The extrapolated time of rise for crystals of zero area is 0.4 μsec., which is of the same order of magnitude as the sum of the time constant of the amplifier ($\sim$0.1 μsec.) and the time required for sound to traverse the thickness of the crystal (0.1 μsec.). The theoretical limitations of the apparatus are thus very nearly reached. The data of Fig. 5 are from the oil-immersed crystals only, at both distances. The plastic-coated crystals indicated a considerably longer time of rise (Fig. 4) than the oil-immersed crystals, perhaps caused by a cushioning of the shock wave in the plastic dielectric.

In view of the results of Fig. 5, showing that the shock front is truly discontinuous, at least a part of the increase with distance of time scale for the plastic-coated gauges must be instrumental. This part is that due to the increase in the time of rise, $+0.8\pm0.5$ μsec.

The peak voltage ratios indicate a small but definite departure from the $1/r$ law. The peak pressure at 1 ft. is $\sim$30 percent larger than its value as given by the $1/r$ law and the observed value at 31.3 ft.

In summary therefore the observations indicate (1) practically no spreading of the explosion wave with propagation, (2) the peak pressure falls off slightly faster than $1/r$, and (3) a shock front which is discontinuous within observational accuracy.

TABLE II. Theoretical and observed values of $(\text{p.v.}\times r)_{1'}/(\text{p.v.}\times r)_{31.3'}$.

1	2	3	4
Theoretical Eq. (15) first factor	Theoretical Eq. (19) Kirkwood and Bethe	Observed oil-immersed crystals	Observed, plastic-coated crystals
1.39	1.34	1.31 ± 0.04	1.25 ± 0.03

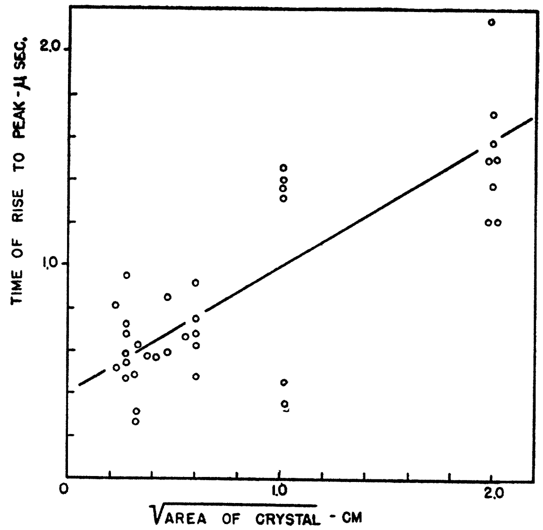

FIG. 5. Time of rise of shock front *vs.* linear dimension of crystal. Oil-immersed crystals.

THEORY. FORMATION OF SHOCK FRONTS

In order to obtain a theoretical estimate of the variation of peak pressure and time scale with distance it is first necessary to determine over what distance an initially continuous wave of finite amplitude must propagate before becoming discontinuous, and second, what changes in shape thereafter obtain. According to the Riemann theory for *plane* shock waves, points of constant pressure propagate with the velocity $(c+u)$, where c is the local velocity of sound for the pressure in question, and u the particle velocity. In order to estimate $c+u$, one can approximate the dependence of c on pressure by a Taylor expansion, and use the acoustic approximation for u. This is done as follows. If the density is a function of the pressure alone, the local velocity of sound is given by

$$c^2=dp/d\rho=(dp/d\rho)_{\rho=\rho_0}+(\rho-\rho_0)(d^2p/d\rho^2)_{\rho=\rho_0}, \quad (1)$$

where $\rho=\rho_0$ at $p=0$. Let $\rho-\rho_0=\Delta\rho$ corresponding to $p-0=\Delta p$ and $(dp/d\rho)_{\rho=\rho_0}=c_0^2$. Then (1) can be written

$$c=c_0[1-(c_0^2/2)(d^2\rho/dp^2)_0\Delta p]=c_0(1+k'\Delta p). \quad (2)$$

The value of k' from various estimates (experiment, second differences of ρ *vs.* p data) will be taken as 1.1×10^{-10} c.g.s. units. From the acoustic approximation u is $\Delta p/\rho c_0$ so that finally

$$c+u=c_0(1+1.6\cdot10^{-10}\Delta p)=c_0(1+k\Delta p). \quad (3)$$

Consider two points of given pressure p_A, p_B at points on the front side of a *plane* wave of finite amplitude, separated by a distance Δs, Fig. 6. The rate of change of Δs at any instant is given by

$$d(\Delta s)/dt=(c+u)_A-(c+u)_B=-kc_0\Delta p. \quad (4)$$

If one makes the change of variable $r=c_0t$ then

$$d(\Delta s)/dr=-k\Delta p. \quad (5)$$

If in addition one makes the assumption that, for spherically diverging waves, Δp is given by its acoustic approximation $\Delta p=\Delta p_0r_0/r$ where Δp_0 is the value at some given distance r_0 from the explosion, then

$$d(\Delta s)/dr=-k\Delta p_0r_0/r, \quad (6)$$
$$\Delta s=-k\Delta p_0r_0\ln(r/r_0)+\Delta s_0. \quad (7)$$

Δs_0 is the value at $r=r_0$. It should be noted that in the Riemann theory for *plane* shock waves the points A and B on the profile are *identified* as points of the same pressure as the wave progresses. In the case of spherically diverging waves, they are identified initially at a particular distance r_0 and subsequently identified at pressures diminished by r_0/r—this constituting an approximation.

Equation (7) can be used to make an interesting estimate of the distance over which a wave of finite slope on its front propagates before becoming discontinuous. Let p_0 be the height of the peak at distance r_0 from the source and Δs_0 the distance of rise when the wave is at r_0

$$r/r_0=\exp\left[-(\Delta s-\Delta s_0)/kp_0r_0\right]. \quad (8)$$

At $\Delta s=0$,

$$r_{\text{disc}}=r_0\exp\left[\Delta s_0/kp_0r_0\right], \quad (9)$$

r_{disc} being the distance at which the wave front becomes discontinuous. Thus it seems that a spherically diverging wave always becomes discontinuous eventually, though it can be readily verified that, for ordinary acoustic amplitudes, r_{disc} is of more than astronomical magnitude. Evidently the critical quantity which determines whether or not a given wave will become dis-

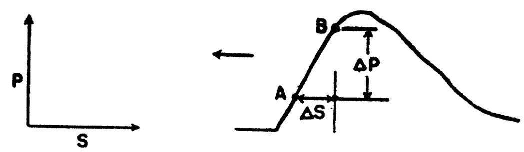

FIG. 6. Formation of shock fronts in a continuous wave of finite amplitude.

continuous in any reasonable stretch of propagation is the exponent in Eq. (9). If the exponent is of the order of or less than unity, the wave front becomes discontinuous after an interval of propagation not much greater than r_0. However if the exponent is very much greater than unity, the distance at which the wave front becomes discontinuous is a perfectly enormous factor times r_0. Two examples will illustrate the force of these statements.

By use of figures commonly realized in laboratory practice, a one-megacycle wave of double amplitude 10^3 dynes/cm² at a distance 1 ft. (30 cm) from the source ($p_0=10^3$, $\Delta s_0=0.075$ cm, $r_0=30$ cm) will become "sawtoothed" at $r_{\text{disc}}\simeq10^{6800}$ cm, a colossal distance.

On the other hand, for an explosion wave, it would seem reasonable to use as the initial thickness of the shock front an average dimension of the charge. It is over such a distance that the pressure difference at the shock front must exist at the instant of detonation. For r_0 one can also use this same average dimension as the "distance from the center of the explosion." A lower limit to the corresponding p_0 can be obtained by extrapolating backward using the $1/r$ law and some *observed* pressure at a safe distance from the explosion. The observed pressure at a distance of 1 ft. was 71×10^6 dynes cm, the equivalent spherical radius of the cap (see below) was 0.436 cm. Thus the distance from the cap at which the explosion wave must develop a shock front is approximately

$$r_{\text{disc}}=0.436e^{1.28}=1.56\text{ cm}.$$

The actual value must be less than this, since the pressure obtained from the $1/r$ law was a lower limit.

If instead of the $1/r$ law, one assumes a $1/r^n$ law, $n>1$, Eq. (6) can be integrated to show that a wave front may or may not become discontinuous, depending on the initial amplitude.

CHANGE OF PROFILE WITH PROPAGATION

There is nothing in the derivation of Eq. (6) which requires that Δp_0 or Δs be restricted to small values, so long as the Taylor expansion for $c+u$ is good. One can write therefore

$$s=-kp_0r_0 \ln (r/r_0)+s_0. \tag{10}$$

In this expression the distance coordinate s in the profile is expressed in terms of p_0, s_0, and r. This is a type of Lagrangian solution. p_0 and s_0 are initial conditions, r is a field variable. For the particular case of the explosion wave about to be discussed, p_0 is given as a function of s_0 so that there are only two independent variables, s_0 and r.

For an explosion wave, $p_0=P_0 \exp(-s_0/\sigma)$, where P_0 is the maximum pressure when the wave has traveled to a distance r_0 from the explosion. σ is the "space constant" observed at this distance r_0, i.e., it is the distance from the shock front ($s_0=0$) over which the pressure wave profile falls to an eth of its maximum. Then

$$s=-kP_0 \exp(-s_0/\sigma)r_0 \ln (r/r_0)+s_0. \tag{11}$$

This expression indicates that s becomes negative for $r>r_0$ and s_0 small. s is measured from the real shock front, negative values of s refer to those "points" (*identified* as above) near the shock front or head of the wave when it was at r_0 which have progressed forward through the shock front. Another way of expressing this is that when the wave is at $r>r_0$, the shock front has "eaten back" to a point ($s=0$) corresponding to a point $s_0=s_0'$ on the wave when it was at $r=r_0$, i.e., $s=0$ at $s_0=s_0'$ in Eq. (11). The value of s_0' can give an estimate of the departure of the peak pressure from the $1/r$ law. To determine s_0':

$$\begin{aligned} s(s_0')&=0\\ &=-kP_0 \exp[-s_0'/\sigma]r_0 \ln (r/r_0)+s_0'. \end{aligned} \tag{12}$$

For values of $s_0'\ll\sigma$ (numerical estimates verify such to be the case), one can expand the exponential factor and get

$$s_0'(r)=kP_0r_0 \ln (r/r_0)/[1+kP_0r_0 \ln (r/r_0)/\sigma]. \tag{13}$$

Note that s_0' is a function of r.

It is now desired to determine the way in which observed time intervals 0–$\frac{3}{4}$, 0–$\frac{1}{2}$, 0–$\frac{1}{4}$ should vary with distance, on the basis of the above theory. By initial assumption the pressure at any point (identified as above) in the pressure wave in terms of its distance s_0 from the shock front when the wave was at r_0 is

$$p=P_0r_0 \exp(-s_0/\sigma)/r. \tag{14}$$

Since $s_0\equiv s_0'(r)+s_0-s_0'(r)$,

$$\begin{aligned} p=(P_0r_0 &\exp[-s_0'(r)/\sigma]/r)\\ &\times(\exp[-(s_0-s_0'(r))/\sigma]). \end{aligned} \tag{15}$$

Again note that the independent variables are s_0 and r. The first factor in Eq. (15) represents the variation of the peak pressure with distance, the second the shape of the profile behind the real shock front $[s_0>s_0'(r)]$. The corresponding values of s_0, $s_{0\eta}$ at which the pressure has fallen to a specified fraction $\eta=\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ of the value at the peak are therefore given by solving

$$\exp[-[s_{0\eta}-s_0'(r)]/\sigma]=\eta \tag{16}$$

for $s_{0\eta}$. The solution is

$$s_{0\eta}=s_0'(r)-\sigma \ln \eta. \tag{17}$$

The values of s, s_η corresponding to $s_{0\eta}$, therefore are, by use of (11) and (12),

$$\begin{aligned} s_\eta=-kP_0 &\exp[-[s_0'(r)-\sigma \ln \eta]/\sigma]\\ &\times r_0 \ln (r/r_0)+s_0'(r)-\sigma \ln \eta, \end{aligned} \tag{18}$$

$$s_\eta=s_0'(r)(1-\eta)-\sigma \ln \eta$$

$-\sigma \ln \eta$ is the distance interval 0–$\frac{3}{4}$, 0–$\frac{1}{2}$, 0–$\frac{1}{4}$ as measured at $r=r_0$ so that $s_0'(r)(1-\eta)$ is the *increase* in these intervals measured at r over those measured at r_0. The corresponding values of the time are $s_0'(1-\eta)/c_0$. If $r_0=1$ ft., $r=31.3$ ft., $P_0=71\times10^6$ dynes/cm², $\sigma=2.34$ cm corresponding to a time constant σ/c_0 as indicated on Fig. 3 then the values of the time intervals $s_0'(1-\eta)/c_0$ are as given in Table I, column 3.

As previously stated, the observed values indicate that the spreading is zero, or at least a good deal smaller than that required by theory. Part of the increase indicated by the plastic coated gauges is instrumental. If the intervals 1–$\frac{3}{4}$, 1–$\frac{1}{2}$, 1–$\frac{1}{4}$ are used, i.e., if the interval 0–1 or time of rise is subtracted from the observed values above, the observed spreading is even less. This corresponds to using the peak, rather than foot, of the shock wave as origin of time interval measures, or subtracting 0.8 ± 0.5 from all the entries in the last column.

No explanation could be found for this disagreement between theory and observation for the spreading. All known or imagined sources of experimental error would have added to rather than diminished this spreading, and would have also affected the peak volts. Air around the crystal was a potent source of false spreading.

DEPARTURE OF PEAK PRESSURE FROM $1/r$ LAW

The factor $P_0 r_0 \exp[-s_0'(r)/\sigma]/r$ in Eq. (15) gives the variation of the peak pressure with distance. The numerical predictions from this factor for the ratio (peak volts$\times$dist.)$_{1'}$/(p.v.$\times$dist.)$_{31.3'}$ agree fairly well with the observations, as indicated in Table II. σ, P_0, r_0, and r take the same values as before.

COMPARISON WITH OTHER THEORIES

J. G. Kirkwood and H. A. Bethe of Cornell have derived a theory (as yet unpublished) which gives the variation of the pressure with time, measured from the shock front, and with distance r from the explosion. They give

$$p = \text{const.}_1\, r_0 \exp[-t/\theta(r)]/r[\ln(r/a_0)]^{\frac{1}{2}}, \quad (19)$$

where

$$\theta(r) = \text{const.}_2\, [\ln(r/a_0)]^{\frac{1}{2}}.$$

a_0 is the equivalent spherical radius of the charge taken as $(3V/4\pi)^{\frac{1}{3}}$ the volume V being measured as 0.346 cm^3 for a No. 6 cap.

The numerical predictions of this formula are given in column 4 of Table I and column 2 of Table II. It will be seen that it predicts slightly larger values for the time scale increase and slightly smaller values for the peak volts ratio. Mathematically the two expressions for the peak pressure are practically equivalent as can be shown by expanding them in powers of $\ln(r/r_0)$. They agree to the first and second order of small quantities, and do not differ appreciably in the third. Equation (15) gives, where $a = kP_0 r_0/\sigma$,

$$\begin{aligned} p_{\max} &= P_0 r_0 \exp[-s_0'(r)/\sigma]/r \\ &= (P_0 r_0/r)\{1 - a\ln(r/r_0) \\ &\quad + (3a^2/2)[\ln(r/r_0)]^2 \\ &\quad - (13a^3/6)[\ln(r/r_0)]^3 + \cdots\}. \end{aligned} \quad (20)$$

On the other hand, Eq. (19) gives

$$\begin{aligned} p_{\max} &= c_1 r_0/r[\ln(r/a_0)]^{\frac{1}{2}} \\ &= c_1 r_0/r[\ln(r_0/a_0)]^{\frac{1}{2}} \\ &\quad \times\{1 - (1/2)[\ln(r/r_0)/\ln(r_0/a_0)] \\ &\quad + (3/8)[(\ln(r/r_0)/\ln(r_0/a_0)]^2 \\ &\quad - (15/48)[\ln(r/r_0)/\ln(r_0/a_0)]^3\}. \end{aligned} \quad (21)$$

Note that by (19), $c_1/[\ln(r_0/a_0)]^{\frac{1}{2}} = P_0$, the peak pressure at $r = r_0$. Let $2\ln(r_0/a_0) = b$, and (21) becomes

$$\begin{aligned} p_{\max} &= (P_0 r_0/r) \\ &\quad \times\{1 - \ln(r/r_0)/b + (3/2b^2)[\ln(r/r_0)]^2 \\ &\quad - (15/6b^3)[\ln(r/r_0)]^3 + \cdots\}. \end{aligned} \quad (22)$$

Thus it is seen that if one identifies the two constants $a = 1/b$, the form of the two expressions (22) and (20) including term of the second order in $\ln(r/r_0)$ are the same and they differ only by about 15 percent in terms of the third order.