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Theory of a Microwave Spectroscope*

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The measurement of the absorption of centimeter radiation in water vapor described by Becker and Autler (preceding paper) is based on the assumption that the thermocouple readings are proportional to the Q of the cavity and its contents. The conditions for this are investigated theoretically. An expression for the Q of a hole in the wall of the cavity is derived for use in the measurements of Q's on an absolute basis.

INTRODUCTION

IT is to be expected that the wartime development of strong microwave sources will greatly increase the opportunities for spectroscopic studies in the wave-length range about one centimeter. Besides the physical interest in problems of molecular structure, the absorption of microwaves in atmospheric gases is of basic importance in radar and communication applications.

Among the methods which are available for such studies are the following: (1) Absorption of the radiation from a monochromatic source in a relatively short path of highly absorbing vapor. This method was used by Cleeton and Williams[1] in their study of the ammonia inversion. The fine structure due to the various rotational levels was not resolved due to the large pressure broadening. (2) Absorption in a long atmospheric path of the continuous microwave radiation emitted by the sun (Southworth[2]). (3) Measurement of the thermal radiation emitted by a long column of atmospheric gases (Dicke[3]). (4) Field studies using the atmosphere. Either point-to-point or reflected signalling is used (Bender[4]). (5) Molecular beam and other resonance methods.[5] (6) Measurements of attenuation of a given density of absorbing vapor in a wave-guide (Beringer[6]). (7) Use of a tuned resonant cavity. This method has been used by Bleaney[7] and others to resolve the ammonia fine structure under reduced pressure. (8) Reverberation time measurements in a large untuned cavity or echo box. (9) Steady-state response of a large untuned cavity. The last two methods will be discussed in this paper.

The analogs of some of these methods have been used in acoustics in order to measure the absorption of sound in gases.[8,9] A recent survey article "Sound Waves in Rooms" by Morse and Bolt[10] will be found helpful in this connection.

The problem proposed to the Columbia Radiation Laboratory was a determination of the wave-length dependence of the absorption of centimeter radiation in atmospheric water vapor. An account of the experimental procedure and results is given in the accompanying paper by Becker and Autler.[11] The first method suggested was a measurement of the exponential decay of the radiation in an untuned echo box between pulses. The attenuation coefficient could then be calculated from the time constant of the decay. This method was soon abandoned on account of difficulties which will be discussed in

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[1] C. E. Cleeton and N. H. Williams, Phys. Rev. **45**, 234 (1934).

[2] G. C. Southworth, J. Frank. Inst. **239**, 285 (1945).

[3] R. H. Dicke, Radiation Lab. Report No. 787 (1945). R. H. Dicke, E. R. Beringer, R. L. Kyle, and A. B. Vane. Radiation Lab. Report No. 1002, (1946).

[4] R. W. Bender, "Atmospheric Attenuation at K-Band," Radiation Lab. Report No. 41, (10/21/44).

[5] I. I. Rabi, S. Millman, P. Kusch, J. R. Zacharias, Phys. Rev. **55**, 526 (1939). Also recent resonance methods developed by F. Bloch, W. W. Hansen, and M. Packard; by E. M. Purcell, R. V. Pound, and H. C. Torrey; and by A. Roberts, Y. Beers, and A. G. Hill.

[6] E. R. Beringer, "The Absorption of ½ Cm. e.m. Waves in O_2." Radiation Lab. Report No. 684 (1945). Also, W. D. Hershberger, J. App. Phys. **17**, 495 (1946).

[7] B. Bleaney and R. P. Penrose, Nature **157**, 339 (1946). Also W. E. Good, Phys. Rev. **69**, 539 (1946) and C. H. Townes, Bull. Am. Phys. Soc. **21**, No. 3, page 15 (1946).

[8] V. O. Knudsen, J. Acous. Soc. Am. **3**, 126 (1931), **5**, 112 (1933), Method 8.

[9] V. O. Knudsen, J. Acous. Soc. Am. **7**, 249 (1936). V. O. Knudsen and E. F. Fricke, J. Acous. Soc. Am. **10**, 89 (1938), Method 9.

[10] P. M. Morse and R. H. Bolt, Rev. Mod. Phys. **16**, 69 (1944).

[11] G. E. Becker and S. Autler, Phys. Rev. **70**, 300 (1946).

Appendix 1. The method ultimately chosen was the determination of the steady-state response of the resonator. In view of the object of the experiment, this cavity was filled mostly with air at atmospheric pressure. However, for more fundamental research in which pressure broadening should be kept at a minimum, it would be possible to design a cavity, perhaps somewhat smaller, which could be evacuated of all gaseous substances except the desired density of absorbing vapor. It would appear that this method is one of the most sensitive ones available for a study of weak radiative transitions under conditions of small pressure broadening. It would be limited on the short wave-length side by the need for a strong source of nearly monochromatic power, and on the high wave-length side by the difficulty in constructing a cavity whose linear dimensions must be perhaps a hundred wave-lengths.

In the experiments of Becker and Autler,[11] pulsed magnetron power is fed into the cavity and a measure of the space-time average energy level in the cavity is obtained by the sum of the electromotive forces generated in a large number[12] of thermocouples placed "at random" in the box. As described below, such measurements can be used to obtain the attenuation of the radiation in the water vapor. The theory of the measurement in acoustical terms dates back to Sabine.[13] A derivation of the fundamental equation, based on elementary considerations and stated in microwave terminology, is given in the text. Use is made of a photon argument. A more rigorous treatment of the subject based on Maxwell's equations is given in Appendix 2.

DERIVATION OF FUNDAMENTAL EQUATION USING PHOTON MODEL

The success of the method depends on the proportionality of the thermocouple response to the "Q" of the cavity and its contents. As follows from the wave acoustic theory of Morse and Bolt,[10] a necessary condition for this is that all of the modes of the cavity which are excited should have essentially the same degree of excitation and of absorption. They obtain correction formulas for the case in which the modes can be classified as oblique, tangential and axial. We can see the need for such corrections even within the framework of the classical theory.

We consider that the radiation in the vessel at any time may be replaced by photons. To represent the different classes of modes, these may be of several types which may be distinguished by a subscript k. Let there be $N_k(t)$ photons of type k at time t. Let $M_k(t)$ be the rate of creation of photons by the source. The photons will eventually be absorbed either by the walls or in the gas. The rates of these two kinds of absorption will be proportional to the density of photons, and to the total wall area S or to the volume V of the cavity. The following differential equations may now be written

$$\frac{dN_k}{dt} = M_k(t) - \alpha_k S N_k - \beta_k V N_k, \tag{1}$$

where the α_k, β_k are constants of proportionality. It is convenient to introduce the Q's for wall and gas losses using the definition derived from circuit theory

$$\frac{1}{Q} = \frac{\text{rate of loss of photons}}{\omega\ (\text{number of photons})} \tag{2}$$

for each photon type and for each kind of absorption. Then

$$\alpha_k S = \omega_k / Q_B(k), \tag{2a}$$

$$\beta_k V = \omega_k / Q_V(k). \tag{2b}$$

The differential Eq. (1) are then

$$dN_k/dt + \omega_k N_k / Q(k) = M_k(t), \tag{3}$$

where

$$1/Q(k) = 1/Q_B(k) + 1/Q_V(k) \tag{4}$$

defines the resultant Q for photons of type k. We must expect that the Q for the wall loss may depend on the mode type, while that for the gas loss is much more nearly independent of k. The frequency variation of either type of loss may be neglected for the band width associated with the pulses.

By averaging the differential Eq. (3) over a number of pulses, we obtain

$$\bar{N}_k = \frac{Q(k)}{\omega_k} \langle M_k(t) \rangle_{\text{Av}}. \tag{5}$$

[12] See Appendix 4.
[13] W. C. Sabine, *Collected Papers on Acoustics* (Harvard University Press, Cambridge, Massachusetts, 1922), p. 43.

Averaging Eq. (5) for all k, we find

$$\bar{N}=\frac{1}{n}\sum_{k=1}^{n}\bar{N}_k=\frac{1}{n}\sum_{k=1}^{n}\frac{Q(k)\bar{M}_k}{\omega_k}, \tag{6}$$

where n is the number of modes with appreciable excitation. If all the excited modes have the same rates of excitation and of absorption as well as frequency, we obtain

$$\bar{N}=(Q/\omega)\bar{M}, \tag{7}$$

i.e., the almost obvious result that the mean level of energy excitation of the box is $Q/2\pi$ times the average energy introduced per cycle, irrespective of the shape of the pulse or the repetition rate.

Without this direct proportionality, actual or sufficiently approximate, the experiment cannot be made to give accurate values for Q_V and hence for the attenuation in the gas unless very detailed assumptions are made about the values of $Q_B(k)$ and $\bar{M}_k$ for the various modes. The presence of thermocouples, dew point apparatus, wall irregularities, etc., make these values impossible to find. Fortunately, the proportionality of the thermocouple reading to the resultant Q is to some extent capable of direct experimental confirmation. As described by Becker and Autler, this can be done by introduction of a variable known loss. We will therefore treat all classes of photons alike in the subsequent discussion.

IDEALIZED FORM OF EXPERIMENTS

The experiment consists in principle of a comparison of the small probability of absorption of a photon at each collision with the wall[14] and the small probability of absorption by the gas during each crossing of the box. The thermocouple reading for constant average power input and varying amounts of an absorber in the box is proportional to the Q of the box and its contents. To idealize, let us imagine that we could obtain a thermocouple reading for the box with all water vapor removed. In this case, the steady state would be given by

$$\bar{N}_1=Q_B\bar{M}/\omega. \tag{8a}$$

Then we would introduce such a density of water vapor as to cut the thermocouple reading in half. The steady state would then be

$$\bar{N}_2=Q\bar{M}/\omega=\tfrac{1}{2}\bar{N}_1, \tag{8b}$$

whence $Q=\frac{1}{2}Q_B$ and hence $Q_B=Q_V$. If Q_B were known, so would be Q_V. The attenuation coefficient of the radiation in the water vapor at the wave-length used could then be determined by the following argument. If all other losses were negligible, the radiation would decay accordingly to the law $\exp(-\omega t/Q_V)$. Since in time t the photons traverse a total distance of $x=ct$, the exponential factor may be written as $\exp(-\omega x/cQ_V)$ so that cQ_V/ω is the mean free path for photons in the absorbing vapor. The same result also follows from the wave theory of Appendix 3. Hence a value for Q_B is needed.

The walls are largely of copper, and it is possible to derive a theoretical value of the Q for a very large cavity with metallic walls (Appendix 5). Unfortunately, there are a number of obstacles to the use of this theoretical result for the determination of the Q's on an absolute basis, i.e., the presence of other lossy substances in the box: solder, flux, thermocouples, glass, dew point apparatus, perspiration, oxygen, etc. Also the attenuation in copper wave guide for 1.25 cm is believed to disagree with theory by something like 10 percent. Hence an absolute measurement of Q_B is required. One method, of course, would be a measurement of the reverberation time, but the gurgle phenomenon interferes (Appendix 1). Another way, used in the corresponding acoustical problem,[15] is to make a hole in the wall of the box. If no reflecting material is outside the hole, all of the photons striking its area will escape from the box and lower the resultant Q as measured by the thermocouples. We may accordingly define a Q_A for the hole, and write for the total Q

$$1/Q=1/Q_B+1/Q_V+1/Q_A. \tag{9}$$

Suppose we again start with a dry box and measure the thermocouple response. We then introduce a hole into the side of the box, and increase its size until the thermocouple e.m.f. is halved. For this hole size, $Q_B=Q_A$. If Q_A can be calculated, we can determine Q_B and hence Q_V and

[14] The probability of absorption for normal incidence on copper is 4.3×10^{-4} at 1.25 cm.

[15] W. C. Sabine, reference 13, page 24.

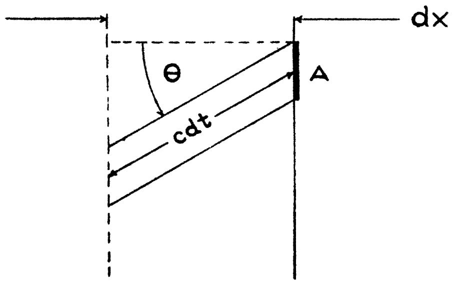

FIG. 1. Figure used in calculation of Q_A.

the attenuation coefficient for the water vapor at the wave-length used.

DERIVATION OF EXPRESSION FOR Q_A

The photon model offers a very simple derivation for the value of Q_A. According to Eq. (2) it is necessary to calculate the number of photons which escape through the hole per second. To do this, we make the usual assumptions of the ideal kinetic theory of gases. Consider those photons moving in a direction which will hit the area A of the hole in a very short time dt. These will all be found in a prism whose base area is A, and whose height is $dx = c\cos\theta dt$. (See Fig. 1.) The density of photons, and their angular distribution of velocities, may be taken as uniform in the body of the vessel, but near the hole these quantities may be quite seriously distorted by the presence of the hole, since many molecules escape which would otherwise be reflected. This is the feature which makes difficult the problem of the efflux of a real gas through an aperture whose dimensions are comparable to the mean free path. However, we must sum only over the photons which are moving toward the hole, and in the present case, these have been moving in a straight line since their last collision with a distant part of the wall or a molecule of the absorbing gas. Provided only that $A \ll S$, we may assume uniformity of their angular distribution of velocities for

$$0 \leqslant \theta \leqslant 90°$$

and a value for the number of photons per unit volume moving between θ and $\theta + d\theta$ of

$$\frac{\bar{N}}{4\pi V} 2\pi \sin\theta d\theta.$$

(The chance that a photon is absorbed in any one trip across the box is too small to influence the angular distribution appreciably.) The number of photons escaping through A per second is then

$$\frac{\bar{N}Ac}{2V}\int_0^{\pi/2} \cos\theta \sin\theta d\theta = \frac{NAc}{4V},$$

and hence

$$Q_A = \frac{\omega\bar{N}}{(\bar{N}Ac/4V)} = \frac{8\pi V}{\lambda A}. \tag{10}$$

A calculation using the wave picture gives the same answer.

A necessary condition for the validity of this derivation is that a sufficiently large and representative selection of the modes of the cavity should be excited. Otherwise, one might find Eq. (10) in error, as for example, if the source were so directional that an appreciable fraction of the energy were to escape without many reflections. Another condition is that the dimensions of the window should be large compared to a wave-length so that diffraction effects can be neglected. The limiting case $A \ll \lambda^2$ has been treated rigorously by Bethe[16] who shows that very small holes radiate considerably less than is given by Eq. (10). On the other hand, if the area A is too large, the assumptions regarding uniformity of the photon distribution will be invalid. The dependence of $1/Q_A$ on the area A of the hole should be linear over the greater part of the range $0 < A < V^{\frac{2}{3}}$, with oscillations for $A \sim \lambda^2$, and a gradual change of slope from the law of Eq. (10) for large A. In practice, only the intermediate straight part of the curve is likely to be obtained because (1) a small hole gives a very small change in thermocouple e.m.f., and (2) a large hole would ruin the box.

APPENDIX 1. FLUCTUATIONS DURING REVERBERATION

In the decay-time method mentioned in the text, some of the r-f was taken out of the box through a wave guide and detected by a superheterodyne receiver. The video-frequency output of this was fed into an oscilloscope. According

[16] H. A. Bethe, Phys. Rev. 66, 163 (1944).

to one authority,[17] the observed decay curve should look something like that shown in Fig. 2 with the relative amplitude of the "squiggles" given by $1/n^{\frac{1}{2}}$.

The number of modes n appreciably excited by the pulse of duration Δ is given by

$$n = 16\pi V/\nu\Delta\lambda^3.$$

For the box used at Columbia, $\nu = 2.4\times 10^{10}$ sec^{-1}, $\lambda = 1.25$ cm, $\Delta = \frac{1}{4}\times 10^{-6}$ sec, $V = 8'\times 8' \times 8\frac{1}{2}'$, $n = 66{,}100$ and the relative amplitude should be 0.4 percent. This would be increased somewhat by degeneracy of the mode spectrum, and by variations in the Q's of the different modes, but the discrepancy with the observations (100 percent fluctuations) seems to great.

On the other hand, R. C. Jones[18] has treated the same problem (in acoustics) and finds that the relative fluctuations are of order unity and independent of V for sufficiently large volumes. It is therefore necessary to find the source of the difference in the two results.

After the pulse has passed, the electric field at a point x, y, z in the box is given as a superposition of the free damped normal vibrations in the form

$$\mathbf{E}(xyzt) = \sum_k a_k \mathbf{u}_k(xyz) \exp(i\omega_k t - \omega_k t/2Q_k), \quad (11)$$

where the $\mathbf{u}_k$ are the vector eigenfunctions for the normal modes in the box, $\omega_k/2\pi$ are the normal frequencies, and $Q(k)$ is the measure of the persistence of the kth mode. As these latter do not differ very much from one another for a rectangular box, they will here be taken equal. The amplitudes a_k can be exactly calculated from the theory of the forced oscillations of a cavity resonator if the current distribution due to the magnetron pulse is assumed to be known. However, this would give rise to integrals impossible of evaluation in practice. Instead, it is possible to ask for statistical information about the values assumed by $\mathbf{E}$ at some point x, y, z as a function of time t, i.e., what is the probability that $\mathbf{E}(xyz, t)$ will lie in a certain range of magnitude and direction if all values of t in a certain small range of t are equally likely? Because of the factors $\exp(i\omega_k t)$, the expression (11) is the sum of a large number of two-dimensional vectors

[17] Radiation Laboratory Text T-2, page 12–31.
[18] R. C. Jones, J. Acous. Soc. Am. 11, 324 (1940).

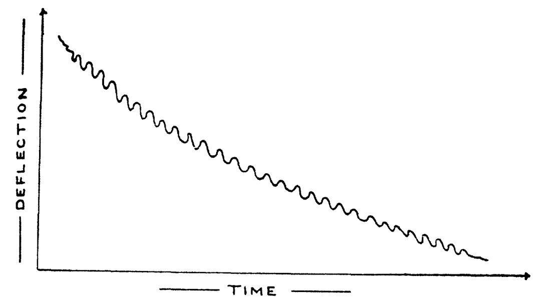

FIG. 2. Expected decay curve of the cavity.

whose phase angles at any time are essentially oriented at random. The relation to the two-dimensional random walk problem is obvious. Jones used its known solution for the probability distribution of the resultant displacement vector, which in our case is the electric field $\mathbf{E}(xyz, t)$ whose magnitude, or perhaps its square is indicated on the oscilloscope screen.

A simpler insight into the reason for the large relative fluctuations may be obtained if the phase angles of the vector summands in Eq. (11) are restricted to the two values 0° and 180°. The problem then becomes that of the one-dimensional random walk problem: a man walks left or right with equal probabilities, and each time through distances distributed in magnitude according to a certain law. What is the probability that he is displaced by X from his starting point after n walks? The result is well known:[19] His average displacement is zero, and the average magnitude of his displacement is proportional to the square root of the number of walks. However, the relative fluctuation in the magnitude of the displacement is large:

$$(X)_{\text{Av}} = 0, \quad (12a)$$

$$|X|_{\text{Av}} = \alpha n^{\frac{1}{2}}, \quad (12b)$$

$$(X^2)_{\text{Av}} = \beta n, \quad (12c)$$

$$[(X^2)_{\text{Av}} - \{|X|_{\text{Av}}\}^2]^{\frac{1}{2}}/|X|_{\text{Av}} = \frac{(\beta-\alpha^2)^{\frac{1}{2}}}{\alpha}, \quad (12d)$$

independent of n. (A similar result, differing only in the numerical coefficients α and β was obtained by Jones in the two-dimensional problem.)

[19] S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943).

The question now arises: How could one obtain the $1/n^{\frac{1}{2}}$ law for the relative fluctuations? One way would be to evaluate the *sum* of the *magnitudes* of the distances walked. This is

$$\sigma=\sum_i|X_i|, \quad (13a)$$

$$\bar{\sigma}=n\langle|X_i|\rangle_{\mathrm{Av}}, \quad (13b)$$

$$\sigma^2=\sum_i\sum_j|X_i|\,|X_j|=\sum_i|X_i|^2+\sum_{i\neq j}\sum|X_i|\,|X_j|, \quad (13c)$$

$$\langle\sigma^2\rangle_{\mathrm{Av}}=n\langle|X_i|^2\rangle_{\mathrm{Av}}+n(n-1)(\langle|X_i|\rangle_{\mathrm{Av}})^2, \quad (13d)$$

$$\langle\sigma^2\rangle_{\mathrm{Av}}-(\bar{\sigma})^2=n\{\langle|X_i|^2\rangle_{\mathrm{Av}}-(\langle|X_i|\rangle_{\mathrm{Av}})^2\}. \quad (13e)$$

The relative fluctuation in σ is

$$\left[\frac{\langle\sigma^2\rangle_{\mathrm{Av}}-(\bar{\sigma})^2}{(\bar{\sigma})^2}\right]^{\frac{1}{2}}=\frac{1}{\sqrt{n}}\left[\frac{\langle|X_i|^2\rangle_{\mathrm{Av}}-(\langle|X_i|\rangle_{\mathrm{Av}})^2}{(\langle|X_i|\rangle_{\mathrm{Av}})^2}\right]^{\frac{1}{2}}, \quad (13f)$$

expressing the $1/n^{\frac{1}{2}}$ law. Jones, on the other hand, discusses the magnitude of the vector sum of the displacements. This is, of course, the correct procedure, for actually the magnitude of the total electric field at a point determines the measured quantity, not the sum of the magnitudes or magnitudes squared of the contributions of the various modes.

The higher frequency components in the gurgle could be avoided if one used a video circuit which eliminated them. (This is apparently what happens in the acoustical case when the human ear is used to estimate reverberation time.[20]) However, the exponential decay itself would thereby be distorted, especially for the stronger attenuations. Even if the gurgle could be eliminated, the spacial interference of the modes would cause trouble if the walls of the box were slightly deformable (as they were!). All in all, the thermocouple method seems preferable.

Kroll[21] has suggested the use of a large number of randomly located crystal rectifiers connected in series. This would give a measure of the total energy in the box as a function of the time, and would not suffer from the sluggishness of the thermocouples. It can be shown that during the time between the pulses the total energy in the box does not have squiggles, so that the Q's could be measured absolutely. This method seems feasible, but has not been tried due to the difficulty of obtaining enough crystals.

[20] For example, P. M. Morse and R. H. Bolt, reference 10, p. 79, Fig. 6.

[21] N. M. Kroll, private communication. The author is indebted to Mr. Kroll for several interesting discussions.

APPENDIX 2. FIELD THEORY OF THERMOCOUPLE RESPONSE

Maxwell's equations for the interior of the box are

$$\begin{aligned}\mathrm{curl}\,\mathbf{E}+\mu\frac{\partial\mathbf{H}}{\partial t}&=0, \quad \mathrm{div}\,\mathbf{H}=0,\\ \mathrm{curl}\,\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}&=\mathbf{J}, \quad \mathrm{div}\,\mathbf{E}=0,\end{aligned} \quad (14)$$

where m.k.s. units are used, $\mathbf{J}$ is the current density exciting the cavity, $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}$, and $\mathbf{P}=\chi\epsilon_0\mathbf{E}$ where χ is the complex polarizability of the medium. Since χ depends on the frequency, it must be applied as a factor to each time fourier component of $\mathbf{E}$ separately. The wall losses are taken into account by the imposition of a homogeneous boundary condition at the walls of the cavity.

The simplest proof using the field equations that the steady state energy density in the cavity is proportional to Q is based on the energy conservation integral which can be derived from Eqs. (14).

$$-\int_V\mathbf{E}\cdot\mathbf{J}d\tau=\int_V(\mathbf{E}\cdot\dot{\mathbf{D}}+\mathbf{H}\cdot\dot{\mathbf{B}})d\tau+\int_S(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a}. \quad (15)$$

This can be rather plausibly interpreted as equivalent to Eq. (1). The surface integral represents the wall losses, the integral on the left the magnetron power input, and the volume integral on the right both the energy storage and the losses in the gas.

A more explicit proof assuming excitation of the cavity by a given current density $\mathbf{J}(xyz, t)$ will also be given. Elimination of $\mathbf{H}$ from (14) gives the inhomogeneous wave equation

$$\nabla^2\mathbf{E}-\frac{(1+\chi)}{c^2}\ddot{\mathbf{E}}=\mu\dot{\mathbf{J}}. \quad (16)$$

The homogeneous wave equation, i.e., $\mathbf{J}=0$, has solutions of the form

$$\mathbf{E}(xyzt)=\mathbf{u}(xyz)e^{-ipt}, \tag{17}$$

representing damped normal modes, if $\mathbf{u}$ satisfies

$$\nabla^2\mathbf{u}+k^2\mathbf{u}=0 \tag{18}$$

with

$$k^2=p^2(1+\chi)/c^2. \tag{18a}$$

The requirement that $\mathbf{u}$ should satisfy the correct boundary conditions at the walls determines a discrete set of complex values of $\mathbf{k}$, and hence of p. Since both boundary conditions and polarizability χ depend on the frequency p, the above solutions of Eq. (18) are not orthogonal. A complete set of appropriate orthonormal functions may however be obtained by considering the solutions of Eq. (16) for a real constant ω equal to some frequency in the magnetron band width. This set of functions may then be used in the expansion of the steady-state function of Eq. (16). In the present applications, unlike the acoustic, there will be no appreciable difference between the two sets of functions, as the magnetron pulses are so nearly monochromatic that χ and the wall impedance are essentially constant over the band width.

For each damped normal mode, the wall losses may be "shifted" into the gas by assigning a different imaginary part of χ, in such a way that the Q of the mode has the correct value allowing for both kinds of loss. The amount of the wall contribution to χ will depend somewhat on the nature of the mode, but for large cavities of irregular shape, relatively few of the modes have a Q_B sensibly different from the ergodic value. The eigenfunctions which will be used then obey Eq. (18) and have vanishing tangential components at all walls. The eigenvalues k^2 are then real.

Writing χ, p in terms of their real and imaginary parts

$$\chi=a+ib,\quad p=\omega+i\eta, \tag{19}$$

we find that

$$k^2=(\omega+i\eta)^2(1+a+ib)/c^2 \simeq\frac{\omega^2}{c^2}+i\left(\frac{2\omega\eta}{c^2}+\frac{b\omega^2}{c^2}\right), \tag{20}$$

since a, $b\ll 1$. Hence k^2 will be real if

$$\omega=ck \tag{21}$$

and

$$\eta=-\omega b/2. \tag{22}$$

For positive damping, η must be negative, and hence b positive.

To solve Eq. (16), we make a space-time expansion of $\mathbf{J}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$. If T is the repetition time,

$$\mathbf{J}(\mathbf{r},t)=\sum_{\mathbf{k}}\sum_{n=-\infty}^{\infty}J_{n\mathbf{k}}e^{-2\pi int/T}\mathbf{u}_{\mathbf{k}}(\mathbf{r}), \tag{23}$$

where the summation index $\mathbf{k}$ runs over all wave vectors which are eigenvalues of Eq. (18) for perfectly conducting boundaries of the box. If the $\mathbf{u}_{\mathbf{k}}$ are normalized so that

$$\int d\tau|\mathbf{u}_{\mathbf{k}}(\mathbf{r})|^2=V, \tag{24}$$

the expansion coefficients $J_{n\mathbf{k}}$ are given by

$$J_{n\mathbf{k}}=\frac{1}{VT}\int_0^T dt\int_V d\tau\mathbf{J}(\mathbf{r},t)\cdot\mathbf{u}_{\mathbf{k}}(\mathbf{r})e^{2\pi int/T}. \tag{25}$$

Likewise

$$\mathbf{E}(\mathbf{r},t)=\textstyle\sum_{\mathbf{k}}\sum_n E(\mathbf{k},n)\mathbf{u}_{\mathbf{k}}(\mathbf{r})e^{-2\pi int/T}. \tag{26}$$

Insertion of these expansions into Eq. (16) and comparison of the coefficients of

$$\mathbf{u}_{\mathbf{k}}(\mathbf{r})e^{-2\pi int/T}$$

gives

$$E(\mathbf{k},n)\left\{-k^2-\frac{1}{c^2}(1+\chi_k)\left(\frac{2\pi in}{T}\right)^2\right\} =-\frac{2\pi in}{T}\mu J_{n\mathbf{k}} \tag{27}$$

whence, using Eq. (26)

$$\mathbf{E}(\mathbf{r},t)=\mu\sum_{\mathbf{k}}\sum_{n=-\infty}^{\infty}\frac{2\pi in}{T}\times\frac{J_{n\mathbf{k}}\mathbf{u}_{\mathbf{k}}(\mathbf{r})e^{-2\pi int/T}}{\mathbf{k}^2-(1+\chi_k)(2\pi n/cT)^2} \tag{28}$$

where the subscript on $\chi_{\mathbf{k}}$ denotes that it is a function of the mode $\mathbf{k}$, but that any frequency dependence, i.e., on n is neglected.

It would be a hard task to evaluate the sums giving $\mathbf{E}(\mathbf{r}, t)$ even if $\mathbf{J}(\mathbf{r}, t)$ and the $\mathbf{u_k}$ were known, and the result would be exceedingly dependent on their exact values. The time average of $\mathbf{E}(\mathbf{r}, t)$ is of course zero. A single thermocouple, owing to its long relaxation time, would give an essentially steady e.m.f. proportional to the time average of $|\mathbf{E}(\mathbf{r}, t)|^2$ at its position. The reading would be very dependent on the location of the thermocouple and sensitive to small deformations of the walls. Instead, an attempt is made to space average $|\mathbf{E}|^2_{\text{time average}}$ over the entire box by using a large number of "randomly" located thermocouples. A further average over the possible standing wave patterns is achieved by "stirring" up the modes by rotation of large copper fans.

The expression for $|\mathbf{E}|^2$ averaged over space and time is fortunately rather simple because of the orthogonality of the expansion functions, and one finds

$$\frac{1}{TV}\int\int|\mathbf{E}|^2 d\tau dt=\mu^2\sum_{\mathbf{k}}\sum_{n=-\infty}^{\infty}\left(\frac{2\pi n}{T}\right)^2 \times\frac{|\mathbf{J}_{n\mathbf{k}}|^2}{|k^2-(1+\chi_k)(2\pi n/cT)^2|^2}. \quad (29)$$

Recalling that

$$1+\chi_k=1+a_k+ib_k\backsimeq 1+ib_k \text{ since } a_k\ll 1,$$

the expression (29) becomes

$$(|\mathbf{E}|^2)_{\text{Av}}=\mu^2\sum_{\mathbf{k}}\sum_n \times\frac{|J_{n\mathbf{k}}|^2(2\pi n/cT)^2}{[k^2-(2\pi n/cT)^2]^2+(2\pi n/cT)^4 b_k^2}. \quad (30)$$

We consider the summation over n. Because of the smallness of b_k in the denominator the summand has a sharp maximum of half-width $\delta n = n_k b_k$ for

$$n=n_k=cTk/2\pi=\omega_k T/2\pi. \quad (31)$$

(Thus n_k is equal to the number of cycles of the wave k during the repetition period T. For the pulsed source used in the experiments of Becker and Autler this is of the order of $10^{10}\cdot 10^{-3}=10^7$.) The numerator contains the factor $|J_{n\mathbf{k}}|^2$ which, as we shall see, is rapidly variable for pulses long compared to the reverbation time of the cavity, but not for short pulses. It is therefore necessary to distinguish between the two cases.

We may calculate the n dependence of $J_{n\mathbf{k}}$ on the assumption of rectangular pulses of duration Δ by setting

$$\mathbf{J}(\mathbf{r}, t)=\mathbf{J}(\mathbf{r})f(t), \quad (32)$$

where

$$f(t)=\begin{cases} e^{-i\omega t} & 0<t<\Delta \\ 0 & \Delta<t<T \\ \text{Periodic with period } T. \end{cases} \quad (33)$$

Then $J_{n\mathbf{k}}=J_{\mathbf{k}}f(n)$ where $J_{\mathbf{k}}$ are the expansion coefficients of $\mathbf{J}(\mathbf{r})$ and the expansion coefficients of $f(t)$ are given by

$$|f(n)|^2=\frac{4\sin^2\left(\omega-\dfrac{2\pi n}{T}\right)\dfrac{\Delta}{2}}{(\omega T-2\pi n)^2}. \quad (34)$$

This function is well known in connection with the spectrum of the pulsed magnetron, and has a maximum when $n=\omega T/2\pi$, i.e., nearly at $n=n_k=\omega_k T/2\pi$ for all the modes which will be appreciably excited. Its half-width is

$$\delta n=2(1.39)(T/\pi\Delta)=0.885T/\Delta. \quad (35)$$

The condition that the half-width of the maximum provided by the numerator is very large compared to that provided by the denominator is that

$$0.885T/\Delta\gg b_k n_k \quad (36)$$

or in terms of Q

$$Q\gg(\omega_k/2\pi)(\Delta/0.885). \quad (36a)$$

For $\Delta=\frac{1}{4}\times 10^{-6}$ sec and $\omega/2\pi=2.4\times 10^{10}$ sec^{-1} this requires that

$$Q\gg 6800. \quad (36b)$$

The values of Q used in the experiments have exceeded 10^5. A more physical statement of the condition (36a) is that the pulse duration should be short compared to the ring time Q/ω of the cavity.

Pulsed Source

In this case, the sum $\sum_n$ may be replaced by an integration and the factor $n|J_{n\mathbf{k}}|^2$ removed from the integral and evaluated at $n=n_{\mathbf{k}}$. The resulting integral is elementary and gives

$$|\mathbf{E}|^2_{\mathrm{Av}}=\frac{\mu^2 cT}{4}\sum_{\mathrm{k}}\frac{|J_{\mathrm{k}}f(n_{\mathrm{k}})|^2}{kb_{\mathrm{k}}} \tag{37}$$

or in terms of $Q(k)=1/b_{\mathrm{k}}$

$$|\mathbf{E}|^2_{\mathrm{Av}}=\frac{\mu^2 c^2 T}{4}\sum_{\mathrm{k}}\frac{Q_{\mathrm{k}}|J_{\mathrm{k}}f(n_{\mathrm{k}})|^2}{\omega_k}. \tag{38}$$

which is of the same form as Eq. (6), and when the conditions for the ergodic state are met, is directly proportional to the Q of the box and its contents.

Case of a Monochromatic (CW) Source

In the case of a CW source, the numerator of the integrand in Eq. (33) is the more rapidly variable factor. In this case, the pulse length Δ is equal to the repetition period T, and equation (34) may be written

$$|f(n)|^2=\left[\frac{\sin\pi(N-n)}{\pi(N-n)}\right]^2 \tag{39}$$

where $N=\omega T/2\pi$ is a positive integer. This has a half-width in n of order unity, and is essentially a delta-function with the property

$$|f(n)|^2=\begin{cases}1 \text{ if } n=N\\ 0 \text{ if } n\neq N.\end{cases} \tag{40}$$

The sum over n in Eq. (30) then reduces to

$$\textstyle\sum_n \rightarrow \frac{c^2\omega^2|J_{\mathrm{k}}|^2}{(\omega^2-c^2k^2)^2+\omega^4 b_{\mathrm{k}}^2} \tag{41}$$

independent of N as it should be. Then

$$|\mathbf{E}|^2_{\mathrm{Av}}=\mu^2c^2\omega^2\textstyle\sum_{\mathrm{k}}\frac{|J_{\mathrm{k}}|^2}{(\omega^2-c^2k^2)^2+\omega^4/Q_{\mathrm{k}}^2} \tag{42}$$

a result which can, of course, be obtained without the time Fourier analysis, since all of the forced normal modes oscillate with the same frequency. Under ergodic conditions, this sum will also be proportional to the Q of the cavity. The number of terms involved in the sum (42) is the number of cavity modes within the half-width of a cavity resonance. The expression for this number is

$$n=8\pi V/\lambda^3 Q. \tag{43}$$

For the box used at Columbia, using the theoretical $Q=1.465\times 10^6$ for copper losses only, n is only 135, to be compared with the 66,100 terms in Eq. (38). Hence it would appear that the ergodic state is more likely to be realized when a pulsed source is used.

Resolving Power of the Spectroscope

The water vapor line studied by Becker and Autler[11] was so highly pressure broadened that the spectral width of the source ($\sim 1/\Delta$ cycles $\sim 4\times 10^6$ cycles) was much less than the width of the absorption line. ($\sim 5\times 10^9$ cycles.) If the total gas pressure were reduced until the line breadth became comparable to $1/\Delta$ for the pulsed source, it would be necessary to correct for the lack of resolution. The resolving power could be increased by lengthening the pulse, but only at the cost of decreasing the number n of modes excited. As explained above, n given by Eq. (43) is still appreciable even for a CW source. In this case, the resolving power would be infinite except for the probable appearance of "ghost" lines due to failure to realize the ergodic state with the limited number of modes excited. These questions are worthy of further experimental study.

APPENDIX 3. CONNECTION BETWEEN Q_V AND THE ATTENUATION

We next establish the connection between the Q of the gas and the attenuation. The homogeneous wave equation (18) has spacially damped monochromatic running wave solutions

$$e^{iKx-i\omega t}$$

where

$$K^2=(1+\chi)(\omega/c)^2. \tag{44}$$

Separating K into real and imaginary parts

$$K=\frac{2\pi}{\lambda}+\frac{i}{2\Lambda} \tag{45}$$

and comparing with the exponential decay factor $e^{-x/\Lambda}$ involving mean free path Λ in kinetic theory, one obtains a mean free path

$$\Lambda=Qc/\omega \tag{46}$$

as stated in the text. The absorption cross section σ of a molecule is then defined by the equation

$$\sigma = 1/n_1\Lambda \quad (47)$$

where n_1 is the number of absorbing molecules per unit volume.

APPENDIX 4. NUMBER OF THERMOCOUPLES NEEDED

Granted that the value of $|\mathbf{E}|^2_{\text{Av}}$ is directly proportional to the Q of the box and its contents, the question arises: How well do a large but finite number of thermocouples simulate a genuine space average of $|\mathbf{E}|^2$? If there are n thermocouples located at positions $\mathbf{r}_i$, $i=1, 2, \ldots n$, the e.m.f. will be proportional to

$$R = \sum_i |\mathbf{E}(\mathbf{r}_i)|^2. \quad (48)$$

The average of R over all possible thermocouple positions is

$$\bar{R} = n|\mathbf{E}|^2_{\text{Av}}. \quad (48a)$$

For any actual distribution of the thermocouples, the reading may differ from $\bar{R}$ by a fractional amount of the order

$$\left[\frac{\langle R^2\rangle_{\text{Av}} - \langle R\rangle_{\text{Av}}^2}{\langle R\rangle_{\text{Av}}^2}\right]^{\frac{1}{2}} = \frac{1}{n^{\frac{1}{2}}}\left[\frac{\langle E^4\rangle_{\text{Av}} - \langle E^2\rangle_{\text{Av}}^2}{\langle E^2\rangle_{\text{Av}}^2}\right]^{\frac{1}{2}}, \quad (49)$$

as in Appendix 1, and decreases with the inverse square root of the number of thermocouples. In practice, the error can be made even less by the use of copper fans which stir up the standing wave patterns.

APPENDIX 5. USE OF THE PHOTON MODEL TO DERIVE Q_B FOR A LARGE METALLIC CAVITY

This calculation differs from that of Q_A only in that there is a non-zero reflection coefficient dependent on the polarization and angle of incidence of the photons. For a dielectric, this dependence is so complicated that the averaging over the angles would have to be done numerically for each particular case. In the case of a good conductor, however, the expressions become considerably simpler, and the integrals can be done with sufficient approximation. The reflection coefficients $R_\perp$ and $R_\parallel$ are given by[22]

$$1 - R_\perp = 2\mu x, \quad (50a)$$

$$1 - R_\parallel = \frac{2\mu x}{\mu^2 + \mu x + (x^2/2)}, \quad (50b)$$

for the two kinds of polarizations. Here $\mu = \cos\theta$ and the parameter $x = (2\epsilon_0\omega/\sigma)^{\frac{1}{2}}$ is much less than unity for metals at microwave frequencies.

The number of photons lost per second to an area A of the wall is

$$\frac{2\pi nAc}{4\pi}\int_0^1 d\mu\mu\tfrac{1}{2}\{(1-R_\parallel)+(1-R_\perp)\}$$
$$= \tfrac{1}{4}nAc\int_0^1 \mu\left(2\mu x + \frac{2x}{\mu}\right)d\mu \quad (51)$$
$$= \tfrac{2}{3}nAcx,$$

where the approximation $x \ll 1$ has been made. The Q for the area A of metallic surface is then

$$Q = \omega(nV)/[2nAcx/3] = 3\pi V/A\lambda x, \quad (52)$$

or in terms of skin depth

$$\delta = (2/\omega\mu_0\sigma)^{\frac{1}{2}} \quad (53)$$

one has

$$Q = \tfrac{3}{2}V/A\delta \quad (54)$$

which agrees with the result derived from the field theory.

The author has profited from many interesting discussions with Professors J. M. B. Kellogg and A. Nordsieck and his other colleagues in the Columbia Radiation Laboratory.

[22] J. A. Stratton, *Electromagnetic Theory*, Ch. IX, Eqs. (87), (88).