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The Theory of the Synchrotron

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In accelerators of the type discussed by Veksler and McMillan (e.g., the synchrotron and synchro-cyclotron) the motion of particles can be described in terms of stable oscillations about a synchronous orbit. Expressions are worked out for the frequencies of these oscillations, and for the way in which their amplitudes are damped as the energy is increased. The effect of radiation losses on the damping is discussed. It is shown that the synchrotron can advantageously be operated as a betatron until the electron velocity is close to that of light; the dee voltage is then turned on and the machine works as a synchrotron for the remainder of the acceleration. The transition from betatron to synchrotron operation proved to be quite efficient. Formulae are given for the distortions of the orbits by azimuthal asymmetries of the magnetic field. The results are illustrated in terms of the California synchrotron.

THEORY OF THE SYNCHROTRON

I. Introduction

IN the cyclotron,[1] ions revolve with approximately constant angular velocity in a magnetic field, passing through accelerating gaps across which an alternating electric field is applied. The frequency of the electric field is chosen to match the angular velocity of the ions, so that many successive accelerations can occur. However, when the velocity of the ions becomes appreciable compared to that of light, the angular velocity diminishes; as a consequence of this, the ions fall out of step with the alternating electric field, and only a finite number of accelerations can be achieved.[2]

This paper is concerned with a method of acceleration in which the cyclotron principle is used, but with such modifications that the apparent difficulty mentioned above is converted into an aid, allowing a theoretically unlimited number of successive accelerations. The method was proposed independently by McMillan[3] and Veksler.[4] (Veksler's publication was earlier, but it did not come to the attention of the authors until after the work reported here had been started.)

The basic idea of the method involves the occurrence of what may be called "phase stability" in the motion of a charged particle in a cyclotron-like combination of electric and magnetic fields. The angular velocity of a particle of charge e and mass m in a magnetic field H is given by:

$$\omega = eH/mc = ecH/E. \qquad (1)$$

The mass in the above equation is the total relativistic mass, E is the total energy including the rest energy. Thus ω decreases as the energy increases. Suppose now that the particle is moving with a given energy, and that the frequency of the applied electric field just matches the angular velocity corresponding to that energy. The electric field will either add or subtract from the energy depending on the time of crossing the accelerating gaps with respect to the cycle of field variation. If this time is such that the energy is increased, the angular velocity will start to decrease; the time of crossing the gaps will become later and later until finally the particle is being decelerated; this will continue until the energy is brought back to the original value. The energy will then go through a similar deviation in the opposite direction, after which the cycle will be repeated. Thus the time of gap crossing (to be described later in terms of a phase angle) and the energy can undergo stable oscillations, and the particle will try to stay in or

[1] E. O. Lawrence and M. S. Livingston, Phys. Rev. **40**, 19 (1932).
[2] M. E. Rose, Phys. Rev. **53**, 392 (1937). R. R. Wilson, Phys. Rev. **53**, 408 (1937).
[3] E. M. McMillan, Phys. Rev. **68**, 143 (1945).
[4] V. Veksler, J. Phys. U.S.S.R. **9**, No. 3, 153 (1945). Since this work was submitted for publication, another paper on the synchrotron has appeared: D. M. Dennison and T. H. Berlin, Phys. Rev. **70**, 58 (1946). There are, however, considerable differences in subject matter and method of treatment between this paper and the present paper. See also: N. H. Frank, Phys. Rev. **69**, 689(A) (1946); D. S. Saxon and J. Schwinger, Phys. Rev. **69**, 702(A) (1946).

near the stable orbit if small disturbances are applied to it.

The most significant disturbance is that caused by a small change in the magnetic field or frequency, since these lead to a net change in energy, allowing the attainment of the desired acceleration. Suppose for example that a small increase in H is made; the angular velocity will now be too great, which is the same as if the energy were now too small; following the argument given above, we see that the energy will now oscillate about a new equilibrium value for which m has been increased in the same ratio as H. A parallel argument applies if the frequency is decreased.

Generalizing the above to continuous variations, we see that *acceleration can be accomplished by varying slowly the magnetic field or the frequency in a cyclotron-like combination of electric and magnetic fields.* If the rate of variation is slow enough, phase stability is maintained throughout the acceleration. This stability means that no precise control of field or frequency or their rates of variation is necessary even though the particle may make hundreds of thousands of revolutions during the course of the acceleration.

Application to Electron Acceleration. (Synchrotron)[5]

Equation (1) shows that the ratio of magnetic field to frequency must vary as m during the acceleration. In the acceleration of electrons to high energies, m changes by a large factor; therefore it seems more practical to use magnetic field variation for this case. Since the frequency remains constant, the radius of the synchronous orbit will be proportional to the velocity, and if the electrons are injected at a velocity near c, the radius will remain nearly constant during the acceleration. The magnet must therefore produce a varying field over an annular region, and will be similar to a betatron[6] magnet with the central core omitted. The conditions for radial and axial stability of the orbits will be essentially the same as in the betatron, and therefore the same kind of radial field dependence can be used. The electrodes providing the high frequency accelerating field, and all other metal parts near the orbit, must be designed so that no eddy currents of sufficient magnitude to disturb the field distribution seriously can flow in them.

In the original proposal[3] it was planned to inject electrons at 300 kv directly into the synchronous orbit, but a better method of injection has been suggested independently by Wilson Powell and D. Bohm[7] of this laboratory and by J. P. Blewett and H. C. Pollock[8] of the General Electric Company. This is to allow operation as a betatron up to about 1.5 Mev; the oscillator is then turned on at an appropriate moment, catching the electrons in a synchronous orbit. The central flux necessary for the betatron operation is provided by bars of laminated iron which saturate after their flux change is no longer needed. The "betatron injection" has the advantages that a high voltage injector does not have to be developed, the change of radius during acceleration is reduced, and a larger fraction of the injected electrons can be caught.

One of the difficulties that must be considered in accelerating electrons is the radiation caused by their circular motion.[9] In the synchrotron this is automatically compensated for if the applied high frequency voltage is sufficiently greater than the radiation loss per turn. Therefore the synchrotron should be able to reach higher ultimate energies than the betatron; also the magnet requires less laminated iron for a given energy, because of the absence of the central core.

Application to Heavy Particles (Synchro-Cyclotron)[5]

If a deuteron is given a kinetic energy of 200 Mev, its mass increases by only 10 percent. Therefore it is practical to consider frequency variation for the deuteron and other heavy particles, which has the great advantage of avoiding very massive structures of laminated iron (a

[5] In the original letter (reference 3) the name "Synchrotron" was meant to apply to all modifications of this principle of acceleration. This, however, can be confusing, and it is proposed to limit the term "Synchrotron" to the case of magnetic field variation applied to electrons, and to introduce the term "Synchro-Cyclotron" or "Frequency Modulated Cyclotron" to describe the case of frequency variation applied to heavy particles. The mathematical treatment in this paper, up to and including Eq. (23), is valid for both cases or for any combination of them.

[6] D. W. Kerst and R. Serber, Phys. Rev. **60**, 53 (1941); D. W. Kerst, Phys. Rev. **60**, 47 (1941).

[7] Unpublished.

[8] H. C. Pollock, Phys. Rev. **69**, 125 (1946).

[9] D. Iwanenko and I. Pomeranchuk, Phys. Rev. **65**, 343 (1944). E. M. McMillan, Phys. Rev. **68**, 145 (1945); L. I. Schiff, Rev. Sci. Inst. **17**, 6 (1946).

200-Mev deuteron would have an orbit radius of 2 meters at 14,000 gauss). The radius of the orbit will vary roughly as the square root of the kinetic energy, and therefore the magnetic field will have to extend to the center. It is apparent that the machine is then just like a cyclotron with the added feature of frequency modulation,[10] which can be accomplished, for example, by means of a rotating condenser.[11] The simplest, and probably the best, way of introducing the ions is to let them start from rest near the center, as is normally done in the cyclotron. A detailed discussion of this modification will be given in a later paper.

Other Modifications

Other arrangements in which magnetic field and frequency are varied together are also possible. For instance, frequency modulation could be applied to the synchrotron to vary the radius of the orbit as an aid in bringing the beam out, or field variation could be added to the synchrocyclotron to extend its range beyond what may turn out to be a practical limit to the amount of frequency variation attainable.

Removal of the Beam

Many of the desired experiments can be done with an internal target, as for example the production of x-rays and mesotrons, and studies of induced radioactivity. However, if it is desired to bring the beam out, difficulty is encountered because of the close spacing of successive turns in these devices. In spite of this difficulty, calculations by Crittenden and Parkins[12] indicate that a large part of the ions can be brought out in a collimated beam by suitable arrangements of fields which will not be discussed in detail here.

II. Description of the Motion

A particle in a machine of the kind we have been describing, if started at the proper radius and with the proper velocity, will move in a circle with a frequency given by (1). The relation between the proper radius and momentum is the familiar one,

$$p = eHr/c. \tag{2}$$

For a suitably designed magnetic field,[6] the orbit described by (2) is stable, in that all neighboring orbits execute rapid horizontal and vertical oscillations about it, with an oscillation frequency comparable to the frequency of rotation. In consequence, the orbit will follow any slow change of the magnetic field with time in such a way that (2) continues to hold. The condition for a slow, or adiabatic, variation is that the field should change by only a small fraction of itself during a period of oscillation.

A similar situation exists with respect to the phase oscillations which have been discussed in the preceding section. The "synchronous" orbit, about which oscillations occur, is described by (2), and (1) with $\omega = \omega_s$, the angular frequency of the applied electric field. A particle starting in the proper phase with respect to the electric field will move in this orbit. If the phase is wrong, it will oscillate about the equilibrium phase, with a frequency small compared to the frequency of rotation of the particle.

Since the energy, and therefore the momentum, varies during the phase oscillation, the radius of the orbit will, according to (2), also suffer an oscillation which is coupled to the phase oscillation. However, just because the period of phase oscillation is long compared to that of rotation, this motion is essentially independent of the "free" radial and vertical oscillations previously described: the phase can be considered constant during a free oscillation, while, conversely, the effect of the rapid free oscillations on the phase motion averages to zero.

The energy of a particle in the synchronous orbit follows from (1),

$$E_s = \frac{ecH}{\omega_s}, \tag{3}$$

and the energy (and motion) will follow any variations of H/ω_s which are slow compared to the period of phase oscillation.

The Free Oscillations

The free oscillations have been studied by Kerst and Serber[6]; we need only quote their re-

[10] J. R. Richardson, K. R. MacKenzie, E. J. Lofgren, and B. T. Wright, Phys. Rev. 69, 669 (1946).
[11] To be published.
[12] E. C. Crittenden, Jr., and W. E. Parkins, J. App. Phys. 17, 444 (1946); see also L. S. Skaggs, G. M. Almy, D. W. Kerst, and L. H. Lanzl, Phys. Rev. 70, 95 (1946).

sults here. They find for the frequencies of radial and vertical oscillation

$$\omega_r=(1-n)^{\frac{1}{2}}\omega, \quad \omega_v=n^{\frac{1}{2}}\omega, \tag{4}$$

where ω is the frequency of rotation of the particle and

$$n=-d(\ln H)/d(\ln r). \tag{5}$$

In either case the amplitude of oscillation decreases as the magnetic field is increased:

$$A\sim H^{-\frac{1}{2}}. \tag{6}$$

The Synchronous Orbit

The synchronous orbit is defined by (3) which gives its energy, (2) which gives its radius, and by its phase relative to the electric field. We suppose that the potential on the ith gap, located at azimuth θ_i, is

$$V_i \sin\left(\int_0^t \omega_s dt+\alpha_i\right).$$

If the azimuth of the particle in its orbit is θ, we define the phase to be

$$\phi=\theta-\int_0^t \omega_s dt+\alpha. \tag{7}$$

On crossing the ith gap the particle gains an energy $eV_i \sin(\theta_i+\alpha_i+\alpha-\phi)$, and the total energy gain per turn is obtained by summing over i. A suitable choice of α will reduce the expression for the energy gain per turn to the form

$$eV \sin\phi,$$

where $V=-\Sigma V_i \cos(\theta_i+\alpha_i+\alpha)$ is the maximum possible gain.

The phase, ϕ_s, of the synchronous orbit is determined by the energy balance; the energy gain per turn, $\Delta E_s=2\pi\dot{E}_s/\omega_s$, required by (3), plus any radiation loss, L_s per turn, must be supplied by the electric field, ϵ_s, due to the changing magnetic flux, plus the contribution of the applied voltage. Thus we must have

$$eV\sin\phi_s=\Delta E_s+L_s-2\pi r_s e\epsilon_s. \tag{8}$$

Since $\dot{E}_s=v_s\dot{p}_s$ and $\omega_s=v_s/r_s$, an alternative form of this relation is

$$(eV/2\pi)\sin\phi_s=r_s\dot{p}_s+L_s/2\pi-er_s\epsilon_s. \tag{9}$$

The Phase Equation

The equation for the rate of change of angular momentum of a particle is

$$\frac{d}{dt}\left(rp+\frac{e}{c}rA\right)=\frac{eV}{2\pi}\sin\phi-\frac{L}{2\pi}, \tag{10}$$

where A is the θ component of the vector potential. The magnetic and electric fields, and the total flux enclosed by a circle of radius r, are given in terms of A by

$$H=\frac{1}{r}\frac{\partial}{\partial r}(rA), \quad \epsilon=-\frac{1}{c}\frac{\partial A}{\partial t}, \quad \Phi=2\pi rA. \tag{11}$$

To solve (10), we write $r=r_s+\Delta r$, and suppose that $\Delta r/r_s$ is small. The associated deviation of p from p_s can be computed from (2),

$$\frac{\Delta p}{p_s}=\frac{\Delta r}{r_s}+\frac{\Delta H}{H_s}=(1-n)\frac{\Delta r}{r_s}, \tag{12}$$

since, according to (5), $\Delta H/H_s=-n\Delta r/r_s$. Similarly, we find from (1)

$$\Delta\omega/\omega_s=\Delta H/H_s-\Delta E/E_s. \tag{13}$$

We also have

$$\frac{\Delta E}{E_s}=\frac{v_s\Delta p}{E_s}=(1-n)\frac{v_s p_s}{E_s}\frac{\Delta r}{r_s}=(1-n)\frac{v_s^2}{c^2}\frac{\Delta r}{r_s}, \tag{14}$$

and, differentiating (7), and remembering $\dot{\theta}=\omega$,

$$\dot{\phi}=\Delta\omega.$$

Thus (13) becomes

$$\dot{\phi}/\omega_s=-K\Delta E/E_s, \tag{15}$$

with

$$K=1+\frac{n}{1-n}\frac{c^2}{v_s^2}. \tag{16}$$

Returning to (10), we expand the left-hand side, using (11),

$$rp+\frac{e}{c}rA=r_s p_s+\frac{e}{c}r_s A_s+r_s\Delta p+\left(p_s-\frac{e}{c}H_s r_s\right)\Delta r.$$

The last term vanishes, in virtue of (2). Differ-

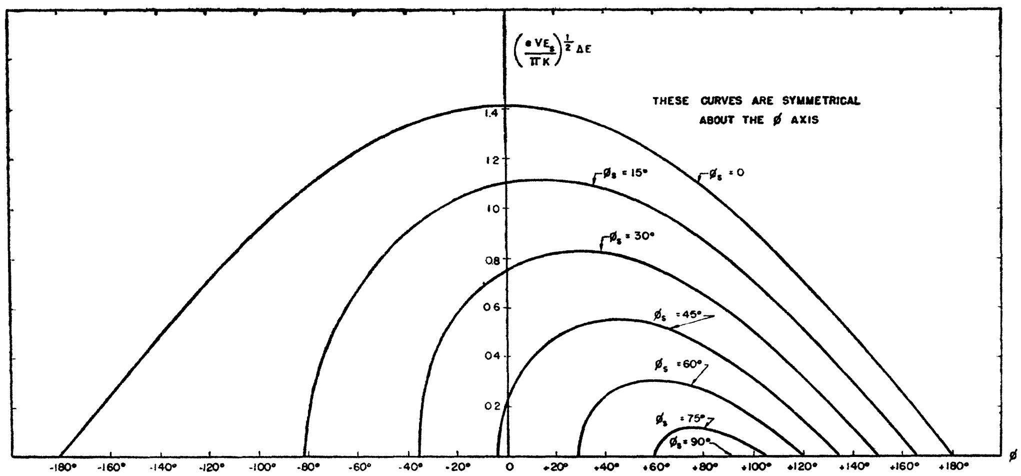

FIG. 1. Difference between particle energy and synchrotron energy, as a function of phase angle, for the limiting case of stable motion.

entiating with respect to time, and again using (11),

$$\frac{d}{dt}\left(rp+\frac{e}{c}rA\right)=r_s\dot{p}_s-er_s\epsilon_s+\left(p_s-\frac{e}{c}H_sr_s\right)\dot{r}_s+\frac{d}{dt}(r_s\Delta p).$$

With the aid of (9) and the relation $\Delta E=v_s\Delta p=\omega_sr_s\Delta p$, (10) becomes

$$\frac{d}{dt}(\Delta E/\omega_s)=\frac{eV}{2\pi}(\sin\phi_s)-\frac{L_s'}{2\pi}\Delta r-\frac{eV}{2\pi}\sin\phi,$$

where $L'=\partial L/\partial r$. Finally, using (14) and (15),

$$\frac{d}{dt}\left(\frac{E_s}{\omega_s^2K}\phi\right)+\frac{r_sL_s'c^2}{2\pi(1-n)Kv_s2\omega_s}\phi+\frac{eV}{2\pi}\sin\phi=\frac{eV}{2\pi}\sin\phi_s. \quad (17)$$

This is identical with the equation given by McMillan[3] if we omit the L' term and set $K=1$, corresponding to $n=0$. The connection between McMillan's variable θ and our t is $\omega dt=d\theta$.

The Phase Oscillations

Aside from the radiation loss term, the phase equation, (17) is that of a pendulum of moment of inertia $I=E_s/\omega_s^2K$, and restoring torque $=(eV/2\pi)\sin\phi$, which is acted on by a constant torque $(eV/2\pi)\sin\phi_s$. The rest point of the pendulum is at $\phi=\phi_s$, and beyond $\phi=\pi-\phi_s$ the pendulum goes into accelerated circular motion due to the action of the constant torque.

If we neglect the slow variation of I and ϕ_s (and possibly V) with time, we find from the energy equation of the pendulum.

$$\phi^2=\frac{eV\omega_s^2K}{\pi E_s}[U(\phi_m)-U(\phi)], \quad (18)$$

where

$$U(\phi)=-[\cos\phi+\phi\sin\phi_s]$$

gives the angular dependence of the pendulum's potential energy, and ϕ_m is the maximum amplitude of the oscillation.

During the phase oscillation the particle energy will vary about the synchronous energy by an amount given by (15) and (18),

$$\Delta E=-(eVE_s/\pi K)^{\frac{1}{2}}[U(\phi_m)-U(\phi)]^{\frac{1}{2}}. \quad (19)$$

For small amplitudes of oscillation this gives for the amplitude of energy variation

$$\Delta E=-(eVE_s\cos\phi_s/2\pi K)^{\frac{1}{2}}\psi_m, \quad (19a)$$

with $\psi_m=\phi_m-\phi_s$.

In Fig. 1 the values of the dimensionless quantity, $(\pi K/eVE_s)^{\frac{1}{2}}\Delta E$, for the limiting cases of stable motion, $\phi_m=\pi-\phi_s$, are plotted against ϕ for several values of ϕ_s. Any point within a curve belongs to a stable motion. The corresponding radial amplitudes can be obtained from (14).

The effect on the motion of adiabatic changes of I, ϕ_s, and V can be determined from the action integral, $J=\int Id\phi$, which is invariant under such changes. If we evaluate the integral by expanding U in powers of $\psi=\phi-\phi_s$, we find

$$J=\left(\frac{\pi eVE_s\cos\phi_s}{2\omega_s^2K}\right)^{\frac{1}{2}}\psi_m^2\times\left[1-\frac{1}{3}\tan\phi_s\psi_m-\frac{5}{96}\psi_m^2+\cdots\right]. \quad (20)$$

The frequency of phase oscillation is

$$\omega_P=\left(\frac{eVK\cos\phi_s}{2\pi E_s}\right)^{\frac{1}{2}}\omega_s[1-\tfrac{1}{16}\psi_m^2]. \quad (21)$$

For small oscillations

$$\phi=\phi_s+\left(\frac{2\omega_s^2K}{\pi eVE_s\cos\phi_s}\right)^{\frac{1}{4}}J^{\frac{1}{2}}\sin\left[\int\omega_pdt\right], \quad (22)$$

$$\frac{\Delta E}{E_s}=-\left(\frac{eV\cos\phi_s}{2\pi^3\omega_s^2E_s^3K}\right)^{\frac{1}{4}}J^{\frac{1}{2}}\cos\left[\int\omega_pdt\right]. \quad (23)$$

and $\Delta r/r_s$ is given by (14). In these equations J is a constant, and the explicit dependence of the amplitudes on E_s, ω_s, V, and K is exhibited. For the synchrotron at relativistic velocities (ω_s =const., $v_s^2/c^2\sim1$) the amplitudes are proportional to:

$$\psi_m\sim[(1-n)VE_s]^{-\frac{1}{4}},$$
$$\Delta E/E_s\sim[(1-n)V/E_s^3]^{\frac{1}{4}}, \quad (24)$$
$$\Delta r/r_s\sim[V/(1-n)^3E_s^3]^{\frac{1}{4}}.$$

These results, together with the corresponding ones, (6), for the free radial oscillations, demonstrate the principal characteristic of such machines; the stability of the synchronous orbit. A particle once trapped is trapped for good; its orbit shrinks down on the synchronous orbit. Moreover errors in the fields tend to be automatically compensated, and, because of the positive damping, their effects tend to disappear.

An Example: The California Synchrotron

To give an idea of the magnitudes involved, we shall quote some figures for the synchrotron at present being planned by McMillan.

The magnet will operate at 60 cycles, with a maximum field $H_f=10{,}000$ gauss. The synchronous radius at high energies is $r_s=1$ meter. The maximum energy is $E_s=300$ Mev. The r-f voltage will be $V=10$ kv, at a frequency of 47.75 mc ($\omega_s=3\times10^8$). The value of n is $\frac{2}{3}$.

The electrons are injected with 70-kev kinetic energy at a radius $r_I=93.5$ cm. The initial operation is as a betatron, with an equilibrium radius $r_\beta=96.8$ cm. There is thus about 3.3 cm clearance between the injector and the equilibrium radius. Since electrons which are injected so early that their instantaneous radius is still outside the equilibrium radius are unlikely to clear the injector, this gives the maximum amplitude of radial oscillation of the electrons which can be caught. Free radial oscillations of an initial amplitude of 3.3 cm, will by the end of the acceleration be damped, according to (6), to a final amplitude of 0.1 cm.

The dee voltage is turned on when the electrons reach a kinetic energy of 1.5 Mev ($E_s=2$ Mev). At this point $r_s=r_\beta$. The magnetic field is now 67 gauss. During the betatron operation, the rate of increase of the magnetic field is slowed down by the presence of the betatron flux bars. The effective frequency is reduced from 60 to 21.5 cycles. At this rate of increase of H, the magnetic field, as we shall see in the following section, is supplying energy to the electrons at a rate of 800 ev per turn. The additional energy required of the electric field is only 17 ev per turn. If it takes 1000 cycles to complete turning on the r-f field, the energy of the electrons will have increased to $E_s=2.8$ Mev, and the synchronous radius to $r_s=98.4$ cm. There will then be 5-cm clearance between the synchronous orbit and injector.

According to (19) and (14) the amplitudes of energy and radial oscillation will be

$$\Delta E_m=-54.5(1-\cos\phi_m)^{\frac{1}{2}}\text{ kev}$$
$$\Delta r_m=-6.14(1-\cos\phi_m)^{\frac{1}{2}}\text{ cm}.$$

Since we must have $|\Delta r_m|<5$ cm, the maximum amplitude of phase oscillation allowable is

$\phi_m = 70°$. The energy spread in the beam is then $\Delta E_m = \pm 44$ kev. At this point the frequency of phase oscillation is $\omega_p = 0.041\omega_s$.

The betatron flux bars begin to saturate when the field in the gap is 80 gauss. When the rate of rise of the field again corresponds to 60 cycles, the energy gain per turn is $\Delta E_s = 2400$ ev. The electric field must supply about three-quarters of this, which gives $\phi_s = 10°$. At the end of the acceleration, ϕ_s is determined by the radiation loss. Taking $L_s = 1{,}000$ ev, we find $\phi_s = 6°$.

The final magnitudes of phase oscillations can be calculated from the values just after the transition to synchrotron operation, and the damping laws, (24). At 300 Mev we find $\psi_m = 19°$, an energy spread in the beam $\Delta E_m = \pm 164$ kev, and a radial spread $\Delta r_m = \pm 0.15$ cm. Adding the spread due to free radial oscillations we have $\Delta r = \pm 0.25$ cm. The frequency of phase oscillation is $\omega_p = 0.004\omega_s$.

Damping Due to Radiation Loss

At very high energies, the radiation loss may introduce an additional damping of the phase oscillations. The energy loss per turn due to incoherent radiation is, at high energy,

$$L = \frac{4\pi}{3}\frac{e^2}{r}\left(\frac{E}{mc^2}\right)^4.$$

Differentiating, we find, with the aid of (14)

$$r_s L'_s / L_s = 3 - 4n. \tag{25}$$

For $n < \frac{3}{4}$, the radiation loss of a particle with too large an r (or E) is greater than that of a particle in the synchronous orbit; there is a consequent damping which is described by the second term in (17). Inclusion of this term has the effect of multiplying the amplitude of phase oscillation give in (22) by a factor

$$\exp\left\{-\frac{1}{2}\int \frac{r_s L_s' c\omega_s}{2\pi(1-n)E_s K v_s^2}\right\} dt.$$

Using (25), and setting $v_s/c = 1$, we find for the decrement in amplitude per turn

$$\Delta\psi_m/\psi_m = \tfrac{1}{2}(3-4n)L_s/E_s.$$

By the end of the acceleration, the phase oscillations which exist during the early part of the acceleration are reduced, in addition to the damping described in (22), by a factor

$$\exp\left\{-\frac{2}{3}\left(1-\frac{4}{3}n\right)\frac{r_0}{r_s}\left(\frac{E_f}{mc^2}\right)^3\left(\frac{\omega_s}{\Omega}\right)\right\},$$

where $r_0 = e^2/mc^2$, Ω is the angular frequency of the magnetic field ($H = H_f \sin \Omega t$), and E_f is the final electron energy.

The damping is not important for the Berkeley machine, the amplitudes being reduced only by 3.6 percent; however it is an effect which might be used to advantage in a very high energy accelerator.

Transition from Betatron to Synchrotron Operation

The primary purpose of the initial betatron acceleration of the electrons is to avoid the increase in r_s involved in reaching relativistic velocities, since $r_s = \omega_s v_s$. During the betatron operation the orbits are pulled clear of the injector and well out into the doughnut. One might therefore expect that if the transition from betatron to synchrotron operation is reasonably efficient, a larger range of phase oscillation will be available than with pure synchrotron operation, and in consequence higher currents could be obtained. The central question in determining the efficiency of the transition is whether it is possible to turn on the radiofrequency field sufficiently rapidly to catch particles in the synchronous orbit. We shall see that this is not at all difficult.

The radius of the synchronous orbit, r_s, starts out smaller than the equilibrium betatron radius, r_β, increases, and eventually becomes larger. The transition to synchrotron operation takes place when $r_s = r_\beta$ (and $E_s = E_\beta$). In terms of the pendulum model, the phase during betatron operation is being uniformly accelerated by the constant torque; ϕ is initially negative, is reduced to zero at $r_s = r_\beta$, and subsequently (if the r-f voltage were not turned on) would continue to increase. While $\phi \sim 0$, it is necessary to turn on the voltage sufficiently rapidly to trap the particle in the trough of the restoring torque potential.

The dee voltage necessary to make up the difference between $\dot{E}_s$ and $\dot{E}_\beta$ is actually quite small.

For the synchronous orbit $\omega_s = v_s/r_s$ is constant; thus $\dot{r}_s/r_s = \dot{v}_s/v_s$, or in terms of $\dot{E}_s$,

$$\frac{\dot{r}_s}{r_s} = \frac{c^2}{v_s^2}\left(\frac{mc^2}{E_s}\right)^2 \frac{\dot{E}_s}{E_s}. \tag{26}$$

From (3), $\dot{E}_s/E_s = [(\partial H/\partial t) + (\partial H/\partial r)\dot{r}_s]/H$, whence, using (26),

$$\frac{\dot{E}_s}{E_s} = \frac{1}{1 + n\dfrac{c^2}{v_s^2}\left(\dfrac{mc^2}{E_s}\right)^2} \frac{\partial H/\partial t}{H}$$

$$\sim \left[1 - n\left(\frac{mc^2}{E_s}\right)^2\right] \frac{\partial H}{\partial t} \Big/ H, \tag{27}$$

the latter form being valid for $mc^2/E_s \ll 1$. For the betatron energy we have, from (2), and remembering that r_β is constant,

$$\frac{\dot{E}_\beta}{E_\beta} = \frac{v_\beta \dot{p}_\beta}{E_\beta} = \frac{v_\beta^2}{c^2} \frac{\partial H}{\partial t} \Big/ H.$$

At the time of transition ($E_s = E_\beta$), the energy per turn which must be supplied by the applied field is

$$eV \sin \phi_s = \frac{2\pi}{\omega_s}(\dot{E}_s - \dot{E}_\beta) = (1-n)\left(\frac{mc^2}{E_s}\right)^2 \Delta E_\beta, \tag{28}$$

where $\Delta E_\beta = (2\pi/\omega_s)\dot{E}_\beta$ is the energy supplied per turn by the betatron flux. In the California synchrotron, the transition takes place when the electron kinetic energy is 1.5 Mev. Using the figures for the California synchrotron, we find $\Delta E_\beta = 800$ ev, $eV \sin \phi_s = 17$ ev. Since it is quite feasible to reach the full r-f voltage of 10 kv in 1000 cycles, only two turns elapse before the requisite voltage is available, and a few more turns suffice to make ϕ_s small.

Thus if the field is turned on at just the right time, an electron is easily trapped in the synchronous orbit. The pertinent question now is the effect of errors of timing, and of the energy spread of electrons in the beam. It is expected that timing errors will be kept less than ± 1 microsecond. This corresponds to 50 turns, or an energy spread of 50×16 ev $= 850$ ev. The remaining spread of energies in the beam can be estimated in the following way. The spread arises from the finite interval of injection times: this interval lasts from the time the radius, r_1, of the instantaneous orbit[6] for electrons of the energy of injection crosses r_β to the time it crosses r_I, the injector radius. Electrons injected when $r_1 = r_\beta$ remain at this radius. The instantaneous radius of electrons injected at $r_i = r_I$ approaches r_β, while the field is increasing from its initial value H_I to a value H, according to the law given by Kerst and Serber,

$$\frac{\delta r_i}{\delta r_I} = \frac{r_i - r_\beta}{r_I - r_\beta} = \frac{H_I}{H} = \frac{p_I}{p_i},$$

where p_I is the initial momentum. The difference in energy between a particle at r_1 and one at r_β is, for $v/c \sim 1$,

$$\delta E/E_\beta = \delta p/p_\beta = (1-n)\delta r_1/r_\beta,$$

as in (12). Thus

$$\delta E = (1-n)\frac{p_I E_\beta}{p_\beta}\frac{\delta r_I}{r_\beta} = (1-n)(2mc^2T)^{\frac{1}{2}}\frac{\delta r_I}{r_\beta},$$

with T the kinetic energy of injection. With the previously given specifications, we find $\delta E = 3.1$ kev. The spread about the mean energy is $\frac{1}{2}\delta E = \pm 1.65$ kev. The requirements are thus that we trap particles of an energy spread of ± 2.5 kev.

Consider a particle which, at the time of turning on of the r-f voltage, has a phase ϕ_0 and an energy discrepancy ΔE_0 which, according to (15), determines a value of $\dot{\phi}_0$. For definiteness, let us suppose ϕ_0 and $\dot{\phi}_0$ are positive. The effect of the applied voltage is to reduce $\dot{\phi}$. If $\dot{\phi}$ is brought to zero before $\phi = \pi$, the particle will be caught, since subsequently the amplitude of phase oscillation will be damped, according to (24), by a factor $V^{-\frac{1}{4}}$. (For large amplitudes, we see from the sign of the correction terms in (20) that the damping will be even more rapid.) The amplitude of radial oscillation, however, continues to increase, with $V^{\frac{1}{4}}$, until V reaches its final value. A particle which passes $\phi = \pi$ is not necessarily lost; since V is still increasing it may still be caught, say between π and 3π. However, such orbits will, for the most part, end up with large amplitudes of phase oscillation, and in consequence such large radial amplitudes that they may well be lost for this reason. In any event, we shall not overestimate the ease of catching by ignoring such possibilities.

In order to make an estimate of the magnitudes involved let us suppose that the voltage rises linearly with time, and reaches its final value, V_f in S_f cycles. If S is the number of cycles after the voltage is turned on, $V=V_fS/S_f$. In terms of $dS=\omega_s dt/2\pi$, the phase equation, (17), becomes

$$\frac{d^2\phi}{dS^2}+\frac{2\pi S}{S_f}\frac{eV_fK}{E_s}\sin\phi=0. \tag{29}$$

The radiation loss term is negligible at these low voltages, and as we have already remarked, $eV\sin\phi_s$ is small. It can easily be verified that it may be omitted. Also, during the short times under consideration, E_s and ω_s can be taken constant. To study the initial motion, just after the voltage is turned on, we replace $\sin\phi$ by a suitable average, $\langle\sin\phi\rangle_{Av}$, the nature of which we shall determine later. The solution of (29) is then

$$\phi=-\frac{\pi eV_fK\langle\sin\phi\rangle_{Av}}{3S_fW_s}S^3+\frac{2\pi}{\omega_s}\dot{\phi}_0S+\phi_0. \tag{30}$$

When $\dot{\phi}_0>0$, ϕ reaches its maximum,

$$\phi_m=\frac{4}{3}K\left(\frac{2S_f(-\Delta E_0)^3}{eV_fE_s^2\langle\sin\phi\rangle_{Av}}\right)^{\frac{1}{2}}+\phi_0, \tag{31}$$

when $S=S_m$,

$$S_m=\left(\frac{2S_f(-\Delta E_0)}{eV_f\langle\sin\phi\rangle_{Av}}\right)^{\frac{1}{2}}. \tag{32}$$

In (31) and (32) $\dot{\phi}_0$ has been expressed in terms of ΔE_0 by means of (15). It should be remarked that, for $\phi_0>0$, the case $\dot{\phi}_0>0$ (i.e., $\Delta E_0<0$) is the critical one. If $\dot{\phi}_0<0$, ϕ is initially decreasing, rather than increasing, and more time is available for catching.

We now ask under what conditions particles with phase angles of initial magnitude less than $\pi/2$ are caught. We must have $\phi_m-\phi_0<\pi/2$. If for $\langle\sin\phi\rangle_{Av}$ we take its linear average,

$$\langle\sin\phi\rangle_{Av}=(2/\pi)\int_{\pi/2}^{\pi}\sin\phi d\phi=2/\pi,$$

(31) gives us

$$-\Delta E<\left[\frac{9}{64\pi}\frac{eV_fE_s}{K^2S_f}\right]^{\frac{1}{3}}. \tag{33}$$

With $eV_f=10$ kev, $E_s=2$ Mev, $K=3$ (corresponding to $n=\frac{2}{3}$), $S_f=1000$, we find $-\Delta E<5.8$ kev. The number of turns required to eliminate an initial energy discrepancy of this amount is $S_m=43$.

Although this treatment is somewhat rough if ϕ is allowed to approach π, it is fairly accurate for phase angles between $\pi/4$ and $3\pi/4$, since in this range $0.7<\sin\phi<1$, so an average $\langle\sin\phi\rangle_{Av}=0.85$ is never far off. The maximum phase angle reached is determined by (31); the subsequent motion is an oscillation with an amplitude damped[13] by a factor $V^{-\frac{1}{4}}$ or $S^{-\frac{1}{4}}$. The final amplitude of oscillation will be

$$\phi_f=\left(\frac{S_m}{S}\right)^{\frac{1}{4}}\phi_m=\left[\frac{2(-\Delta E_0)}{\sin\phi eV_fS_f}\right]^{\frac{1}{8}}\phi_m. \tag{34}$$

The corresponding amplitude of radial oscillation can be determined from (13) and (14). It is approximately proportional to $(V/E_s)^{\frac{1}{2}}\phi_f$.

A primary advantage of a large value of E_s at the time of transition to synchrotron operation is to reduce the amplitude of this radial oscillation.

The argument leading to (34) fails if ϕ_m is reached while V is still too small for the adiabatic theorem to be applied. It is easily seen that the condition for the validity of the adiabatic theorem is

$$S>S_A=(S_fE_s/8\pi eV_f)^{\frac{1}{2}}.$$

Thus, for (34) to be right we must have $S_m>S_A$, or

$$-\Delta E_0>\frac{\langle\sin\phi\rangle_{Av}}{8}\left(\frac{E_seV_f}{\pi^2S_f}\right)^{\frac{1}{2}}. \tag{35}$$

For $-\Delta E_0$ smaller than this, a first approximation is obtained by replacing $-\Delta E_0$ in (34) by the minimum value set by (35). A better approximation can be obtained by fitting the asymptotic solutions to (30) for a value of S greater than S_A, rather than at ϕ_m.

Another approximation, good for $|\phi|<60°$, is

[13] For greater accuracy the damping should be obtained from Eq. (20). Even for these large amplitudes, however, the higher terms are not very important and may be neglected.

TABLE I. Values of ϕ_m and ϕ_f for various values of ϕ_0 and ΔE_0.

	$\phi_0=0$			$\phi_0=45\nu$			$\phi_0=90°$		
ΔE kev	ϕ_m radians	ϕ_f	Δr_m cm	ϕ_m	ϕ_f	Δr_m	ϕ_m	ϕ_f	Δr_m
0	0	0	0	0.78	0.285	1.33	1.57	0.570	2.55
1	0.234	0.100	0.045	0.89	0.348	1.66	1.67	0.650	2.91
2	0.468	0.200	0.090	0.99	0.401	1.80	1.76	0.715	3.21
3	0.702	0.300	1.35	1.39	0.590	2.65	2.07	0.880	3.95
4	0.936	0.400	1.80	1.56	0.690	3.10	2.34	1.03	4.60

obtained by replacing $\sin\phi$, in (29), by ϕ. The solution of (29) is then

$$\phi=\Gamma\left(\frac{4}{3}\right)3^{1/3}\left(\frac{2\pi V_fK}{S_fE_s}\right)^{-1/3}\frac{2\pi}{\omega_s}\phi_0\mu^{\frac{1}{2}}J_{1/3}(\tfrac{2}{3}\mu_{3/2})$$
$$+\Gamma(\tfrac{2}{3})3^{-1/3}\phi_0\mu^{\frac{1}{2}}J_{-1/3}(\tfrac{2}{3}\mu^{3/2}), \quad (36)$$

where $\mu=(2\pi V_fK/S_fE_s)^{-\frac{1}{3}}S$. For given initial conditions ϕ_m can be determined from tables of the Bessel functions.[14] The final amplitude of oscillation can be evaluated from the asymptotic forms of $J\pm\frac{1}{3}$,

$$\phi_f=\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\left(\frac{E_s}{2\pi_eV_fKS_f^2}\right)^{1/12}$$
$$\times\left[3^{1/3}\Gamma\left(\frac{4}{3}\right)\left(\frac{S_fE_s}{2\pi eV_fK}\right)^{1/3}\right.$$
$$\times\frac{2}{\omega_s}\phi_0\cos\left(\frac{2}{3}u_f^{3/2}-\frac{5\pi}{12}\right)$$
$$\left.+3^{-1/3}\Gamma\left(\frac{2}{3}\right)\phi_0\cos\left(\frac{2}{3}\mu_f^{3/2}-\frac{\pi}{12}\right)\right]. \quad (37)$$

The final amplitude of radial oscillation is, from (19a) and (14),

$$\frac{\Delta r_f}{r_s}=\left(\frac{eV}{2\pi E(1-n)}\right)^{\frac{1}{2}}\phi_f. \quad (38)$$

In Table I, ϕ_m and ϕ_f are tabulated, for various values of ϕ_0 and ΔE_0. These calculations refer to the 300-Mev synchrotron. The final amplitude of radial oscillation is also tabulated. For $\phi_0=0$, and for $\Delta E_0=0$, the small oscillation approximation was used. The remaining values for $\phi_0=\pi/4$ and $\phi_0=\pi/2$ were computed from (31) and (34).

Even particles with $\Delta E=4$ kev and $\phi_0=90°$ are within the range $\Delta r=\pm 5$ cm.

It thus appears that with the expected energy spread of 2.5 kv in the beam, and a rise time of the r-f voltage of 1000 cycles, at least half the particles accelerated during the betatron phase of operation will be trapped in the synchronous orbit.

Azimuthal Asymmetries of the Magnetic Field

A question of considerable practical importance in the design of an accelerator such as the synchrotron is the effect on the orbits of small azimuthal asymmetries in the magnetic field. In particular asymmetries may be expected to be appreciable at the injection time, when H is small, because of eddy current and hysteresis effects.

The effect of an azimuthally varying field on the radial motion will be to introduce forced oscillations. Since these have the same period as that of rotation, the result is to slightly distort the orbit from a circle. Because there is no resonance with the period of free radial oscillations, the amplitude of the forced oscillations remains rather small.

Suppose that

$$H=H(r)\left[1+\sum_{l=1}^{\infty}h_l\cos(l\theta+\alpha_l)\right]. \quad (39)$$

Let $r=r_1+x$, where r_1 is the radius of the instantaneous orbit, defined by (2), and x is the deviation from the instantaneous circle. The equation of motion for x is

$$\frac{d}{dt}(mx)+(1-n)\omega^2x$$
$$=-m\omega^2r_i\sum_{l=1}^{\infty}h_l\cos(l\theta+\alpha_l). \quad (40)$$

Setting $\theta=\omega t$, we readily find

$$\frac{x}{r_i}=\sum_{l=1}^{\infty}\frac{h_l}{l^2+n-1}\cos(l\theta+\alpha_l), \quad (41)$$

an equation which can be used to estimate the permissible magnitudes of asymmetries.

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[14] G. N. Watson, *A Treatise on the Theory of Bessel Fnnctions*, Cambridge, 1944.