assumes that the *cascade* component can be expressed in the form

$$F(E,l)=F(l)E^{-s}, \tag{1a}$$

$$G(E,l)=G(l)E^{-s}, \tag{1b}$$

where $F(E, l)$ and $G(E, l)$ are the number of electrons and quanta, respectively, with energies greater than E at a distance l (measured in radiation units); and considers the problem as a boundary value problem with the single boundary condition that the energy content of the cascade as a whole cannot be infinite, then it follows that $1<s<2$. The demonstration proceeds along the following lines.

1. To show that $s>1$

Define new functions $f(E, l)$ and $g(E, l)$ such that

$f(E, l)dE$ = number of electrons at l with energies between E and $E+dE$,

$g(E, l)dE$ = number of quanta at l with energies between E and $E+dE$.

Then

$$f(E,l)dE=-\frac{\partial F(E,l)}{\partial E}dE, \tag{2a}$$

$$g(E,l)dE=-\frac{\partial G(E,l)}{\partial E}dE. \tag{2b}$$

The energy content of the cascade as a whole $\bar{E}$ can be expressed as

$$\bar{E}=\int_{E_j}^{\infty}[f(E,l)+g(E,l)]EdE, \tag{3a}$$

where E_j is the "crossover" energy between the shower mechanisms of collisions with nuclei (i.e., twin birth and radiative collision) and the classical mechanisms of energy loss by collisions with electrons (i.e., Compton effect and ionization).

Substituting (1) and (2) in (3) results in

$$\bar{E}=\beta(l)s\int_{E} E^{-s}dE. \tag{3b}$$

Integration and examination of (3b) leads to the conclusion that for $\bar{E}$ to be finite s must be greater than one. This has been pointed out by Heisenberg.[2]

It is even more important to know how fast the energy spectrum can fall off.

2. To show that $s<2$.

On integrating out Eq. (3b) for $s>1$ there results

$$\bar{E}=\beta(l)\frac{s}{s-1}\frac{1}{E_j^{s-1}}. \tag{3c}$$

The definition of E_j is that it is the energy of an electron at which it will lose on the average all of its energy in the next radiation unit of distance by collisions with electrons. That is, $E_j=X_0(dE/dx)_I$ where X_0 is the length of the radiation unit in centimeters

$$\left(\frac{1}{X_0}=4N\frac{Z^2e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\log\frac{183}{Z^{\frac{1}{3}}}\right),$$

and $(dE/dx)_I$ is the rate of energy loss by ionization. In the definition of $1/X_0$, N is the number of nuclei of charge Ze per cubic centimeter. Because of the very slow variation of $1/X_0$ with $\log Z^{-\frac{1}{3}}$, X_0 can be assumed to vary inversely as the product of Z^2 by N. The term $(dE/dX)_I$ is proportional to the number of extra nuclear electrons and thus to N times Z. Therefore E_j is proportional to $1/Z$. Utilizing this fact (3c) can be written

$$\bar{E}=\gamma(l)\frac{s}{s-1}Z^{s-1}. \tag{3d}$$

Equation (3d) means that the energy content of a cascade increases with the Z of the material in which the cascade is formed. The physical reason for this is that the energy at which ionization losses become important is much higher in lead than in water for example, and while a 50-Mev particle would be a cascade particle in lead it would not be in water ($E_j=10$ Mev for lead and $E_j=100$ Mev for water). While it is allowable for the energy content to increase with Z, this increase must stop as Z approaches infinity or the cascade will not have a finite energy content. This means that $d\bar{E}/dZ$ must go to zero when Z approaches infinity if not before. The limiting condition is

$$\lim_{Z\to\infty}(d\bar{E}/dZ)=\lim_{Z\to\infty}(\gamma(l)sZ^{s-2})=0. \tag{4}$$

In order for (4) to be satisfied the exponent of Z must be negative and the quantity s must be less than 2. Observations on the cascade component [1]lead to a value for the exponent "s" of 1.8 ± 0.17. This falls in the range prescribed by the foregoing that $1<s<2$.

---

[1] H. Euler, Zeits f. Physik **116**, 73 (1940).

[2] W. Heisenberg, *Cosmic Radiation* (lectures given at Max Planck Institute). Translated by T. H. Johnson (Princeton University Press, New Jersey).

The Vibrational Levels of an Anharmonic Oscillator

D. TER HAAR
Institute for Theoretical Physics, University of Copenhagen, Denmark

July 2, 1946

MORSE[1] has given an equation for the potential energy function of a diatomic molecule as a function of the distance apart of the nuclei which allows (for S-states) an exact solution. The potential has the following form:

$$U(r)=D\cdot(\exp[-2a(r-r_0)]-2\exp[-a(r-r_0)]), \tag{1}$$

where D is the dissociation energy and r_0 the equilibrium distance of the two nuclei. Morse concludes now that the energy levels are given *rigorously* by the equation:

$$E=-D+\frac{2\sqrt{D}}{A}(n+\tfrac{1}{2})-\frac{(n+\frac{1}{2})^2}{A^2}, \tag{2}$$

where

$$A=(2\mu)^{\frac{1}{2}}/a\cdot\hbar \quad (\mu\text{: reduces mass}). \tag{3}$$

Since Morse's conclusion has been quoted in many textbooks,[2] it is perhaps useful to point out that Eq. (2) is only an approximation, although in every practical case of diatomic molecules a very good approximation, as we shall see in the following.

The Schrödinger equation has the form:

$$\Delta\psi+\frac{2\mu}{\hbar^2}(E-U(r))\psi=0. \tag{4}$$

Introducing polar coordinates we get in the usual way:

$$\psi = P_l^m(\vartheta, x)\cdot\varphi(r)/r, \tag{5}$$

where $\varphi(r)$ has to be solved from the following equation (restricting ourselves to S-states):

$$\frac{d^2\varphi}{dr^2}+\frac{2\mu}{\hbar^2}(E-U(r))\varphi=0. \tag{6}$$

We look for a solution of (6) satisfying the following boundary conditions:

$$\varphi=0 \quad \text{for} \quad r=0, \tag{7a}$$

$$\varphi=0 \quad \text{for} \quad r=\infty. \tag{7b}$$

(At this point Morse introduces a different boundary condition: $\varphi=0$ for $r=-\infty$ instead of (7a); since r is a polar coordinate it will, however, never take negative values. Furthermore, Morse remarks himself that φ should be equal to zero for $r=0$ but he adds that for the eigenvalues of E, which he finds, $\varphi(0)$ is very small. This last fact corresponds just to the outcome that Eq. (2) is such a good approximation for the energy levels.)

Introducing expression (1) for $U(r)$ into Eq. (6) and applying condition (7b) we get the following solution for φ:

$$\varphi(r)=N\cdot z^{Ap}\cdot e^{-z}\cdot M(Ap+\tfrac{1}{2}-A\sqrt{D},\, 2Ap+1;\, 2z), \tag{8}$$

where

$$z=A\sqrt{D}\cdot\exp\left[-a(r-r_0)\right], \tag{9}$$

and $p=(-E)^{\frac{1}{2}}$, N a normalizing factor while $M(\alpha, \beta; x)$ is the confluent hypergeometric series, satisfying):[3]

$$x\frac{d^2M}{dx^2}+(\beta-x)\frac{dM}{dx}-\alpha M=0. \tag{10}$$

Applying now condition (7a) and using (9) we find the energy levels of the closed stationary states from the equation:

$$M(Ap+\tfrac{1}{2}-A\sqrt{D},\, 2Ap+1;\, 2A\sqrt{D}\exp(ar_0)). \tag{11}$$

In all applications to diatomic molecules $2A\sqrt{D}\cdot\exp(ar_0)$ is large so that we can use the asymptotic expression for M:[3]

$$M(\alpha, \beta; x)\sim\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(-x)^{-\alpha}\left(1-\alpha\cdot\frac{\alpha-\beta+1}{x}+\cdots\right)$$
$$+\frac{\Gamma(\beta)}{\Gamma(\alpha)}e^x x^{\alpha-\beta}\left(1+\frac{(1-\alpha)(\beta-\alpha)}{x}+\cdots\right) \tag{12}$$

and even:

$$M(\alpha, \beta; x)\sim\frac{\Gamma(\beta)}{\Gamma(\alpha)}e^x x^{\alpha-\beta}, \tag{13}$$

which gives the following zero points:

$$\Gamma(\alpha)=\infty \quad \text{or} \quad \alpha=-n, \quad n=0, 1, 2, \cdots. \tag{14}$$

Comparing this with (11) we get:

$$p=\sqrt{D}-\frac{n+\frac{1}{2}}{A}, \tag{15}$$

which gives with $p=(-E)^{\frac{1}{2}}$ just Eq. (2). We see, however, that this is an *approximation* and not the rigorous solution. The deviations are, however, so small as to be negligible in every case of diatomic molecules. From (12) it is easily seen that these deviations are only beginning to be appreciable if:

$$\left.\begin{aligned} &A\sqrt{D}\cdot(\exp(ar_0)-ar_0-1)\lesssim 1 \quad (r_0\neq 0)\\ \text{or}\quad &A\sqrt{D}\lesssim 1 \quad (r_0=0).\end{aligned}\right\}(16)$$

Morse, Fisk, and Schiff[4] have used the potential (1) for the interaction between two nucleons and in this case (16) is satisfied and they use indeed Eq. (11) to determine the relation between E and D; they do not, however, refer to Morse's original paper[1] and they do not point out the fact that for all cases of diatomic molecules Eq. (2) is the solution of Eq. (11). In Table I we have collected for different values of r_0 and $A(-E)^{\frac{1}{2}}$ the solution of $A\sqrt{D}$ from (11) according to Morse, Fisk, and Schiff[4] and the solution of $A\sqrt{D}$ from (2) (between brackets). As should be expected, the deviations decrease for increasing $A\sqrt{D}$ and ar_0.

TABLE I. Values of $A\sqrt{D}$.

$A\sqrt{-E}$ / ar_0	0	0.5	1.0
0	0.90 (0.50)	1.48 (1.00)	2.03 (1.50)
0.5	0.73 (0.50)	1.23 (1.00)	1.71 (1.50)
1.0	0.61 (0.50)	1.07 (1.00)	1.54 (1.50)

Finally we may add that a similar point arises for the potential introduced by Rosen and Morse.[5] The boundary conditions in this case should be (using their notation):

$F=0$ for $u\rightarrow 1$ and for $u=\frac{1}{2}$ (and not for $u=0$).

I want to express my sincere thanks to Professor N. Bohr and Professor C. Møller for the interest shown in this note and to the Rask-Ørsted-Foundation for a grant which has enabled me to stay in Copenhagen.

[1] P. M. Morse, Phys. Rev. **34**, 57 (1929).

[2] E.g., H. Hellmann, *Einführung in die Quantenchemie* (Leipzig-Wien, 1937), p. 294; G. Herzberg, *Molecular Spectra and Molecular Structure* (New York, 1939), Vol. I, p. 109. (Herzberg gives Eq. (2) and adds: *without* any higher powers of $n+\frac{1}{2}$ (italics of Herzberg).)

[3] See e.g., E. Jahnke and F. Emde, *Tables of Functions* (New York, 1945), Chap. X.

[4] P. M. Morse, J. B. Fisk, and L. I. Schiff, Phys. Rev. **50**, 748 (1936).

[5] N. Rosen and P. M. Morse, Phys. Rev. **42**, 210 (1932).

Cosmic Radiation Above 40 Miles

S. E. Golian, E. H. Krause, and G. J. Perlow
U. S. Naval Research Laboratory, Washington, D. C.
July 15, 1946

We have obtained cosmic-ray data above the earth's atmosphere by means of an apparatus contained in a German V-2 rocket. The rocket was fired by the Ordnance Department, United States Army on June 28, 1946 in connection with a series of tests being made by the Army at its White Sands, New Mexico, proving grounds.

Data were transmitted back to a receiving station on the ground by means of a multi-channel radio equipment. Difficulties which developed in this and accompanying electronic circuits prevented satisfactory records below 200,000 feet. Forty-one seconds of data were obtainable after this time, all of it on ascent. Maximum altitude obtained was 350,000 feet.