by previous investigators:

exp[7] -14.956; Guillemin and Zener[8] -14.837; Wilson[8] -14.838.

We see that the improvement in the energy value is considerable. The percentage error is reduced almost by a factor 3. Yet it is not as good as one might have expected, for the result still falls short of the accuracy obtained with the simplest Hylleraas functions for two-electron problems. And previous results with $1s2s$ configurations also show that the use of the simple $2s$ function cannot be wholly responsible for the discrepancy. So the inaccuracy is caused probably by the way we have omitted the products of the variable terms in constructing the trial wave function. It is uncertain if it can be improved in such a way as not to add materially to the labor of calculation.

In conclusion it is a pleasure to thank Professor Ta-you Wu for his interest and encouragement during the course of this work.

[7] Bacher and Goudsmit, *Atomic Energy States*.
[8] Condon and Shortley, *The Theory of Atomic Spectra* (1935), p. 352.

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Induction Effects in Terrestrial Magnetism*

Part II. The Secular Variation**

WALTER M. ELSASSER
RCA *Laboratories, Princeton, New Jersey*

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In Part I a method has been developed for the integration of the electromagnetic field equations in the presence of fluid motions in a spherical conductor. This analysis is here applied to an interpretation of the secular geomagnetic variations. A very brief survey of some of the observed features of the secular variation is first given. It is pointed out that not only the phases but also the magnitudes of the harmonic components, including the main dipole, are subject to large changes at the present time. There follows a brief study of the skin effect which indicates that the observed variations of the dipole terms originate in a layer adjacent to the core's boundary several hundred kilometers deep; those of the higher terms originate in a layer no more than 200 km deep. Next, the "coupling matrix" introduced in Part I is evaluated in form of a table of all matrix elements that contain vectors of dipole and quadrupole type but no higher harmonics. It is shown that a zonal fluid motion (zonal toroidal flow in the terminology adopted here) produces rotation of the tesseral magnetic dipole terms and also oscillatory changes in amplitude of these terms. There is one and only one type of matrix element that represents an interaction of the main magnetic dipole with itself; the corresponding fluid motion is a meridional flow (poloidal flow) of quadrupole symmetry. With this term amplification or de-amplification occurs, depending on the sign of the velocity. The theory thus can account for all the observed components of the secular variation.

IN Part I of this paper the mathematical treatment was brought close to the point where a comparison with observational data is feasible. We shall now first give a brief glance to some of the observed features of the secular magnetic variation.

OBSERVATIONAL DATA

There is no need for a complete review at this place since a few years ago two extensive reference works[1,2] have appeared in short succession. The reader is referred to these for all details.

It is always possible to separate the fields of internal and external origin and in this paper we consider only the part of the field of internal origin. The separation of the two fields can be carried out by a straightforward mathematical procedure based on the principles of potential theory; but the external field is so small that its contribution to the first few harmonics is negli-

* The completion of this work which was begun prior to the writer's joining the RCA Laboratories, has been delayed because of the war.
** Part I of this paper: W. M. Elsasser, Phys. Rev. **60**, 876 (1941).

[1] J. A. Fleming, ed. *Terrestrial Magnetism and Electricity* (McGraw-Hill Book Company, Inc., New York, 1939).
[2] S. Chapman and J. Bartels, *Geomagnetism* (Clarendon Press, Oxford, 1940), 2 volumes.

gible for most purposes, especially if the external field is averaged over the short period fluctuations of the order of a few hours to a few days.

In Part I we used sets of fundamental vectors expressed in terms of spherical harmonics. While this development is convenient as a mathematical tool, the observed distribution of the magnetic field does not lend itself readily to the same type of analysis. The observational data are in general summarized in the form of magnetic maps, and rightly so since the spherical harmonic series for the actual field converges very slowly. It can therefore hardly replace the maps in practice even for the broad outlines of the field. Moreover, an accurate determination of spherical harmonic coefficients from observation would presuppose data available from all over the globe while in practice data originating in latitudes above 60° are extremely scarce. This fact does not preclude the formal determination of spherical harmonic coefficients from observations, but it introduces a considerable degree of inaccuracy into these coefficients. If nevertheless we have used the method of spherical harmonics in this paper, it is mainly on account of its mathematical convenience.

There exist two extensive analyses of the field in terms of spherical harmonics, that of Schmidt[3] for the year 1880 and that of Dyson and Furner[4] for the year 1922. Both include coefficients up to $n=6$, but the sixth-order terms involve too large errors to be significant. The distribution in magnitude of these coefficients is apparently random except for an over-all decrease with increasing n. It can be shown that a rather regular convergence obtains for the series whose coefficients are the root-mean-square values of all the coefficients having the same zonal index n. If we assume that there are no important sources of the magnetic field above the boundary of the core, it may be concluded from the convergence of this series that the bulk of the sources of the higher harmonics (quadrupoles and above) is most probably located in the top strata of the core.[5]

Although the two analyses quoted are some 40 years apart, it is not advisable to use them for the purpose of finding the secular variation of individual components except for a few leading terms, because of the errors inherent in this method. The material available for the dipole terms alone is somewhat more ample.[6,7] In Fig. 1 are plotted the relative magnitudes of the dipole terms parallel and perpendicular to the earth's axis, from eight separate determinations. The large scatter of the points is indicative of the inaccuracies of the method of computation, but the general trend is unmistakable. The relative values have been adjusted so that both coefficients are unity in the year 1900; the straight lines have been drawn from a visual estimate. The slope of the parallel dipole term represents a variation of -4.3 percent per century, that of the perpendicular dipole term, -3.6 percent per century. Similarly, the phase of the perpendicular dipole can be plotted against time. The resulting points scatter somewhat like those of Fig. 1; a mean curve corresponds to a rotation of the dipole, in the same sense as the earth itself, at a rate of 4.5 degrees per century, corresponding to a period of revolution of 8000 years. This is one of the slower components of the secular variation.

The objection may be raised that the primary data vary in time with the progressively greater

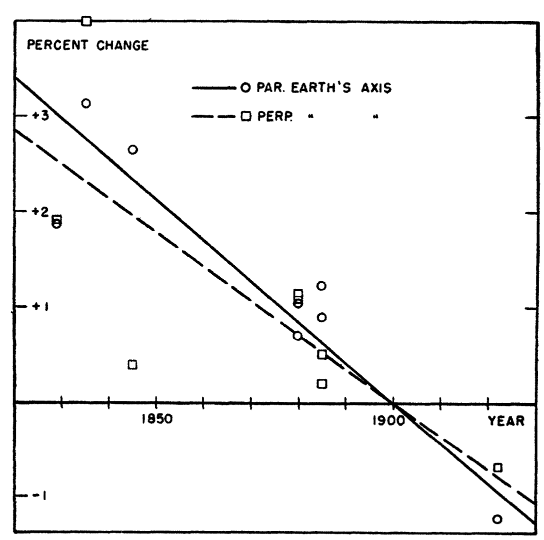

FIG. 1. Relative magnitude of dipole terms from eight different determinations.

[3] Ad. Schmidt, Abh. Bayer. Akad. Munich 19, 1 (1895).
[4] F. W. Dyson and H. Furner, M.N.R.A.S., Geophys. Suppl. 1, 76 (1923).
[5] W. M. Elsasser, Phys. Rev. 60, 876 (1941).
[6] See the article by McNish in reference 1.
[7] See Chapter 18 of reference 2.

extent of the observations, and that nearly simultaneous observations, although made by different authors, are likely to be based on nearly the same material, thus introducing a systematic error into the results. This is probably true to a certain extent, but only a fraction of the observed decrease of the dipole field could be attributed to this cause.*

The data found in the literature[6, 7] indicate that for the quadrupole and higher terms the rates of relative change and the corresponding rates of rotation are even larger and that periods of the order of a thousand years or less are not uncommon. As has been pointed out before, the spherical harmonic series is not well adapted to the evaluation of the more detailed variations. Instead, a glance upon charts showing lines of constant annual secular variation[8, 9] is instructive. Most of these lines consist of several closed curves that surround some seven or eight principal "foci" of geomagnetic activity. The regions of intense change around the foci have diameters of several thousand kilometers, so that their appearance on the surface of the earth is well compatible with the assumption that the sources of these variations are located at or below the boundary of the core which is 2900 km beneath the earth's surface. The rate of change of the field in such regions is of the order of $0.5\text{–}1.5\times 10^{-3}$ gauss per year, as compared to an absolute magnitude of the field of 0.3–0.5 gauss (both values referring to the earth's surface). Hence Fourier components with periods of the order of several hundred to a few thousand years are indicated.

The regional distribution of the field variations and of their secular change seems to be capable of, at least, a semiquantitative simple interpretation. McNish[10] has shown that the instantaneous non-dipole part of the field can be represented with some accuracy by means of 14 elementary dipoles of suitable intensities located at a depth of one-half of the earth's radius, that is, slightly below the boundary of the core. The same author indicates that the regional secular variations can similarly be represented by a set of dipoles at this depth whose intensities change at certain given rates.

* *Note added in proof:* A harmonic analysis of the field for the year 1942 which has just appeared corroborates the general decrease of the dipole field as expressed in Fig. 1: V. I. Afanasieva, Terr. Mag. **51**, 19 (1946).

[8] See the article by Fleming in reference 1.

[9] See chapter 3 of reference 2.

[10] A. G. McNish, Trans. Am. Geophys. Union **2**, 287 (1940).

THE FIELD EQUATIONS

Resuming now our mathematical analysis we return for a moment to the electromagnetic field equations. In the presence of fluid motions in the conductor they are, by (1) and (31) of Part I:

$$\nabla\times\mathbf{B}-\mu\sigma\mathbf{E}=\mu\sigma\mathbf{v}\times\mathbf{B}, \tag{1}$$

$$\nabla\times\mathbf{E}+\partial\mathbf{B}/\partial t=0,\quad \nabla\cdot\mathbf{B}=0. \tag{2}$$

On taking the curl of (1) and substituting from (2) we obtain

$$-\nabla^2\mathbf{B}+\mu\sigma\partial\mathbf{B}/\partial t=\mu\sigma\nabla\times(\mathbf{v}\times\mathbf{B}). \tag{3}$$

Subsequently we shall assume, in addition

$$\nabla\cdot\mathbf{v}=0. \tag{4}$$

On using (4) and the second equation (2), the right-hand side of (3) can be transformed by a known vector identity into:

$$\partial\mathbf{B}/\partial t=(\mu\sigma)^{-1}\nabla^2\mathbf{B}-(\mathbf{v}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{v}. \tag{5}$$

If in this equation we put $\mathbf{v}=0$, we are left with an equation for the magnetic vector which has the form of a diffusion equation. This equation describes the gradual march of the field across a conductor and gives rise to the familiar phenomenon of the skin effect. It should be pointed out that the analogy with diffusion is rather limited since the boundary conditions at the surface of a conductor are different from the boundary conditions of a diffusion problem; the analogy with diffusion applies, therefore, mainly in the interior of an extensive conductor where the effect of the boundaries is small.

Assume now, in the second instance, that the velocities are very large so that the "diffusion" term can be neglected. The second term on the right hand side of (5) represents the convection of the field with the fluid. The field equations can now be written

$$d\mathbf{B}/dt=(\mathbf{B}\cdot\nabla)\mathbf{v}, \tag{6}$$

where the total derivative has the usual significance, indicating the rate of change of the field with respect to a moving particle.

The term, finally, which appears on the right-hand side of (6), represents the induction effects proper. Equation (6) is capable of a simple kinematical interpretation. Introduce a cartesian system of coordinates whose x axis has the direction of $\mathbf{B}$ at a given point and instant. We then have, by (4),

$$\frac{d\mathbf{B}}{dt}=-B\left(\frac{\partial \mathbf{v}}{\partial y}+\frac{\partial \mathbf{v}}{\partial z}\right). \tag{7}$$

Hence the rate of change of $\mathbf{B}$ for a fluid particle is equal to the instantaneous two-dimensional *convergence* of the velocity in a plane normal to the direction of $\mathbf{B}$.

Following now the analysis of Part I further, we set

$$\mathbf{B}=\nabla\times\mathbf{A}=\nabla\times(\textstyle\sum_\gamma c_\gamma \mathbf{T}(\gamma)), \tag{8}$$

and for the velocity

$$\mathbf{v}=\textstyle\sum_\alpha (v_\alpha \mathbf{S}(\alpha)+w_\alpha \mathbf{T}(\alpha)). \tag{9}$$

By standard procedures we arrive at the system of differential equations:

$$dc_\gamma/dt+\Lambda_\gamma c_\gamma=\textstyle\sum_{\alpha\beta} c_\beta R^{-1}(v_\alpha[\mathbf{T}_\alpha\cdot\mathbf{S}_\beta\times\mathbf{T}_\gamma^*] +w_\alpha[\mathbf{S}_\alpha\cdot\mathbf{S}_\beta\times\mathbf{T}_\gamma^*]), \tag{10}$$

where by (36), Part I, a square bracket stands as an abbreviation for the integral over the volume of the conducting sphere. The Λ's, which in Part I have been identified as the coefficients of free decay, arise from the "diffusion" term of Eq. (5). The "diffusion" effects will now be dealt with in a form appropriate to our special problem.

SKIN EFFECT

On applying to the currents flowing in the core harmonic analysis, both in time and in space, we are able to confine our attention to a system of currents with the following properties: It varies in time as $\exp(-i\omega t)$, and it is two-dimensional, the currents flowing in a spherical surface of radius r_0 where they are defined by the following toroidal vector field[11]

$$\Pi_{(\vartheta)}=\frac{f}{\sin\theta}\frac{\partial Y_n{}^m}{\partial\varphi},\quad \Pi_{(\varphi)}=-f\frac{\partial Y_n{}^m}{\partial\vartheta}, \tag{11}$$

where f is a constant. Apart from this surface the magnetic field fulfills the equation

$$\nabla^2\mathbf{B}+i\omega\mu\sigma\mathbf{B}=0,$$

and the same equation applies to the generating scalar, ψ:

$$\nabla^2\psi+iq^2\psi=0,\quad q=(\omega\mu\sigma)^{\frac{1}{2}}. \tag{12}$$

A particular solution of (12) is

$$\psi=\text{const.}\ r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(i^{\frac{1}{2}}qr)\,Y_n{}^m(\vartheta,\varphi).$$

Now for values of ω corresponding to periods of several thousand years or less the argument qr of the Bessel function is numerically large except for points near the center of the sphere. Hence in the upper strata of the core we may replace the Bessel function by its asymptotic expansion. We now set, as a solution for the interior region, $r\leqq r_0$, valid everywhere except near the center,

$$\psi^{(i)}=ar^{-1}\exp(i^{-\frac{1}{2}}qr)\,Y_n{}^m,$$

where a is a constant. In the region intermediate between the current-bearing surface and the outer boundary of the core we must use a combination of two linearly independent solutions of the Bessel differential equation. Introducing at once the asymptotic expressions, we may use two conveniently chosen linearly independent solutions without reference to the special cylinder functions from which they originate. We thus set in the middle region, $r_0\leqq r\leqq R$

$$\psi^{(m)}=r^{-1}[b\cdot\exp(i^{-\frac{1}{2}}qr)+c\cdot\exp(-i^{-\frac{1}{2}}qr)]\,Y_n{}^m,$$

and for the external space, $r\geqq R$, we set

$$\psi^{(e)}=dr^{-n-1}\,Y_n{}^m,$$

where b, c, d are constants.

From the generating function ψ the field vectors are derived by differentiation. The field equations are fulfilled if

$$\mathbf{A}=\mathbf{T},\quad \mathbf{E}=i\omega\mathbf{T},\quad \mathbf{B}=R^{-1}\mathbf{S}.$$

Equations (10)–(14) of Part I give the explicit expressions for these vectors. The boundary conditions are as follows. At the inner boundary, $r=r_0$, we require continuity of $\mathbf{E}$ and $\mathbf{B}_{(r)}$, and further,

$$\mathbf{B}_{(\vartheta)}{}^{(m)}-\mathbf{B}_{(\vartheta)}{}^{(i)}=\mu\Pi_{(\varphi)},$$

$$\mathbf{B}_{(\varphi)}{}^{(m)}-\mathbf{B}_{(\varphi)}{}^{(i)}=-\mu\Pi_{(\vartheta)}.$$

[11] According to the discussion in Part I the magnetic field is purely poloidal, hence the electric field and current density are purely toroidal.

At the outer boundary, $r=R$, we require

continuity of $\mathbf{E}$ and of $\mathbf{B}_{(r)}$, and
continuity of $\mathbf{B}_{(\vartheta)}/\mu$ and $\mathbf{B}_{(\varphi)}/\mu$.

As before, μ will be assumed constant throughout space, so that the latter boundary conditions reduce to the requirement of continuity for both field vectors. The boundary conditions lead in the usual way to a system of linear equations among the coefficients, a, b, c, d, f. If the first three coefficients are eliminated one obtains the following relation for the strength of the field in the external space expressed by the intensity of the two-dimensional currents:

$$d=(n+i^{-\frac{1}{2}}qR)^{-1}R^{n+1}r_0\mu f\cdot\exp\left[-i^{-\frac{1}{2}}q(R-r_0)\right].$$

We may evaluate this expression by comparing it with the strength of the field that would be produced by the same system of surface currents in empty space, that is if we had $\sigma=0$ everywhere, apart from the current-bearing surface. The generating scalar of this field in the external space is given by

$$\psi=d_0r^{-n-1}Y_n{}^m,$$

where, after some simple calculations, we find

$$d_0=(2n+1)^{-1}r_0{}^{n+2}\mu f.$$

Therefore, finally,

$$\frac{d}{d_0}=\frac{2n+1}{n+i^{-\frac{1}{2}}qR}\left(\frac{R}{r_0}\right)^{n+1}\exp\left[-i^{-\frac{1}{2}}q(R-r_0)\right]. \quad (13)$$

If n is not too large the behavior of (13) is swamped by the real part of the exponential, which is

$$\exp\left[-(\tfrac{1}{2}\omega\mu\sigma)^{\frac{1}{2}}(R-r_0)\right].$$

The factor under the square root is known from the usual theory of the skin effect. Assuming $\omega=2.5\cdot10^{-11}$, corresponding to a frequency of one cycle per 8000 years we obtain a "skin depth" of

$$(\tfrac{1}{2}\omega\mu\sigma)^{-\frac{1}{2}}=250\text{ km}.$$

It may safely be concluded that the secular variations of the dipole terms originate within the upper fifth of the core. Any constant part of the field may, of course, come from greater depth. The regional variations, having periods of the order of a thousand years or so, must possess sources that lie entirely in the top strata of the core, no lower than about 200 km below the boundary.

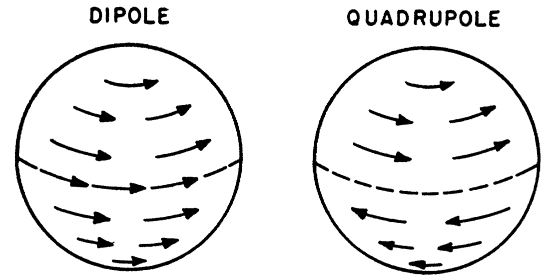

FIG. 2. A toroidal dipole and quadrupole, in perspective.

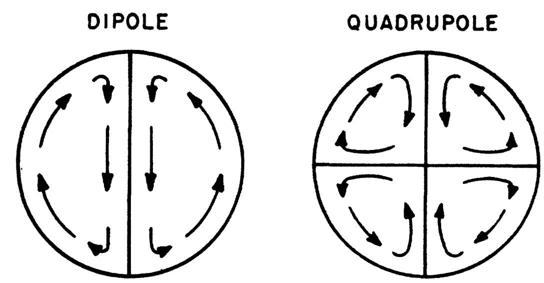

FIG. 3. A poloidal dipole and quadrupole, in meridional cross section, for $s=1$.

A TABLE OF MATRIX ELEMENTS

We pass now to the integration of the equations of motion (10) in simple cases. Below are given explicit expressions for all matrix elements which contain dipoles and quadrupoles but no higher spherical harmonics. The table is divided into two parts according to whether the velocity vector is toroidal or poloidal. In order to enable the reader to visualize readily the geometry of these vector fields some simple patterns are shown in Figs. 2 and 3. The fields shown are of the zonal type, i.e., $m=0$. Figure 2 represents in a perspectivic drawing a toroidal dipole and quadrupole field as it appears on the surface of the sphere. Figure 3 shows in meridional cross section types of flow which have the symmetry of a poloidal dipole and quadrupole. The character of these vector fields may be described as follows. If the zonal index is n there are n "belts" of zonal toroidal flow of alternating directions. In the poloidal case there are, in the meridional plane, n "vortices" along a meridional strip extending from the north to the south pole.

Thus in the dipole case the vortex shown extends from the north to the south pole; in the quadrupole case there are two vortices, each extending from a pole to the equator; and in the octupole case there would be two adjacent vortices in each hemisphere, separated by a null at 45°.

Similarly, the radial index, s, determines the number of reversals of the vectors between the center and the boundary of the sphere. Thus, in the toroidal case, there would be s belts of flow of alternating direction superposed upon each other on going from the center to the boundary at any given angle θ. For the poloidal case our Fig. 3 corresponds to $s=1$; for higher values of s there would be s vortices of alternating sense of circulation superposed upon each other between the center and the boundary, in any one sector of angular width π/n.

As illustrated by Figs. 2 and 3 for $m=0$, toroidal vector fields are symmetrical with respect to the equatorial plane when n is odd, more generally, when $n-m$ is odd; they are antisymmetrical with respect to the equatorial plane when $n-m$ is even. Poloidal vector fields are symmetrical with respect to the equatorial plane when $n-m$ is even and antisymmetrical when $n-m$ is odd. In a first approximation, disregarding turbulence, the fluid motion should be symmetrical about the equatorial plane and should therefore only contain odd toroidal and even poloidal terms.

The following are the matrix elements that involve vectors of toroidal flow:

$$
\begin{aligned}
&[\mathbf{T}_1 \cdot \mathbf{S}_1{}^1 \times \mathbf{T}_1{}^{-1}]=4\pi i(2/3)NF,\\
&[\mathbf{T}_1{}^1 \cdot \mathbf{S}_1 \times \mathbf{T}_1{}^{-1}]=-4\pi i(2/3)NF,\\
&[\mathbf{T}_1 \cdot \mathbf{S}_2{}^1 \times \mathbf{T}_2{}^{-1}]=4\pi i(6/5)NF,\\
&[\mathbf{T}_1 \cdot \mathbf{S}_2{}^2 \times \mathbf{T}_2{}^{-2}]=4\pi i(12/5)NF,\\
&[\mathbf{T}_1{}^1 \cdot \mathbf{S}_2{}^{-1} \times \mathbf{T}_2] =4\pi i(6/5)NF,\\
&[\mathbf{T}_1{}^1 \cdot \mathbf{S}_2 \times \mathbf{T}_2{}^{-1}]=-4\pi i(6/5)NF,\\
&[\mathbf{T}_1{}^1 \cdot \mathbf{S}_2{}^1 \times \mathbf{T}_2{}^{-2}]=-4\pi i(6/5)NF,\\
&[\mathbf{T}_1{}^1 \cdot \mathbf{S}_2{}^{-2} \times \mathbf{T}_2{}^1] =4\pi i(6/5)NF,\\
&[\mathbf{T}_2 \cdot \mathbf{S}_1{}^1 \times \mathbf{T}_2{}^{-1}]=4\pi i(2/5)NF,\\
&[\mathbf{T}_2{}^1 \cdot \mathbf{S}_1 \times \mathbf{T}_2{}^{-1}]=-4\pi i(2/5)NF,\\
&[\mathbf{T}_2{}^1 \cdot \mathbf{S}_1{}^1 \times \mathbf{T}_2{}^{-2}]=4\pi i(2/5)NF,\\
&[\mathbf{T}_2{}^2 \cdot \mathbf{S}_1 \times \mathbf{T}_2{}^{-2}]=-4\pi i(4/5)NF,
\end{aligned}
$$

Here, N stands as an abbreviation for the product of the normalization factors

$$N=N(\alpha)N(\beta)N(\gamma),$$

where $N(\alpha)$, etc., is the normalization factor given by the square root in (22), Part I. F stands for the integral over the radial functions as given by (39), Part I.[12] F is a function of n and of indices, s, of the radial functions; the sundry indices, n and s, of each individual F have been omitted for the sake of simplicity.

The matrix elements of poloidal flow are:

$$
\begin{aligned}
&[\mathbf{S}_1 \cdot \mathbf{S}_1 \times \mathbf{T}_2] =4\pi(4/5)NG^-,\\
&[\mathbf{S}_1{}^1 \cdot \mathbf{S}_1 \times \mathbf{T}_2{}^{-1}]=4\pi(2/5)NG^-,\\
&[\mathbf{S}_1{}^1 \cdot \mathbf{S}_1{}^{-1} \times \mathbf{T}_2] =-4\pi(2/5)NG^-,\\
&[\mathbf{S}_1{}^1 \cdot \mathbf{S}_1{}^1 \times \mathbf{T}_2{}^{-2}]=4\pi(2/5)NG^-,\\
&[\mathbf{S}_2 \cdot \mathbf{S}_1 \times \mathbf{T}_1] =4\pi(4/5)NG^+,\\
&[\mathbf{S}_2 \cdot \mathbf{S}_1{}^1 \times \mathbf{T}_1{}^{-1}]=-4\pi(2/5)NG^+,\\
&[\mathbf{S}_2 \cdot \mathbf{S}_2 \times \mathbf{T}_2] =4\pi(36/35)NG^-,\\
&[\mathbf{S}_2 \cdot \mathbf{S}_2{}^1 \times \mathbf{T}_2{}^{-1}]=4\pi(18/35)NG^-,\\
&[\mathbf{S}_2 \cdot \mathbf{S}_2{}^2 \times \mathbf{T}_2{}^{-2}]=-4\pi(36/35)NG^-,\\
&[\mathbf{S}_2{}^1 \cdot \mathbf{S}_1 \times \mathbf{T}_1{}^{-1}]=4\pi(6/5)NG^+,\\
&[\mathbf{S}_2{}^1 \cdot \mathbf{S}_1{}^{-1} \times \mathbf{T}_1] =4\pi(6/5)NG^+,\\
&[\mathbf{S}_2{}^1 \cdot \mathbf{S}_2{}^{-1} \times \mathbf{T}_2] =4\pi(18/35)NG^-,\\
&[\mathbf{S}_2{}^1 \cdot \mathbf{S}_2{}^1 \times \mathbf{T}_2{}^{-2}]=4\pi(54/35)NG^-,\\
&[\mathbf{S}_2{}^2 \cdot \mathbf{S}_1{}^{-1} \times \mathbf{T}_1{}^{-1}]=4\pi(12/5)NG^+,\\
&[\mathbf{S}_2{}^2 \cdot \mathbf{S}_2{}^{-2} \times \mathbf{T}_2] =-4\pi(36/35)NG^-\\
&[\mathbf{S}_2{}^2 \cdot \mathbf{S}_2{}^{-1} \times \mathbf{T}_2{}^{-1}]=4\pi(36/35)NG^-.
\end{aligned}
$$

Here, N has the same significance as before, and

$$
\begin{aligned}
G^+&=G_\alpha+G_\beta,\\
G^-&=G_\alpha-G_\beta,
\end{aligned}
$$

where G_α is the integral over the radial functions given by (43), Part I. The sundry indices, n and s, of the individual G's have again been omitted.

In using these formulas it might be remembered that

$$P_n{}^{-m}(\cos\vartheta)=\frac{(n-m)!}{(n+m)!}P_n{}^m(\cos\vartheta).$$

[12] In this formula read rdr in place of r^2dr. Other corrections: In formula (41) the right-hand side should be negative. In formula (44) insert a factor G_α in front of the second integral.

For the *normalized* vectors, however, we find readily $(\mathbf{T}_n{}^m)^* = \mathbf{T}_n{}^{-m}$ and similarly for the vectors $\mathbf{S}$.

There are twelve elements in the first table and sixteen in the second. From these, additional matrix elements are derived by the following two operations: First, permutation of any two or three of the vectors in the bracket, on giving the bracket the positive or negative sign according to whether the permutation is of even or odd order. Secondly, a change of sign of all the upper indices in any one bracket. The total number of matrix elements of toroidal flow thus obtained is forty, that of elements of poloidal flow is forty-six. There are, in addition, forty-six matrix elements of scaloidal flow which differ from those of poloidal flow only in the form of the integrals over the radial functions. All matrix elements involving dipole and quadrupole vectors only which are not contained in this extended table, vanish in accordance with the selection rules derived in Part I.

MAGNETIC DIPOLE INTERACTIONS

In order to analyze the physical effects corresponding to the individual matrix elements we may classify the various vectors according to their order of magnitude. This is readily done for the magnetic field, that is for the "primary" magnetic vector of the matrix elements. The zonal dipole term is dominant, next comes the tesseral dipole term whose magnitude is about 20 percent of the former. The quadrupole terms are much smaller; the root-mean-square value of their potential over the earth's surface amounts to about 3.7 percent of the zonal dipole;[5] the fraction at the boundary of the core is about 6.5 percent.

It is much more difficult to estimate the relative magnitude of the components of the fluid motion. We must distinguish between the mean motion obtained from the time-average over the large scale turbulent deviations on the one hand, and the instantaneous pattern of the motion on the other. As far as the *mean* motion is concerned it may safely be assumed, on general principles of geophysical and astrophysical hydrodynamics, that the toroidal terms are predominant; they must be odd, $\mathbf{T}_1$, $\mathbf{T}_3$, $\mathbf{T}_5$, etc. and they represent a flow directed along the circles of latitude. Such a mean motion must, however, be accompanied by a subsidiary mean motion of the poloidal type; these vectors must be even, $\mathbf{S}_2$, $\mathbf{S}_4$, $\mathbf{S}_6$, etc. and they represent circulations with streamlines confined to the meridional planes. The magnitude of this meridional circulation may safely be assumed to be smaller than that of the zonal toroidal flow.

If we consider the instantaneous motion, including large-scale turbulence, it is still probable that the zonal flow predominates, but the turbulent velocities might occasionally be of comparable magnitude. As far as the harmonic components of the turbulent motion other than $\mathbf{T}_1$, $\mathbf{T}_3$, $\mathbf{T}_5$, etc. are concerned, it is difficult to make general statements, except that most of them are probably of about the same order of magnitude and that the development of the turbulent velocity field in terms of spherical harmonics converges extremely slowly.

We shall now assume that the "primary" magnetic field appearing in the matrix elements is given by the dipole terms, $\mathbf{S}_1$ and $\mathbf{S}_1{}^1$, $\mathbf{S}_1{}^{-1}$ alone. For any given term of the "secondary" magnetic field we can then find a number of matrix elements in the above table by which this term of the secondary field is produced through the action of specific components of the fluid motion. In the following discussion we shall limit ourselves to those cases where the secondary magnetic field is also a dipole. The preceding tables contain seven matrix elements in which both the primary and the secondary field are dipoles.

We begin with the first element in the table of toroidal flow; it represents an interaction of the tesseral dipole with itself.[13] The velocity vector, $\mathbf{T}_1$, appearing in this matrix element is presumably the largest term of the mean fluid motion. Consider now a fluid motion which contains this term only. Even in this case there is still an infinite set of such matrix elements distinguished from each other by the index, s of the radial eigenfunction of the primary field and by the corresponding index of the secondary field. As far as the radial function of the fluid motion is concerned, we have pointed out in

[13] It should be remembered that the last vector in the bracket is conjugate complex to the component actually appearing in the secondary field.

Part I that there is no need for a development into orthogonal components, and we may, therefore, assume that there is only one radial eigenfunction of the velocity vector under the radial integrals.

As has been shown in Part I, the coupling matrix of toroidal flow is anti-symmetrical with respect to the primary and secondary field. It may readily be shown that this particular anti-symmetry comes about by the interchange of the indices n, m alone and that the coupling matrix is symmetrical with respect to an exchange of s_β and s_γ. We may therefore now assume that the matrix has been brought to its diagonal form with respect to the index s, and we can select a single diagonal element for which the equations of motion (10) become

$$\begin{aligned} dc_1^1/dt &= (8\pi/3)^{-\frac{1}{2}}(w_1F/R)ic_1^1, \\ c_1^1 &= \text{const.}\exp\left[(8\pi/3)^{-\frac{1}{2}}(w_1F/R)it\right]. \end{aligned} \tag{14}$$

The integration is performed here under the assumption of stationary fluid motion. The constant, w_1, has the dimension of a velocity and is the coefficient of $\mathbf{T}_1$ in the development (9) of the velocity field. F is the radial integral, of the form (39), Part I, which contains those linear combinations of the radial eigenfunctions of the field that make the matrix diagonal with respect to the index s. The motion of the tesseral dipole described by (14) is a rotation about the earth's axis. On assuming for this rotation the value of 4.5 degrees per century quoted before we can compute the velocity, w_1. We need only estimate F; this, being an integral over normalized functions, cannot exceed unity. On putting $F=0.3$ we find[14]

$$w_1 \sim 0.01 \text{ cm/sec.}$$

Next, we consider the coupling between the zonal and tesseral dipole terms. One such interaction is obtained from the first matrix element of the table of toroidal flow by a transposition, namely

$$[\mathbf{T}_1^{-1}\cdot\mathbf{S}_1^1\times\mathbf{T}_1] = -4\pi(2/3)NF,$$

and the inverse action is given by the second element of the table. On performing again the reduction to diagonal form with respect to the radial index, s, we arrive at the following set of equations

$$\begin{aligned} dc_1/dt &= -(8\pi/3)^{-\frac{1}{2}}(w_1^{-1}F_1^{-1}/R)ic_1^1, \\ dc_1^1/dt &= -(8\pi/3)^{-\frac{1}{2}}(w_1^1F_1^1/R)ic_1, \end{aligned} \tag{15}$$

where, for instance $F_1^{-1}=F({}_1{}^{-1}{}_1{}^1{}_1{}^0)$ and similarly for F_1^1. The solutions are harmonic functions of the time; they are of opposite phase for the two coefficients. The existence of the effect depents on the presence of the components $\mathbf{T}_1^1$ and $\mathbf{T}_1^{-1}$ in the fluid motion; the simplest way of producing these components is to let the fluid rotate about an axis different from that of the earth. As indicated in the appendix, there exist some rather serious reasons against the casual assumption of a change in the rotational axis of the fluid, so that one would not dare draw any further conclusions without a more far-reaching hydrodynamical basis.

The interactions just analyzed exhaust the dipole couplings caused by toroidal flow. On going now to the second table, of poloidal flow, we find there the element, fifth in the list

$$Q=[\mathbf{S}_2\cdot\mathbf{S}_1\times\mathbf{T}_1], \tag{16}$$

representing a coupling of the main, zonal dipole with itself. This matrix element is the only one in which both the primary and the secondary field are equal to the main dipole. To show this, note first that the fluid motion in such a matrix element must obviously have rotational symmetry, that is, $m=0$. If now the velocity vector were toroidal, the corresponding matrix elements would vanish because $\mathbf{T}_n\times\mathbf{T}_m=0$, identically for any two zonal toroidal vectors. Again, if the velocity vector is poloidal, the matrix elements involving higher harmonics, $\mathbf{S}_4$, $\mathbf{S}_6$, etc., vanish by virtue of the selection rules. Hence we are left with the matrix elements of the type (16) as the only one expressing an interaction of the main dipole with itself.

There is, in reality, again a twofold infinity of such matrix elements labeled by the radial indices s_β and s_γ, but now the matrix is no longer simply symmetrical or simply anti-symmetrical with respect to these two indices, as may be seen from the form of the radial integrals given by

[14] Erratum: On the lower right-hand side of page 111, Part I, read $v\sim 2$ mm/hour, and at the end of this paragraph read: "of the order of several decimeters per hour."

(43), Part I. The equations of motion (10) may now be written, if we set $s_\beta = s$, $s_\gamma = s'$

$$dc_s/dt + \Lambda_s c_s = v_2 R^{-1} \sum_{s'} Q(s, s') c_{s'}, \quad (17)$$

where v_2 is the coefficient of $\mathbf{T}_2$ in the development (9) of the velocity field. The matrix elements, $Q(s, s')$ are all real quantities; therefore the coefficients c_s vary in magnitude, roughly speaking, in an exponential manner. Since the matrix in (17) cannot, in general, be reduced to diagonal form, a stationary mechanism of amplification cannot be assumed to exist, as has already been stated at the end of Part I. Instead, it must be assumed that the representative point of the system describes an irregular curve in the multi-dimensional space subtended by the coefficients c_s. This irregularity of the magnetic amplification or de-amplification is superposed upon the statistical behavior arising out of the turbulent character of the fluid motion. There might also exist a stratification of the fluid near the boundary so that the velocity pattern changes appreciably with increasing depth. If this condition holds the secular variation of the dipole terms shown in Fig. 1 might be attributed to a transient flow pattern in the topmost layer. The numerical values of the velocity to be computed from this type of interaction will be of the same general order of magnitude as given before for toroidal flow.

We now proceed to the next matrix element that involves an interaction between magnetic dipole terms. This is the sixth in the list of matrix elements of poloidal flow and describes an interaction of the perpendicular dipole with itself. This element contains the same component of the velocity, $\mathbf{S}_2$, as the preceding one; it differs from the latter only in so far as the coefficients of the radial eigenfunctions of the perpendicular dipole field may be different from those of the parallel dipole field. If these coefficients are alike, the two sets of matrix elements are equal in magnitude and opposite in sign. This action will tend to de-amplify the perpendicular dipole when the parallel dipole is amplified. In order to account for the presence of a perpendicular dipole one must rely on other interaction terms, a conclusion which seems obvious from the rotational symmetry of the velocity vector $\mathbf{S}_2$ appearing in these matrix elements.

There are two more matrix elements in the table of poloidal flow that represent dipole interactions. The tenth element of this table has the main dipole for its primary and a perpendicular dipole for its secondary field. The eleventh element represents the reverse interaction. Both couplings are produced by the velocity component $\mathbf{S}_2^1$. Finally, the third element from the bottom of the list describes an interaction of the perpendicular magnetic dipole with itself, engendered by the velocity component $\mathbf{S}_2^2$.

LIMIT OF AMPLIFICATION

In what precedes we have been concerned with an interpretation of the secular variation in terms of fluid motions. The question arises naturally of whether the amplifying mechanism which we have shown to exist is of such a nature that it could maintain the earth's field over an interval of time long compared with the periods of free decay. From the mathematical viewpoint this question is connected with the symmetry of the coupling matrix, as we have seen in Part I. If the coupling matrix is antisymmetrical the field decays; if the coupling matrix were symmetrical, on the other hand, a mechanism of indefinite amplification could at once be constructed. Now we have seen that the actual coupling matrix is in no case purely symmetrical. It seems difficult to demonstrate whether the symmetry of the coupling matrix is not only sufficient, but also a necessary condition of indefinite amplification.

Some further study makes it highly probable, however, that the amount of amplification is in any case limited,* so that the field would decay in the average over a long time. So long as the free decay is neglected the relative amplification can be estimated from Eq. (7) which shows it to be equal to the convergence of the flow normal to the magnetic lines of force. The changes in field intensity, therefore, represent a compression or expansion, as the case may be, of the magnetic flux rather than a creation of new flux. Thus, the

* This has been pointed out by Dr. T. G. Cowling to whom the author is greatly indebted for the communication of his results. This note and the above remarks have been added in proof.

problem of the long term maintenance of the earth's magnetic field involves phenomena different from the amplification mechanism studied here; we hope to return to this question in a future paper.

It is interesting to remark that there exists a limit of amplification which is quite independent of the mechanism of amplification itself; it is found in the magneto-mechanical forces exerted by the field upon the fluid. These forces become significant when they are comparable in magnitude with the purely mechanical forces which engender or control the fluid motion; in the earth's core this condition is nearly fulfilled. It is well known that the magneto-mechanical forces are directed so that they will tend to counteract the amplification of the field. Thus, for a sufficiently strong field, these forces will slow down and eventually prohibit further amplification. Whether or not this effect is significant for the earth can hardly be said at present, but it will be of interest to carry out the numerical estimate which is simple. The mechanical force produced by the field is given by the formula

$$\mathbf{F}=\mathbf{J}\times\mathbf{B}.$$

Here, $\mathbf{J}$ is the current density which can be written

$$\mathbf{J}=\sigma\mathbf{v}\times\mathbf{B}-\sigma\partial\mathbf{A}/\partial t.$$

We may confine ourselves to the first term on the right-hand side for an estimate of the order of magnitude. Thus, apart from numerical factors the mechanical force is of the order

$$F\sim\sigma vB^2,$$

where the italics stand for the magnitude of the corresponding vectors. We can expect that the magnetic field approaches a limiting value when this force becomes comparable to the forces in the fluid which engender the motions. As explained in the appendix these forces are in absolute magnitude very nearly equal to the Coriolis force,

$$F_c\sim 2\omega\rho v.$$

On equating the two expressions for the forces we obtain for this critical field strength

$$B_{\mathrm{crit}}=(2\omega\rho/\sigma)^{\frac{1}{2}},$$

and numerically,

$$B_{\mathrm{crit}}\sim 12 \text{ gauss}.$$

APPENDIX

In this appendix a few physical data on the earth's core which have immediate significance for our problem are collected. These include density, pressure, temperature, and electric conductivity. Last, not least, some elementary applications of hydrodynamics to motions in the core must be considered.

The direct information about the earth's core comes from seismology.[15,16] There seems to be fairly general agreement about the fluidity of the core, based upon the fact that transverse seismic waves are never transmitted by the core. There is, however, accessory evidence for the fluidity derived from the analysis of the tides of the solid earth.[17] Not so long ago Bullen[18] has made exhaustive determinations of the variation of density, pressure, and compressibility in the core. These are based on a critical analysis of seismological data by Jeffreys.[19]

The core has a radius, R, equal to 0.55 of the earth's radius, and the discontinuity of seismic velocity at the boundary of the core is perhaps the most outstanding single result of seismological observation. According to Bullen the density of the core at its boundary is 9.43 g/cm^3; from there it increases steadily and smoothly to 11.54 g/cm^3 at $0.40R$. Below $0.40R$ the density begins to increase very rapidly, and there appears to be a surface of discontinuity at $0.36R$ where the density changes from 14.2 to 16.8; thereafter it increases very slowly to about 17.2 at the earth's center. It is believed that the data for the upper half of the core are accurate to within a few percent while those for the central part of the earth are much less reliable. The hydrostatic pressure in the core is 1.37×10^{12} dyne/cm^2 (1.4×10^6 atmospheres) at the boundary, increasing to 3.17×10^{12} dyne/cm^2 at $0.40R$.

We shall pass without discussion over the various arguments propounded by chemists to the effect that the predominant chemical component of the core is metallic iron, perhaps with a slight admixture of nickel. The internal structure of the core as revealed by the work of Jeffreys and Bullen makes it more probable that the main constituents of the upper part of the core, between R and $0.40R$, are the metals of the iron group. If the density curve as found between these limits is extrapolated to lower pressures it passes smoothly into the density curve for iron at high laboratory pressures measured by Bridgman.[20] If the

[15] See reference 1 of Part I.

[16] H. Jeffreys, *The Earth* (The Macmillan Company, New York, 1929), second edition.

[17] H. Jeffreys, M.N.R.A.S., Geophys. Suppl. **1**, 376 (1926).

[18] K. E. Bullen, Bull. Am. Seismol. Soc. **32**, 19 (1942).

[19] H. Jeffreys, M.N.R.A.S. Geophys. Suppl. **4**, 594 (1939).

[20] P. W. Bridgman, Proc. Nat. Acad. Sci. **8**, 361 (1922); Proc. Am. Acad. Sci. **58**, 163 (1923).

curve is extrapolated to the side of high pressures it passes into a theoretical curve for extremely compressed iron computed by Jensen.[21]

The temperature of the core is not known, but we have two independent estimates of an upper limit. Jeffreys[22] states that the temperature at the core's boundary cannot exceed the melting point of the overlying solid; on estimating the latter he arrives at a temperature of 10,000° abs. for the boundary. Eucken[23] points out that separation of the solid and liquid phases in the earth would not have taken place unless the temperature of the mixture was less than the critical temperature of the liquid which he estimates, for iron, as 9000° abs., the critical pressure being 7300 atmos.

The electrical conductivity of the core, more precisely of the upper part, between R and $0.40R$, may be estimated as follows, assuming the matter to be mostly iron. According to the theory of electronic conductivity, σ is inversely proportional to the absolute temperature, T, and directly proportional to the square of the Debye temperature, Θ. The latter, in turn, is, apart from a slowly varying factor, proportional to the velocity of sound, known from seismic observations to be about twice as large as in ordinary iron. Taking $T=30T_0$ and $\Theta=2\Theta_0$ where the index 0 refers to laboratory conditions we have, apart from some other slowly varying factors that are difficult to evaluate, $\sigma=0.133\sigma_0$ which is slightly larger than the value, $\sigma=0.1\sigma_0$, adopted here. Since this value is relatively small as electronic conductivities go, it may be assumed that such impurities as are present have a relatively minor effect on the conductivity. It might be noted here that Cowling[24] has recently calculated the electric conductivity of the sun's interior; his figures are, very crudely, of the same general order as ours, larger for the central part and smaller for the upper strata of the sun.

Next we shall deal with motions in the core. Considering first pure inertial effects, we must mention an important paper by Poincaré.[25] The problem studied by him is whether the fluid of the core lags behind the precession of the earth's solid body which, as is well known, has a period of 11,000 years. Poincaré succeeds in showing that there should be no lag. If the fluid is enclosed in an ellipsoidal vessel the degree of lag depends on the numerical excentricity of the boundary. If the latter number is large compared to the ratio of the frequency of precession to the frequency of revolution, the fluid follows the precession of the vessel as if it was a rigid body. Now it is known from the theory of the earth's figure that the eccentricity of the core's boundary is not much less than the eccentricity at the earth's surface, whence Poincaré's result follows. In its derivation the assumption is made that the fluid is incompressible, but this should hardly be too serious a restriction.

The hydrodynamical equations for the fluid in the core are

$$d\mathbf{v}/dt+2\boldsymbol{\omega}\times\mathbf{v}=\rho^{-1}(\mathbf{F}-\nabla p),$$

where $\boldsymbol{\omega}$ is the vectorial angular velocity of the earth, p the pressure, ρ the density, and $\mathbf{F}$ the combined gravitational and centrifugal forces. Now consider small deviations from equilibrium and put

$$p=p_0+p_1 \text{ where } \mathbf{F}=\nabla p_0.$$

The equations of motion then become

$$d\mathbf{v}/dt+2\boldsymbol{\omega}\times\mathbf{v}=-\rho^{-1}\nabla p_1.$$

The first term on the left-hand side, representing the accelerations, is small compared to the second term, the Coriolis term. The methods by which this is proved are extensively discussed in textbooks on dynamic meteorology or oceanography. The hydrodynamical conditions in the core resemble somewhat those in the ocean; the densities differ by a factor of ten and the molecular viscosities should be of the same general order of magnitude if the core consists of liquid metal. We now have the approximate equation

$$2\boldsymbol{\omega}\times\mathbf{v}=-\rho^{-1}\nabla p_1 \tag{18}$$

as the condition of quasi-stationary flow. From this relation we may estimate the magnitude of the deviations from equilibrium pressure. For two points at a distance L from each other we get

$$\Delta p_1\sim 2\omega v\rho L.$$

Taking $v=0.03$ cm/sec. and $L=3.10^8$ cm,

$$\Delta p_1\sim 1.3\times 10^4 \text{ dyne/cm}^2.$$

This is a pressure difference of only about 1 cm of mercury, or one part in 10^8 of the total hydrostatic pressure. These deviations from hydrostatic equilibrium are so minute that it seems of little use to speculate at present about their possible causes. The smallness of these pressure differences is a direct expression of the prevalence of the gyroscopic forces; the fluid reacts upon disturbances of its static equilibrium by setting up a flow pattern that balances these disturbances through the action of the Coriolis force.

In deriving the preceding relationships we have ignored the action of the magneto-mechanical force on the fluid motion. So long as this force is neglected our argument is in complete analogy to the conventional analysis for meteorological and oceanographical problems of this type. The magneto-mechanical force gives rise to an additional term on the left-hand side of (18) which is linear in the velocity. We have previously shown that with the magnitude of the field actually found in the core this term is comparable in magnitude to the Coriolis term. It follows that the estimated order of magnitude of the deviations from hydrostatic equilibrium pressure remains substantially the same in the presence of this term.

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[21] H. Jensen, Zeits. f. Physik **111**, 373 (1938).
[22] H. Jeffreys, M.N.R.A.S., Geophys. Suppl. **3**, 6 (1932).
[23] A. Eucken, Nach. Gött. Akad. 1944, No. 1; Naturwiss. **32**, 112 (1944).
[24] T. G. Cowling, M.N.R.A.S. **105**, 166 (1945).
[25] See H. Lamb, *Hydrodynamics* (Cambridge University Press, England, 1932), sixth edition, Sec. 384.