work, experiments were done with mercury as a target material. Barber and Champion[5] have found about 6 times too much inelastic scattering for mercury. For this purpose, the target holder shown in Fig. 1 was altered and the gold replaced by a thin steel window 0.003 inch thick which allowed the electron beam to bombard the mercury in the calorimeter, the voltage in this case being raised to 2.3 Mev to compensate for the energy lost in the window. As in the case of gold, it was observed that within the experimental error there was no energy carried away by penetrating radiation.

It thus appears that the large energy losses which have been previously reported cannot be accounted for by the suggested emission of neutrinos or other extremely penetrating radiation. As has been referred to in a previous footnote, this result is in accord with the experiments of Ivanov, Walter, Sinelnikov, Taranov, and Abramovich[11] who, employing lead and aluminum targets and a different calorimeter arrangement, find no evidence of neutrino emission and that the radiation losses of electrons in this general energy range are adequately accounted for by the Bethe-Heitler theory.

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The Stability of Electron Orbits in the Synchrotron

N. H. FRANK
Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts

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The motion of an electron in a synchrotron (or betatron) is analyzed as its motion in an axially symmetric magnetic field under the action of external torques provided by an r-f field and radiation reaction acting about the axis of symmetry. The process of transition from betatron to synchrotron action is examined in detail and a criterion is established for the condition of "locking-in" of the electrons to the r-f wave which drives them synchronously. The typical stages of synchrotron operation are discussed and it is shown that the requirements for successful, stable operation should not be difficult to realize.

1.

THE theory of the stability of electron orbits in the synchrotron with respect to angular, radial, and vertical oscillations has been treated by a number of authors. The original calculations of McMillan,[1] Veksler,[2] and Blewett[3] are essentially equivalent and comprise what may be termed the quasi-stationary theory. More recently, Schwinger and Saxon[4] have examined the problem in a more refined treatment, providing a quantitative basis for the fundamental assumptions underlying the quasi-stationary theory.[5]

Consider an electron moving in a magnetic field increasing with time, the field being symmetrical about an axis, the z axis. This is the arrangement for a betatron and, as is well known, stable circular motion in a circle of radius r_0 is possible if the magnetic flux linking this orbit is $2\pi r_0^2 B$, where B is the z component of the magnetic field at the orbit. If the fractional increase of the magnetic field in the time of one revolution

[1] E. M. McMillan, Phys. Rev. **68**, 143 (1945).
[2] V. Veksler, J. Phys. U.S.S.R. **9**, 153 (1945).
[3] Unpublished calculations.
[4] Unpublished calculations.

[5] D. M. Dennison and T. H. Berlin have also investigated the stability of electron orbits in a synchrotron operating with a frequency modulated r-f field. Phys. Rev. **70**, 58 (1946).

is very small, a condition satisfied to a high degree of approximation in both the betatron and the proposed synchrotrons, the period of the electronic motion can be written in m.k.s. units as

$$T=(2\pi/ec^2)(E/B), \tag{1}$$

where E is the total energy of the electron (including its rest energy), and B is the z component of the magnetic field at the orbit at the time of this rotation. The radius of the orbit is related to the period by

$$r=\frac{\beta cT}{2\pi}=\frac{\beta E}{ecB}=\frac{E}{ecB}[1-(m_0c^2/E)^2]^{\frac{1}{2}}, \tag{2}$$

where β is the ratio of the electron speed to that of light. In the betatron, the equilibrium radius r_0 is constant and hence the period of electronic motion decreases very slowly with time, approaching a constant value at high energies as β approaches unity. In this high energy range the ratio of energy to magnetic field becomes very nearly constant. The energy is gained, of course, from the e.m.f. induced around the orbit by virtue of the changing magnetic flux through it.

In synchrotron operation, the energy is obtained largely from a radiofrequency voltage existing across one or more gaps. The electrons are driven synchronously by the r-f so that their average rotation period equals that of the r-f. When this condition is established, the orbit radius will swell, according to Eq. (2), as the energy increases. Thus, if any electron gets "locked" into the r-f when its energy is about 2 Mev, $\beta=0.97$, and is then driven synchronously, its mean radius will increase by about 3 percent from its initial to its final high energy value. In addition to this very slow increase of the orbital radius, there will be departures from the mean radius and by (2) from the mean period of rotation. One of the objectives of the theory is the study of these deviations in radius and phase to determine the stability of synchrotron operation. Another is the investigation of the transition from betatron to synchrotron action to establish criteria for the "locking-in" of the electrons relative to the r-f.

The motion of an electron in either a betatron or a synchrotron can be described to a high degree of approximation as its motion in an axially symmetric magnetic field under the action of external torques about the axis of symmetry, the z axis, since both the applied r-f field and radiation forces will exert negligible torque action about any other axis. Using a cylindrical coordinate system, r, θ, z, the equations of motion can be written in the form:

$$\left.\begin{aligned}&\frac{d}{dt}(m\dot{r})=mr\dot{\theta}^2-er\dot{\theta}B_z,\\&\frac{d}{dt}\left(mr^2\dot{\theta}-\frac{e\Phi}{2\pi}\right)=Q_\theta,\\&\frac{d}{dt}(m\dot{z})=er\dot{\theta}B_r.\end{aligned}\right\} \tag{3}$$

Here $m=m_0/(1-v^2/c^2)^{\frac{1}{2}}$ is the relativistic mass, $-e$ the charge on an electron. $\Phi=2\pi\int_0^r rB_z\,dr$ is the magnetic flux linking a circle of radius r and Q_θ is the external torque, defined as the generalized force by $Q_\theta=dW/d\theta$, where dW is the work done by this torque during an angular displacement $d\theta$ of the electron. If the time rate of change of the magnetic field is slow enough to be considered quasi-stationary, the radial and vertical components of the magnetic field are related by $\partial B_r/\partial z=\partial B_z/\partial r$. The energy equation

$$\frac{d}{dt}(mc^2)=\dot{\theta}\left(\frac{e}{2\pi}\frac{\partial\Phi}{\partial t}+Q_\theta\right) \tag{4}$$

follows from (3) by multiplying those equations by $\dot{r}$, $\dot{\theta}$, and $\dot{z}$, respectively, and adding.

For the interval during which the synchrotron is run as a betatron, prior to the application of an r-f field, $Q_\theta=0$ since radiation is negligible in the low energy range, and the stable motion occurs in a circle of constant radius r_0 with an angular velocity given by the first of Eqs. (3) as $\omega=\dot{\theta}_{r=r_0}=eB/m=ec^2B/E$. This is the same as Eq. (1). The second of Eqs. (3) gives the familiar betatron flux relation. The oscillations about the equilibrium orbit are obtained by setting

$$r=r_0+\rho;\quad \dot{\theta}=\omega-\dot{\phi}\text{ and }z=z, \tag{5}$$

inserting these into Eqs. (3), and treating ρ, $\dot{\phi}$, and z small enough so that second and higher order terms in these quantities can be ignored. If the variation of B_z in the neighborhood of r_0

is given by

$$B_z = B(r_0/r)^n \tag{6}$$

and the consequent variation of B_r used, there follow the familiar results of betatron theory. For our purposes, it is sufficient to note that the angular frequencies of the oscillations are of the same order of magnitude as the angular velocity of the electron, and that the vertical z-motion is independent of the radial and angular motions to this order of approximation.

Now let the external torque Q_θ be provided by an r-f field $\mathcal{E}_\theta$ extending over a gap of negligible angular opening at $\theta = 0$. A Fourier analysis of this field, which is periodic in θ with a period 2π, gives

$$Q_\theta = -\frac{eV}{2\pi} \sum_{k=-\infty}^{+\infty} \sin(\omega_1 t - k\theta), \tag{7}$$

where V is the peak voltage across the gap and ω_1 the angular radiofrequency. If ω_1 is very nearly equal to the angular velocity $\dot{\theta}$ of the electron, the only term in the sum which can be effective in doing work on the electron over a time comprising many cycles is that for which $k=1$. Thus, the resolution of the r-f field into traveling waves—and this will be true in general whether the r-f field be created across a single or multiple gaps—brings out the essential fact that only one component wave is essentially in step with the electron motion and all the others give rise to torques which have a rapid variation with time, frequencies of the order of ω_1 and higher, and hence will do no work on the average. The deviations from exact synchronism of the electron motion from the component wave

$$Q_\theta = -\frac{eV}{2\pi} \sin(\omega_1 t - \theta) \tag{7a}$$

will give rise to slow variations in phase and radius of the electronic motion. As will be shown later, the natural frequency of these oscillations is very much smaller than any produced by the remaining component waves or than that of free betatron oscillations. Thus the resultant electron motion can be considered as a superposition of two independent motions:

(a) A relatively high frequency variation of radius and phase, of the order of ω_1, or higher; and

(b) A low frequency, slow change of radius and phase caused by that wave which can drive the electrons synchronously.

For the slow motion (b), one can neglect the acceleration term on the left-hand side of Eq. (3) (and this is the basic assumption of the quasi-stationary theory), insert the expressions (5), (6), and (7a) into this and the second equation of (3) and there follows

$$\frac{\dot{\phi}}{\omega} = \frac{\omega^2 r_0^2}{c^2}\left(\frac{\rho}{r_0}\right), \tag{8}$$

and

$$\frac{d}{dt}\left(\frac{B\dot{\phi}}{\omega^3}\right) + \frac{V\sin(\omega_1 t - \theta)}{2\pi c^2(1-n)} = \frac{-r_0^2 \dot{B}}{c^2}\left(\frac{1-\delta}{1-n}\right). \tag{9}$$

Here $\delta = \dot{\Phi}_0 / 2\pi r_0^2 \dot{B}$ is the ratio of the time rate of change of the flux linking the betatron orbit to that required for betatron orbit stability and will equal unity as long as the betatron flux condition is satisfied.

2.

We now consider the transition from betatron motion (motion with constant mean radius and slowly increasing angular frequency) to synchronous driving (motion of constant average angular frequency and slowly changing radius). If the betatron angular velocity ω differs at all from the radiofrequency ω_1, there will be a slip of the electron motion relative to the traveling r-f wave and evidently the average energy gain from the r-f wave, averaged over one slip cycle, will be zero. Even if the frequencies differ by as little as one percent it will take only one hundred revolutions to complete such a slip cycle. To obtain synchronous driving, the r-f voltage must be sufficient to stop the slip and "lock" the electrons into synchronism with the wave. Furthermore, this synchronous "locking" should occur before the betatron flux condition is destroyed to insure proper transition from betatron to synchrotron action.

If we set $\psi(=\omega_1 t - \theta)$ equal to the angle between the electron and the r-f wave, the relative angular velocity is, using (5),

$$\dot{\psi} = \omega_1 - \dot{\theta} = (\omega_1 - \omega) + \dot{\phi}. \tag{10}$$

$(\omega_1 - \omega)$ is the slip angular velocity and $\dot{\phi}$ is the perturbation in the angular velocity caused by the r-f field. Inserting (10) into (9) with $\delta=1$, one obtains

$$\frac{d}{dt}\left(\frac{B\dot{\psi}}{\omega^3}\right)+\frac{V\sin\psi}{2\pi c^2(1-n)}=\frac{d}{dt}\left[\frac{B(\omega_1-\omega)}{\omega^3}\right]. \quad (11)$$

The condition for "locking-in" can be then obtained simply if one remembers the exceedingly slow variation of B and ω with time. As a first approximation, we can consider these constant and Eq. (11) becomes the equation of motion of a physical pendulum. The condition for transition from motion with slip to synchronous driving is then the condition for transition from rotatory to oscillatory motion of the pendulum. This gives as the condition for "locking-in"

$$\left|\frac{\omega_1-\omega}{\omega}\right|\leq\left[\frac{4V}{c\lambda(1-n)B}\right]^{\frac{1}{2}}, \quad (12)$$

where λ is the wave-length corresponding to the frequency ω. For the M.I.T. synchrotron design one gets, with $V=1$ kv, $B=400$ gauss $=4\times10^{-2}$ weber/m^2, $n=\frac{3}{4}$ and $\lambda=\pi$ meters,

$$|(\omega_1-\omega)/\omega|\leq 0.02,$$

so that synchronous driving sets in when the difference between betatron and r-f frequencies becomes the order of 2 percent. The effect of the slow variation of B and ω with time can be then included as a second approximation. This modifies the inequality (12) by multiplying the right-hand side by the factor $(1+\alpha)$, where

$$\alpha<\frac{\pi c^2\dot{B}(1-n)}{V\omega_1^2}\left(\frac{m_0c^2}{E_1}\right)^2$$

where E_1 is the electron energy when $\omega=\omega_1$. This term α is a fraction of a percent for the M.I.T. design and is hence quite negligible in this case.

In the preceding discussion, it has been assumed that the r-f voltage had built up to its maximum value without appreciably modifying the betatron angular velocity ω. This will be very nearly true if the inequality (12) is *not* satisfied during the build-up time. If, however, the radio frequency ω_1 does satisfy (12) during the build-up time, "locking-in" will take place at lower than peak voltage. The case of build-up under conditions of exact synchronism $\dot{\psi}_1=\omega$ (at high enough energies) has been considered in detail by Schwinger and Saxon[6] and they have shown that considerable bunching about zero phase (the equilibrium value of ψ or ϕ) can take place during this time and this bunching is enhanced the longer the build-up time.

3.

Once the electrons have been locked into synchronism with the r-f field, the angular departure ψ of the electron from the traveling r-f wave becomes pure oscillatory. In this synchronous region the total energy of the electron will in general be sufficiently high compared to its rest energy that we can take its speed equal to that of light. Equation (8) can then be replaced by

$$\dot{\phi}\cong\frac{\rho\omega}{r_0}\cong\rho\frac{\omega_1}{r_0} \quad (8a)$$

and Eq. (9) for the phase oscillations takes the form

$$\frac{d}{dt}\left(\frac{B\dot{\psi}}{\omega^3}\right)+\frac{V\sin\psi}{2\pi c^2(1-n)}$$
$$=\frac{-r_0^2\dot{B}}{c^2}\left(\frac{1-\delta}{1-n}\right)+\frac{d}{dt}\left[\frac{B(\omega_1-\omega)}{\omega^3}\right]. \quad (9a)$$

First consider the range of operation for which saturation has not yet started, so that $\delta=1$. If we neglect the small forcing term $(d/dt)(B[\omega_1-\omega]/\omega^3)$ for the time being and take $\omega\cong\omega_1$ as constant, Eq. (9a) becomes the equation for the free oscillatory motion of a physical pendulum of slowly changing moment of inertia corresponding to the slow increase of magnetic field B with time. For this motion the integral $\oint B\dot{\psi}d\psi$ taken over a period is an adiabatic invariant, i.e., is constant, so that the amplitudes of the oscillations decrease slowly as B increases. The integral can be evaluated in terms of complete elliptic functions and the result is:

$$h[\sin(\psi_m/2)]=\text{const.}/B^{\frac{1}{2}} \quad (13)$$

where $h(k)=E(k)-(1-k^2)K(k)$, K and E being complete elliptic functions of the first and second

[6] Unpublished calculations.

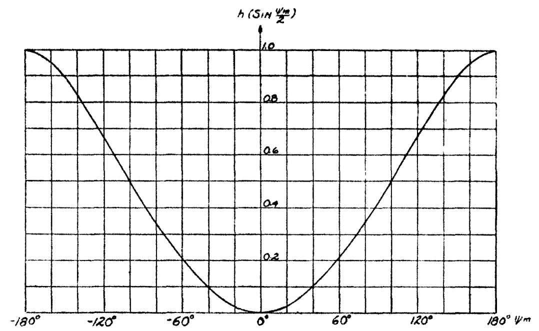

FIG. 1. Plot of $h(\sin\psi_m/2)$ *vs.* ψ_m to be used in conjunction with Eq. (13).

kind, and ψ_m the maximum oscillation amplitude. Figure 1 is a plot of $h[\sin(\psi_m/2)]$ as a function of ψ_m and shows how bunching takes place even for large amplitudes. This may be termed "pre-bunching" as it occurs prior to the onset of saturation.

The neglect of the extremely slow variation of ω at high energies does not change the result appreciably. When included, it indicates a slightly slower rate of decrease of amplitude with increasing magnetic field than is given by Eq. (13) and its effect gets smaller the higher the energy at which this pre-bunching occurs.

The effect of the small forcing term

$$(d/dt)[B(\omega_1-\omega)/\omega^3]$$

is to cause the oscillations to occur about an equilibrium phase slightly different from zero. The position of this equilibrium phase point, which is given to a good degree of approximation by

$$\sin\psi_e=-\frac{2\pi r_0^2(1-n)\dot{B}}{V}\left[2\left(\frac{\omega'-\omega}{\omega'}\right)-\frac{\omega_1-\omega}{\omega_1}\right],$$

with $\omega'=c/r_0$, will change very slowly with time as the betatron frequency ω gradually increases. This slow change occurs coincidently with the slowly changing radius and is accompanied by a small gain of energy of the electrons from the r-f field. Once saturation has started, however, the effect of this forcing term becomes entirely negligible and will be omitted in the remainder of this paper.

During the process of saturation and the consequent destruction of the betatron flux relation, δ changes slowly from unity to something of the order of 20 to 25 percent. During this slow change the equilibrium phase will slowly change according to the relation

$$\sin\psi_e=-(2\pi r_0^2\dot{B}/V)(1-\delta). \qquad (14)$$

For small oscillations about this equilibrium phase, one can readily solve Eq. (9a) and finds further decrease of amplitude of oscillation, i.e., further bunching. From the analogy with the motion of a physical pendulum it is clear that this further bunching will occur for large amplitudes also.

After saturation is complete, the equilibrium phase stays essentially fixed until the energy becomes sufficiently large to have radiation reaction play a significant role. In this region, we shall obtain some of the orders of magnitude of the quantities of interest. For this purpose, it is convenient to change the independent variable in Eq. (9a) from time t to number of revolutions N. This choice of independent variable was originally made by Veksler. To the degree of approximation employed in deriving Eq. (9a), one has

$$dt/dN=T\cong T_1=2\pi/\omega_1,$$

where T and T_1 are the periods of the electron motion and the r-f field, respectively. Equation (9a) takes the form, ignoring the small forcing term in $(\omega_1-\omega)$,

$$\left(\frac{B}{\dot{B}T_1}\right)\frac{d^2\psi}{dN^2}+\frac{d\psi}{dN}+\frac{4\pi^2V\sin\psi}{\lambda^2\dot{B}(1-n)}=-2\pi\left(\frac{1-\delta}{1-n}\right), \qquad (9b)$$

where $\lambda=cT_1$ is the r-f wave-length.

If one now considers the magnetic field a linear function of the time,

$$B=B_0+\dot{B}t=B_0+\dot{B}NT_1.$$

(9b) becomes finally:

$$(N_0+N)\frac{d^2\psi}{dN^2}+\frac{d\psi}{dN}+\Omega^2\sin\psi=-2\pi\left(\frac{1-\delta}{1-n}\right), \qquad (9c)$$

with $N_0=B_0/\dot{B}T_1$ and $\Omega^2=4\pi^2V/(\lambda^2\dot{B}(1-n))$. If now, instead of ψ, we write $\psi+\psi_e$, where ψ_e is the equilibrium phase given by (14) or its equivalent

$$\sin\psi_e=-\frac{2\pi}{\Omega^2}\left(\frac{1-\delta}{1-n}\right),$$

and consider small deviations ψ from this equilibrium phase, ψ satisfies the homogeneous equation

$$(N_0+N)\frac{d^2\psi}{dN^2}+\frac{d\psi}{dN}+(\Omega^2\cos\psi_e)\psi=0. \quad (9d)$$

The solution of this equation is, with $\cos\psi_e\cong 1$ and $x=2(N_0+N)^{\frac{1}{2}}$,

$$\psi=c_1J_0(\Omega x)+c_2N_0(\Omega x).$$

To get the order of magnitudes involved, we take the M.I.T. figures $V=10^3$ volts, $\dot{B}\cong 2\times 10^2$ webers/m²-sec.; $\lambda=\pi$ meters; $n=\frac{3}{4}$ and $B_0\cong 6\times 10^{-2}$ weber/m² corresponding to an energy of about 9 Mev when saturation is complete. Then $\Omega\cong 9$ and $x_0=2\sqrt{N_0}\cong 350$, so that $\Omega x>3000$ and the asymptotic values of the Bessel functions can be used. Then we have

$$\psi=(C/\sqrt{x})\cos(\Omega x-D). \quad (9e)$$

To get the period of small phase oscillations, we have $\Omega(x_2-x_1)=2\pi$, whence

$$\Delta N=N_2-N_1\cong\frac{2\pi}{\Omega}(N_0+N_1)^{\frac{1}{2}}\cong\frac{2\pi}{9}(30{,}000+N_1)^{\frac{1}{2}},$$

so that the number of revolutions per cycle of phase oscillation increases slowly with the total number of revolutions, and is of the order of 200 for $N_1=0$, i.e., when saturation ends.

As Saxon and Schwinger have shown, radiation reaction subsequently results in a further increase in the magnitude of the equilibrium phase and modifies the rate of change of amplitude of oscillation about this point, increasing the bunching action if $n<\frac{3}{4}$ and decreasing it if $n>\frac{3}{4}$.

Because radiation effects become significant only at extremely high energies, the torque Q_θ due to radiation reaction can be written as

$$Q_\theta=-\frac{2}{3}\cdot\frac{e^2}{4\pi\epsilon_0 r}\left(\frac{ecrB_z}{m_0c^2}\right)^4,$$

with $\epsilon_0=(1/36\pi)\times 10^{-9}$ farad/meter, very nearly. Following the same procedure as before, one obtains as the equation for phase oscillations:

$$\frac{2\pi r_0^2(1-n)}{\omega}\frac{d}{dt}(B\dot{\phi})+\frac{e}{3\epsilon_0 r_0\omega}\left(\frac{ecr_0B}{m_0c^2}\right)^4(3-4n)\dot{\phi}$$
$$+V\sin\phi=-2\pi r_0^2\dot{B}(1-\delta)-\frac{e}{3\epsilon_0 r_0}\left(\frac{ecr_0B}{m_0c^2}\right)^4,$$

which leads to an equilibrium phase angle, at very high energies where radiation losses alone practically determine it, given by

$$\sin\phi_e=-\frac{e}{3\epsilon_0 r_0 V}\left(\frac{E}{m_0c^2}\right)^4$$

where $E=ecr_0B$ is the total energy of the electron.

4.

There remains the problem of justifying the neglect of the left-hand side of the first of Eqs. (3) for the slow variations of radius and phase and the assumption that the slow motion can be treated independently of the remaining high frequency motion. To do this simply, consider the motion of an electron of very high energy as given by Eqs. (3) with $\omega=\omega_1=c/r_0$ under the action of the external torque (7a) alone.

By use of (5) and considering only small departures from circular motion, one obtains from the first and second of Eqs. (3) (the z-motion is independent of the radial and angular motion to first order),

$$\frac{d}{dt}\left[\frac{d}{dt}(B\dot{\rho})+B\omega^2(1-n)\rho+\omega^2r_0^2\left(B-\frac{\Phi_0}{2\pi r_0^2}\right)\right]=-\frac{\omega^2V}{2\pi r_0}\sin(\omega t-\theta)$$

and using the relations $\rho=r_0\dot{\phi}/\omega$ and $\phi=\omega t-\theta$, this becomes

$$\frac{d^2}{dt^2}\left(B\frac{d^2\phi}{dt^2}\right)+\omega^2(1-n)\frac{d}{dt}\left(B\frac{d\phi}{dt}\right)+\frac{\omega^3V\sin\phi}{2\pi r_0^2}=-\omega^3\dot{B}(1-\delta) \quad (15)$$

as the equation determining the phase oscilations. Thus there are two frequencies of phase oscillation under the action of the external

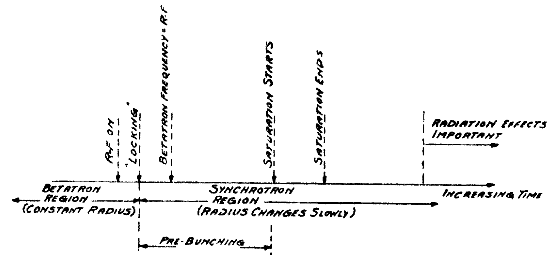

FIG. 2. Schematic sequence of operations of a synchrotron.

torque and they are determined from Eq. (15) by setting B constant and $\sin\phi=\phi$ for small oscillations. These two frequencies ω' and ω'' are then given as the solutions of the biquadratic equation:

$$\omega_0^4-\omega_0^2\omega_r^2+A=0, \tag{16}$$

where $\omega_r^2=\omega^2(1-n)$ and $A=\omega^3 V/2\pi r_0^2 B$. The roots ω' and ω'' are then

$$\left.\begin{aligned}\omega'^2&=\omega_r^2(\tfrac{1}{2}+\tfrac{1}{2}(1-4A/\omega_r^4)^{\frac{1}{2}})\\ \omega''^2&=\omega_r^2(\tfrac{1}{2}-\tfrac{1}{2}(1-4A/\omega_r^4)^{\frac{1}{2}})\end{aligned}\right\}. \tag{17}$$

Oscillatory motion will occur if $4A/\omega_r^4<1$ or

$$B>2V/\pi r_0^2\omega(1-n)^2,$$

a condition well satisfied for almost any reasonable design parameters. In the case of the M.I.T. design, the magnetic field need be greater than only $\frac{2}{3}$ gauss. Since it is of the order of several hundred gauss when the r-f field goes on, $4A/\omega_r^4$ is very small compared to unity and the two frequencies of phase oscillation become

$$\omega'^2=\omega_r^2=\omega^2(1-n),$$

$$\omega''^2=\frac{A}{\omega_r^2}=\frac{\omega V}{2\pi r_0^2 B(1-n)}.$$

Thus it is clear that the root ω'' can be obtained to this high degree of approximation from Eq. (16) with the first term missing. Since this term arose from the radial acceleration term in Eq. (3), the left-hand side of the first of these equations, we have justified the neglect of this term from the outset to get the slow oscillations induced by the r-f field. The high frequency oscillation $\omega'=\omega_r$ is just the betatron radial oscillation frequency and, since this can be obtained from Eq. (16) by neglecting the last term which alone depends on the r-f field, this is unaffected by the presence of this component wave of the r-f field. The higher harmonics of this field whose frequencies are close to ω_r will, of course, modify the high frequency motion but evidently will have no significance for the low frequency motion. Equation (15) with the first term on the left omitted is just Eq. (9a) used to describe the synchronous region of operation.

5.

One can summarize the essential results obtained in this paper schematically with the help of the diagram of Fig. 2. In this the horizontal line represents increasing time but not to any particular scale. The sequence of operations shown in the figure can be described as follows:

Initially, one has pure betatron action during the latter part of which the r-f field is built up. The electrons gain energy by betatron action alone until "locking" into synchronism occurs as indicated. At this point pure betatron action ceases and the synchronous region begins, characterized by a small decrease in radius. Energy is still essentially supplied to the electrons from the betatron induced e.m.f. As this energy increases the radius slowly increases, passing through the betatron equilibrium radius when the betatron frequency equals the radiofrequency. During this time pre-bunching is taking place, and the electrons derive a small additional energy from the r-f field. When saturation begins, the electrons start picking up an appreciable amount of energy from the r-f field, and the equilibrium phase slowly rises to a definite value when saturation is complete. This state of affairs with continually greater bunching persists until radiation effects become important. Then there is a further slow increase of the equilibrium phase, additional energy abstracted from the r-f field to supply radiation losses and a modification of the bunching because of the radiation reaction.