where again

$$G^2=\frac{f^2}{2\pi^2};\quad \mathbf{K}=\mathbf{k}'-\mathbf{k}.$$

The matrix

$$(J, M, K, N; j_I, j_{II}; \mathbf{k}|\, V|\, J, M', K, N; j_I', j_{II}'; \mathbf{k}')$$

is obtained in the same way as was indicated in Appendix I. The non-diagonal elements, with respect to M, are easily derived from the pseudoscalar case since the Ω-term does not contribute to these elements. The diagonal terms are

$$(\pm 1; \tfrac{1}{2}, \tfrac{1}{2}; \mathbf{k}|\, V|\, \pm 1; \tfrac{1}{2}, \tfrac{3}{2}; \mathbf{k}')=\frac{-G^2\sqrt{2}}{9}\left\{\frac{(\mu^2-\kappa^2)[3(K_1^2+K_2^2)+2K_3^2]}{2[\kappa^2+\mathbf{K}^2][\mu^2+\mathbf{K}^2]}-\frac{4\mu^2}{\mu^2+\mathbf{K}^3}\right\},$$

$$(0; \tfrac{1}{2}, \tfrac{1}{2}; \mathbf{k}|\, V|\, 0; \tfrac{1}{2}, \tfrac{3}{2}; \mathbf{k}')=-\frac{G^2\sqrt{2}}{9}\left\{\frac{(\mu^2-\kappa^2)[K_1^2+K_2^2+2K_3^2]}{[\kappa^2+\mathbf{K}^2][\mu^2+\mathbf{K}^2]}-\frac{4\mu^2}{\mu^2+K^2}\right\}.$$

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Resonance Reactions and Anomalous Scattering

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The purpose of the present paper is to give a derivation of the resonance formula for nuclear reactions which is free from artificial assumptions. Mathematically, the method used amounts to a Taylor series development of the wave function with respect to the energy. It is assumed that the first (energy independent) term in this development is, within a region of configuration space where all particles are close together, the same, no matter in which way the compound state is formed and that this is, in that region of configuration space, already a good approximation. The second term in the development of the wave function with respect to the energy difference from the resonance energy can then be calculated very easily and this calculation is carried out in Section II for resonance scattering, in Sections III and IV for resonance reactions. It is assumed in both calculations that the colliding particles have zero orbital angular momentum around their center of mass. The third term in the same expansion is estimated for resonance scattering in Section VI and it is shown that, if there were no other resonances in the neighborhood, the effect of the third term would be negligible over a very wide energy range (several hundred kilovolt). The formulae for the cross sections, as obtained, are of greater generality than the customary ones inasmuch as they contain extra terms which could be interpreted as potential scattering and potential reaction. The existence of such terms has been noticed already by Bethe. However, as discussed in Section V, the extra terms are, particularly in the neighborhood of the resonance, much smaller than the resonance terms so that one is led back, in practice, to the ordinary resonance formulae, as given, e.g., by Bethe. In particular, the disintegration probability is, as function of energy, proportional to the velocity with which the reaction products separate if the orbital angular momentum of the separating particles vanishes. It may be worth while to remark that the resonance part of the collision matrix has a particularly simple form and is, e.g., of rank 1. The case of orbital angular momentum $1\hbar$ is discussed in Section VII. In this case, the disintegration probability of the compound state is proportional to the third power of the velocity of the separating particles so that the scattering is, at very low energies, proportional to the square of the energy. The same holds, in this case, of the "potential scattering" also. Section VII also contains an investigation of the region of validity of the formulae in case of angular momentum $1\hbar$ between the colliding particles and shows that this region will extend to the neighboring resonances.

I. INTRODUCTION

ALTHOUGH there is no experimental evidence available to corroborate the resonance formula[1] with any great accuracy, a number of reformulations of the underlying picture[2] have appeared in the literature.[3] All these formula-

[1] The first four sections of the present article are based on a report, dated April 23, 1945, which the writer prepared for the Uranium Project. He is indebted to the Army Engineers for clearing that report for publication.

[2] G. Breit and E. P. Wigner, Phys. Rev. **49**, 519, 642 (1936); N. Bohr, Nature **137**, 344 (1936).

[3] H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937); F. Kalckar, J. R. Oppenheimer, and R. Serber, Phys. Rev. **52**, 273 (1937); H. A. Bethe, Rev. Mod. Phys. **9**, 71 (1937); P. L. Kapur and R. Peierls, Proc. Roy. Soc. **166**, 277 (1938); and, in particular, A. J. F. Siegert, Phys. Rev. **56**, 750 (1939).

tions endeavor to give a quantum-mechanical equivalent to the picture of the formation of a compound nucleus which disintegrates subsequently and differ from each other in the (more or less openly made) assumptions as to what constitutes a compound state. They have been discussed and critically evaluated by G. Breit.[4] The present attempt borrows more from his results and those of Siegert than from the preceding treatments although some of the underlying ideas are the same as given by Kapur and Peierls.

Because of its greater mathematical simplicity the case of the resonance scattering will be treated separately. However, the basic assumption will be the same for resonance reaction and resonance scattering: that if all the particles of the system are close to each other, thus forming a compound nucleus, it is a good approximation to assume that the wave function is (apart from a constant factor) independent of the energy within the range for which the results are expected to hold and also independent of the origin of the compound state. Thus, the wave function of the compound nucleus is the same no matter whether, for instance, it was obtained from ${}_3Li^7+{}_2He^4$ or from ${}_4Be^9+D$ or from ${}_5B^{10}+$neutron or from ${}_5B^{11}+\gamma$.

Evidently, the above assumption is very rigid and does not permit one, e.g., to treat the energy region just between two resonances. It seems to me, however, that it corresponds more accurately than any other formulation that one can easily think of to the picture of a reaction through one definite compound state. It is also evident how one has to generalize the above picture in order to treat the region between two or even more resonances but this subject will not be taken up at the present time.

In addition to the above assumption, it will be assumed that the amplitude of the compound wave function is very large compared with the amplitude of the wave function at places where the particles which form the system are not all close together. The large amplitude of the wave function corresponds to a high probability, i.e., long lifetime of the compound state. It is believed that no other special assumptions are made in the following considerations.

[4] G. Breit, Phys. Rev. **58**, 506, 1068 (1940).

To anticipate the result: the formulae which will be obtained are of a greater "generality" than the customary ones, i.e., they are less definite inasmuch as they involve more parameters. It can be shown, however, by a very simple example, that this is not the result of the insufficiency of the assumptions but that the customary formulae fail to hold even in very simple cases.

The system will always be described with a wave function in ordinary configuration space. Thus complications which may arise from the inclusion of spin variables are not solved although they are touched upon in the last section. More important, the case of light emission cannot be described by a wave function. It is difficult to believe, however, that the formulae to be obtained should not hold also for light (γ-ray) emission and absorption also. This is explicitly demonstrated in the case of resonance scattering where the corresponding process has been treated by means of Dirac's light theory.[5] On the contrary, it is to be expected that the formulae to be obtained can be further specialized in case of light quanta (i.e., some of the parameters set equal to zero) because of the special properties of these "particles": they do not interact with each other and their interaction with nuclear particles can be considered to be "weak." The expectation that the formulae to be obtained can be specialized in the case of light quanta can be confirmed in the case of resonance scattering where a detailed treatment is available. For real reactions, an extension of the present treatment on the basis of Dirac's light theory is desirable.

Mathematically, the present treatment will be the customary one: it will deal only with stationary states, with a constant flux of the incident particles coming in and a constant flux of the reaction products leaving.

II. RESONANCE SCATTERING

In this, as well as in the following sections, the plane wave which describes a collision will be considered to be decomposed into spherical waves. This decomposition, the value of which has been demonstrated by Faxen and Holtsmark, will not be carried out explicitly as it is described

[5] Cf. e.g., V. Weisskopf and E. Wigner, Zeits. f. Physik **63**, 54 (1930); F. Hoyt, Phys. Rev. **36**, 860 (1930).

in standard textbooks. It will be further assumed in this section that the first spherical wave of this decomposition, i.e., the one with zero angular momentum, is responsible for the scattering so that the scattering cross section can be found by investigating the properties of the spherical wave.

When the two particles are far from each other, the wave function has the form

$$\varphi_E = \frac{1}{(4\pi)^{\frac{1}{2}} r u}(e^{-ikr} - U_E e^{ikr})\psi(i). \tag{1}$$

Herein r is the distance between the colliding particles, u is the square root of their velocity toward each other so that their relative energy is $E=\frac{1}{2}Mu^4$, where M is their relative mass. The first term of (1) corresponds to the incident wave, the second to the scattered wave, k is the wave number $k=Mu^2/\hbar$. The $\psi(i)$ is the normalized, real wave function of the internal coordinates of the two colliding particles. Both waves of (1) are normalized to unit flux so that the conservation law for the number of particles demands $|U_E|=1$. The factor in the bracket of (1) can be written as $-2ie^{i\delta}\sin(kr+\delta)$ so that U_E is, in terms of the "phase shift," equal to $U_E=e^{2i\delta}$. While U_E depends on the energy E with which the particles collide, $\psi(i)$ is independent of E.

If the two particles form a compound nucleus (i.e., in that part of the configuration space which corresponds to all particles being close together), the wave function is in first approximation

$$\varphi_E = \alpha_E \Psi; \tag{2}$$

Ψ is a normalized wave function involving all the coordinates of the system and is, according to out first assumption, within the energy range considered, independent of E. On the other hand, α_E is independent of the variables of the wave function but a function of the energy: $|\alpha_E|^2$ is of the dimension of a time and can be interpreted as the lifetime of the compound state.

We now lay a sphere around the origin of the coordinate system which corresponds to the coincidence of all particles. The radius of this sphere is as small as possible consistent with the requirement that φ shall have already the form (1) on its surface. We then write down the condition that φ_{E_1} is a state with the energy E_1

$$H\varphi_{E_1} = E_1\varphi_{E_1} \tag{3}$$

and a similar equation for another energy

$$H\varphi_{E_2} = E_2\varphi_{E_2}. \tag{3a}$$

In order to simplify the formulae, φ_1 and φ_2 will be written for φ_{E_1} and φ_{E_2} and a similar convention will be adopted with respect to U, and u, k so that the index 1, 2, . . . will stand to indicate the value of the variable $E_1, E_2, \ldots$ As customary, we multiply (3) with φ_2^* and the conjugate complex of (3a) with φ_1, and integrate the difference of the resulting equations over the above sphere. All terms corresponding to the interaction of the particles drop out on the left side; those corresponding to the kinetic energy can be transformed by Green's theorem so that

$$-\frac{\hbar^2}{2M}\int(\varphi_2^* \operatorname{grad}\varphi_1 - \varphi_1 \operatorname{grad}\varphi_2^*)_n dS = (E_1-E_2)\int\varphi_1\varphi_2^* dV; \tag{4}$$

dS denotes integrations over the surface, dV over the volume of "the sphere." The index n denotes the normal component of the vector in parentheses to the surface of integration.

This result appears to contradict our original assumption that φ is, in the region where all particles are close together, independent of the energy. If φ_1 and φ_2 were, apart from a constant, identical within the sphere considered, they would be identical also on the surface of the sphere and the left side of (4) would vanish identically. Actually, (4) corresponds to a one-order higher approximation than an energy independent wave function and can be derived in a more natural way as follows. Let us assume that we know the wave function φ for a certain energy E so that $H\varphi=E\varphi$. We then can try to find the solution of

$$H\varphi_1 = (E+\delta E)\varphi_1 \tag{5}$$

by a perturbation method. We write $\varphi_1=\varphi+\varphi'$ and assume that φ' satisfies the equation

$$(H-E)\varphi' = \delta E\varphi \tag{5a}$$

which is obtained from (5) by neglecting the second-order quantity $\delta E\varphi'$. Multiplying (5a) with φ^* and integrating over the above sphere, one has

$$\int\varphi^*(H-E)\varphi' dV = \delta E\int|\varphi|^2 dV. \tag{6}$$

Since $H-E$ is a Hermitean operator, the first integral would be equal, if it could be extended over the whole space, to $\int[(H-E)\varphi]^*\varphi' dV$ which is zero and one would obtain the paradoxical result that the integral of $|\varphi|^2$ vanishes. It would then follow that (5) has no solution for $\delta E \neq 0$. In our case, the integral over $|\varphi|^2$ diverges if extended over the whole space and the neglect of the $\delta E\varphi'$ term is only justified if the sphere over which the integration is extended is relatively small. For a finite sphere, additional terms will appear if one takes $H-E$ over to the other factor, resulting from the partial integration of the operator of the kinetic energy. One has, as a result,

$$-\frac{\hbar^2}{2M}\int(\varphi^* \operatorname{grad}\varphi' - \varphi' \operatorname{grad}\varphi^*)dS = \delta E\int|\varphi|^2 dV. \quad (6a)$$

This equation will remain valid if one replaces in it φ' by $\varphi_1=\varphi+\varphi'$ and is then identical with (4). One sees that the algebra leading to (4) is only a simplified method for deriving the variation of grad φ/φ with energy. If one were to apply literally the condition of the independence of φ from energy, grad φ/φ would be independent of energy also. This would not prove to be sufficiently accurate.

Introducing (1) for the left side of (4) and neglecting, in the sense of the second assumption of the Introduction, the region where (2) does not hold on the right side, one obtains

$$-\frac{\hbar^2}{2Mu_1u_2}(e^{ik_2a}-U_2^*e^{-ik_2a})ik_1(-e^{-ik_1a}-U_1e^{ik_1a}) + \frac{\hbar^2}{2Mu_1u_2}(e^{-ik_1a}-U_1e^{ik_1a})ik_2(e^{ik_2a}+U_2^*e^{-ik_2a}) = (E_1-E_2)\alpha_1\alpha_2^*; \quad (7)$$

a is the distance of the colliding particles corresponding to the surface of the sphere in configuration space over which we carried out the integration. Derivations similar to that leading to (7) have been described so often that it was not considered worth while to go into more detail in the present case.

There is an alternative way of deriving (7) which involves, perhaps, more algebra but which shows more clearly the fundamental nature of this equation. This is based on the consideration of the wave function

$$\varphi_1\exp(-iE_1t/\hbar)+\varphi_2\exp(-iE_2t/\hbar),$$

which is a non-stationary solution of Schroedinger's time dependent equation. If one calculates the current which flows into a sphere of radius a, one finds (assuming $|U_1|=|U_2|=1$) that this is a periodic function of time and contains terms proportional to $\exp[i(E_2-E_1)t/\hbar]$ and $\exp[i(E_1-E_2)t/\hbar]$. The average value of this current vanishes, as it must. However, there is a periodically changing flow into the sphere and out of it so that the probability of r having a value below a also shows periodic fluctuations with time and contains terms proportional to the above exponentials. The probability in question is, however,

$$|\alpha_1\exp[-iE_1t/\hbar]+\alpha_2\exp[-iE_2t/\hbar]|^2 = |\alpha_1|^2+|\alpha_2|^2+\alpha_1\alpha_2^*\exp[i(E_2-E_1)t/\hbar] + \alpha_1^*\alpha_2\exp[i(E_1-E_2)t/\hbar].$$

Hence $\alpha_1\alpha_2^*$ must be equal to the time integral of the term in the current which is proportional to $\exp[i(E_2-E_1)t/\hbar]$, and this gives Eq. (7). Comparison of the coefficients of the terms proportional to $\exp[i(E_1-E_2)t/\hbar]$ in the last expression and in the integrated current gives the conjugate imaginary to (7). Equation (22) can be derived in a similar way.

If the ka in (7) can be assumed to be small, (7) becomes ($k=Mu^2/\hbar$),

$$\frac{u_1}{u_2}(1+U_1)(1-U_2^*)+\frac{u_2}{u_1}(1-U_1)(1+U_2^*) = \frac{2}{i\hbar}(E_1-E_2)\alpha_1\alpha_2^*. \quad (8)$$

If ka is not small, (8) will still be valid if one substitutes

$$\begin{aligned} U_1&=\bar{U}_1\exp(-2ik_1a);\\ U_2^*&=\bar{U}_2^*\exp(2ik_2a);\\ \alpha_1&=\bar{\alpha}_1\exp(-ik_1a);\\ \alpha_2^*&=\bar{\alpha}_2^*\exp(ik_2a). \end{aligned} \quad (8a)$$

Equation (8) holds for the barred quantities if (7) applies for the unbarred ones. However, the bar will be left off in the following. Evidently, the unbarred quantities, being defined by (1),

are independent of a while the barred quantities are not. On the other hand, Eq. (7) which holds for the unbarred quantities contains a, Eq. (8) which holds for the barred quantities, does not contain a. This somewhat paradoxical situation is partly caused by the approximate nature of these equations. For most of the present paper, the difference between the two kinds of quantities is negligible.

Incidentally, setting $E_1=E_2$ gives, because of $u_1=u_2$, $U_1=U_2$, the equation

$$U_1U_1^*=|U_1|^2=1, \tag{8b}$$

i.e., the conservation law of particles.

The reality condition can be obtained by observing that the conjugate complex of φ_E must be, apart from a constant, equal to φ_E. Since $\psi(i)$ is real, this constant is $-U_E^*$. Hence $\alpha_E^*\Psi^*=-U_E^*\alpha_E\Psi$. This shows that Ψ can be chosen to be real and that, if this is done

$$\alpha_E^*=-U_E^*\alpha_E;\ \alpha_E=-U_E\alpha_E^*. \tag{9}$$

This equation remains unchanged if one replaces the unbarred quantities, for which it is derived, by the barred ones.

Equations (8) are ∞^2 equation for the 2∞ quantities U and α. In order to solve them, one can make the substitution

$$U=\frac{1+ivS}{1-ivS},\quad vS=i\frac{1-U}{1+U}; \tag{10}$$

S will be, of course, just as U is, a function of the energy and velocity. Since the absolute value of U is 1, the S becomes real. Its physical interpretation is that, if multiplied with v, it should give the tangent of the phase shift. The dimension of hS/M is that of a length which will be closely related by (16) to the scattering radius of the colliding particles.

Since the U of (10) is a barred quantity, the S defined by it is also a barred quantity. Hence the above relation for the tangent of the phase shift is only approximate. It should be replaced by the following relation for the square of the sine of the phase shift (which is a more important quantity),

$$\sin^2\delta=\tfrac{1}{4}(2-e^{2i\delta}-e^{-2i\delta})=\tfrac{1}{4}(2-\bar{U}e^{-2ika}-\bar{U}^*e^{2ika}).$$

Substituting for $\bar{U}$ the expression (10) with barred $\bar{S}$, one has

$$\sin^2\delta=\frac{1}{4}\left[2-\frac{1+iv\bar{S}}{1-iv\bar{S}}e^{-2ika}-\frac{1-iv\bar{S}}{1+iv\bar{S}}e^{2ika}\right]$$
$$=\frac{(v\bar{S}\cos ka-\sin ka)^2}{1+v^2\bar{S}^2}. \tag{10a}$$

For small ka, one can write for this

$$\sin^2\delta=\frac{(vS)^2}{1+v^2S^2}, \tag{10b}$$

where

$$S=\bar{S}\cos ka-v^{-1}\sin ka\sim\bar{S}-Ma/\hbar. \tag{10c}$$

One can easily convince oneself that the error made by replacing the denominator of (10a) by the denominator of (10b) is small both if vS is small compared to 1 and also if it is large.

One now divides (8) by $-iu_1(1+U_1)(1+U_2^*)u_2$ to obtain with $u_1^2=v_1$; $u_2^2=v_2$

$$-S_2+S_1=\frac{2}{\hbar}(E_1-E_2)\frac{\alpha_1}{u_1(1+U_1)}\frac{\alpha_2^*}{u_2(1+U_2^*)}. \tag{11}$$

The left side of this is evidently real. It follows that the phase factor ω of

$$\frac{\alpha_1}{u_1(1+U_1)}=b_1\omega \tag{11a}$$

is independent of energy, b_1 being real. This. combined with (9), shows that $\omega=\pm i$. One can assume $\omega=i$, since the sign of b is still at our disposal. Introducing now (11a) into (11)

$$S_1-S_2=(2/\hbar)(E_1-E_2)b_1b_2. \tag{12}$$

If one writes down this equation for the pairs of energy values E_3, E_2 and E_1, E_3, instead of E_2, E_1 and adds the three equations thus obtained, the left side will vanish and one obtains

$$(E_2-E_1)b_1b_2+(E_3-E_2)b_2b_3+(E_1-E_3)b_1b_3=0 \tag{13}$$

or

$$b_1=\frac{(E_2-E_3)b_2b_3}{E_1(b_3-b_2)+E_2b_2-E_3b_3}. \tag{13a}$$

The same equation, with the same coefficients, will be obtained if one substitutes b_4, b_5, etc. for b_1 and E_4, E_5, etc. for E_1. This shows that

$$b_1=\frac{\frac{1}{2}\hbar b}{E_1-E_0} \tag{13b}$$

in which the real b and E_0 are independent of energy. Re-introducing this into (12)

$$S_1-S_2=\tfrac{1}{2}\hbar b^2\frac{E_1-E_2}{(E_1-E_0)(E_2-E_0)}$$

$$=\tfrac{1}{2}\hbar b^2\left(-\frac{1}{E_1-E_0}+\frac{1}{E_2-E_0}\right). \quad (14)$$

Hence

$$S_1=-\frac{\tfrac{1}{2}\hbar b^2}{E_1-E_0}+S_\infty \quad (14a)$$

where again S_∞ is independent of E_1. This gives, finally, with (10)

$$U_1=\frac{(1+iv_1S_\infty)(E_1-E_0)-\tfrac{1}{2}i\hbar b^2v_1}{(1-iv_1S_\infty)(E_1-E_0)+\tfrac{1}{2}i\hbar b^2v_1}, \quad (15)$$

and with (11)

$$\alpha_1=\frac{hibu_1}{(1-iv_1S_\infty)(E_1-E_0)+\tfrac{1}{2}i\hbar b^2v_1}. \quad (15a)$$

The S_1 of (14a) is, strictly speaking, a barred quantity. However, if ka is small, (10c) shows that the unbarred S differs from it only by the constant amount Ma/h which can be absorbed into the S_∞. If ka is large, its energy dependence, as well as that of v, becomes negligible. The U of (15) then becomes the ratio of two linear functions of E which retains its form even if multiplied by e^{-2ika}. One can therefore write for the scattering cross section

$$\sigma(E)=\frac{4\pi}{k^2}\sin^2\delta=\frac{\pi}{k^2}(2-U-U^*)=\frac{4\pi(hS/M)^2}{v^2S^2+1}$$

$$=\frac{4\pi\hbar^2}{M^2}\frac{(S_\infty(E-E_0)-\tfrac{1}{2}\hbar b^2)^2}{(S_\infty(E-E_0)-\tfrac{1}{2}\hbar b^2)^2v^2+(E-E_0)^2}. \quad (16)$$

This goes over into the formula that one would expect

$$\sigma(E)=\frac{\pi}{k^2}\frac{\Gamma^2}{\tfrac{1}{4}\Gamma^2+(E-E_0)^2}, \quad (16a)$$

if $S_\infty=0$, $\Gamma=\hbar b^2v$. The Γ is proportional to v which is an important feature of the resonance formula. Since Γ is[5] $\hbar$ divided by the mean lifetime τ of the compound state, $1/b^2$ can be interpreted as the average distance $l=v\tau$ which the escaping particle could cover during the lifetime of the compound state $l=v\tau=v\hbar/\Gamma=1/b^2$. Inserting for Γ the experimentally observed neutron widths in the n,γ processes, one finds that l is usually more than a million times greater than the nuclear radius.

One may suspect, first, that the occurence of an additional constant S_∞ in (16) is caused by the inadequacy of the two assumptions made in the introduction. It can be shown, however, by means of a simple example[1] that (16) is correct and that its S_∞ does not vanish in general. Another evidence is the simple formula for the proton-neutron scattering. This is obtained from (16) by setting $b=0$ and $\frac{1}{2}M/S_\infty^2$ equal to the binding energy of the deuteron. It corresponds, therefore, to a special case of (16), which is different from (16a), showing that the latter cannot be the generally valid formula. (When comparing (16) with the simple proton-neutron scattering formula, one has to remember that M in (16) is the relative mass of proton and neutron, i.e., half the proton mass, and that E is the energy in the center of mass coordinate system, i.e., half the neutron energy in the laboratory coordinate system.)

There is a way of writing (16) which has the appearance of a resonance formula even if S_∞ does not vanish:

$$\sigma(E)=\frac{\pi}{k^2}\left\{\frac{4S_\infty^2v^2}{1+S_\infty^2v^2}+\frac{\Gamma^2-4\Gamma S_\infty v(E-E_0)/(1+S_\infty^2v^2)}{\tfrac{1}{4}\Gamma^2+(E-E_0-\tfrac{1}{2}\Gamma S_\infty v)^2}\right\}, \quad (17)$$

$$\Gamma=\hbar b^2v/(1+S_\infty^2v^2). \quad (17a)$$

In (17) the first term can be called potential scattering; the second term is then the sum of the resonance scattering and of the interference between resonance scattering and potential scattering. Equation (17) may appear quite artificial. However, the general case of resonance reactions naturally leads to an Eq. (45a) of this kind. Clearly, once one admits energy or velocity dependent Γ and resonance energies, (16) can be written in numerous ways, the plain fact being that the ordinary resonance formula (in which Γ and E_0 are independent of energy) does not hold in this case.

III. RESONANCE REACTIONS WITH ZERO ANGULAR MOMENTUM

The treatment of the resonance reactions differs from the treatment of the resonance scattering mainly because it is necessary to take the different modes of disintegration into account. Strictly speaking, this is necessary also in the case of resonance scattering, and it was avoided in the above treatment only by tacitly assuming that the colliding particles had no spin and that their relative angular momentum was zero. Neither of these assumptions is justified in general although they may be valid under certain conditions; the latter assumption particularly at low velocities and for uncharged particles. It will be adopted also in the present section. However, the cases in which one can disregard the possibility of a nuclear reaction are so rare that the separate consideration of the resonance scattering which was carried out above must be regarded as largely schematic.

According to our assumptions, if the integrals of motion are given, the wave function is, apart from a constant factor, uniquely determined in the region of the configuration space in which all particles are close together. This, however, does not hold for the asymptotic behavior of the wave function, at distant parts of the configuration space. There, the wave function will in general contain both incoming and outgoing waves corresponding to the several possible modes of disintegration or reaction, e.g., both an incoming and an outgoing wave corresponding to the reactions of or into ${}_3\mathrm{Li}^7+{}_2\mathrm{He}^4$, ${}_4\mathrm{Be}^9+\mathrm{D}$, etc. These different possibilities will be distinguished by indices, j, l, etc. In order to specify a stationary state one can stipulate that it contains an incoming wave only of the variety, say, l. The corresponding wave function φ_l may be called the l wave. Its form outside a sphere in the configuration space can be written as

$$\varphi_l=\sum_j (4\pi)^{-\frac{1}{2}}(r_j u_j)^{-1}$$
$$\times(\exp(-ik_j r_j)\delta_{lj}-U_{lj}\exp(ik_j r_j))\psi_j(i_j). \quad (18)$$

In this $\psi_j(i_j)$ is the product of the real normalized wave functions of the two constituents which correspond to the jth type of disintegration, r_j is the distance between these two constituents, u_j the square root of their relative velocity, and k_j the corresponding wave number $k_j=M_j v_j/\hbar=M_j u_j^2/\hbar$ where M_j is the relative mass of the two constituents in the jth type of disintegration. The first term of the bracket corresponds to approaching particles, the δ_{lj} indicates that, in the wave φ_l, only the constituents of the type l do approach each other. The second term in the bracket corresponds to the products of the reaction or, for $j=l$, to scattered particles of the original kind. The U_{lj} are elements of the "collision matrix" introduced by J. A. Wheeler.[6] It is a set of n^2 constants if the number of possible disintegrations is n. The quantity $|U_{lj}|^2$ is the probability of a reaction, yielding the jth type of constituents, if originally the lth type of constituents are present. In (18) u_j, k_j, U_{lj} depend on the energy, $\psi_j(i_j)$ does not.

While (18) is supposed to represent φ_j outside the sphere of interaction of the particles, if all the particles are close, it will be assumed that φ_j has in first approximation the form

$$\varphi_j=\alpha_j\Psi. \quad (19)$$

As in (2), α_j is a function of energy, Ψ is the normalized wave function of the compound state and is, in first approximation, independent of energy. In order to express the energy dependence of the quantities u_j, k_j, U_{lj}, α_j, an additional index will be given to them, 1, 2, 3, . . . indicating that they are coefficients in the expressions (18) or (19) for a φ which corresponds to the values of the energy E_1, E_2, E_3, etc. The matrix $||U_{lj}||$ will also be denoted by $\mathfrak{U}$; if it occurs in the expression for φ_1 with the energy E_1 it will be denoted by $\mathfrak{U}_1$. The quantities α_j, α_l, . . . will be considered to form a vector $\boldsymbol{\alpha}$ with n components; if they occur in the expansion of φ_1 with the energy E_1, the vector will be called $\boldsymbol{\alpha}_1$, etc. In addition, a real positive diagonal matrix $\mathfrak{u}$ will be defined, its diagonal elements are the u_j, u_l, etc., which are the square roots of the relative velocities of the particles in the different modes of disintegration. The significance of the diagonal matrices $\mathfrak{u}_1$, $\mathfrak{u}_2$, . . . etc., will be clear from the preceding.

We now go over to deriving an equation which is analogous to (6). For this purpose we write

[6] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937); W. Heisenberg, Zeits. f. Physik **120**, 513, 673 (1943).

down the two equations

$$H\varphi_{1m}=E_1\varphi_{1m}, \tag{20a}$$

$$H\varphi_{2l}=E_2\varphi_{2l}. \tag{20b}$$

We multiply again (20a) with φ_{2l}^*, the conjugate complex of (20b) with φ_{1j}, subtract the resulting equations and integrate over the sphere which corresponds to all the particles being close together. This gives, after a partial integration, an equation analogous to (4), except that φ_2 is replaced by φ_{2l} and φ_1 by φ_{1m}. On the left side, in the surface integral, (18) will be substituted for φ; in the volume integral (19). Again, no details of the calculation will be given since it is a repetition of a transformation which is carried out several times in any book on quantum mechanics. One obtains

$$\sum_j -(\hbar^2/2M_j u_{1j}u_{2j})(\exp(ik_{2j}a_j)\delta_{lj}-U_{2lj}^*\exp(-ik_{2j}a_j))ik_{1j}(-\exp(-ik_{1j}a_j)\delta_{mj}-U_{1mj}\exp(ik_{1j}a_j))$$
$$+\sum_j(\hbar^2/2M_j u_{1j}u_{2j})(\exp(-ik_{1j}a_j)\delta_{mj}-U_{1mj}\exp(ik_{1j}a_j))ik_{2j}(\exp(ik_{2j}a_j)\delta_{lj}+U_{2lj}^*\exp(-ik_{2j}a_j))$$
$$=(E_1-E_2)\alpha_{1m}\alpha_{2l}^*. \tag{21}$$

In deriving (21) it is necessary to observe that terms with $\psi_j(i_j)$ and $\psi_{j'}(i_{j'})$ are orthogonal on the surface of the sphere if $j\neq j'$ because some of the particles, which are close in the term of ψ_j, are separated into different constituents in the term with $\psi_{j'}$ so that the integral over the product of ψ_j and $\psi_{j'}$ vanishes. This reduces the double sum which should be on the left side of (21) to a single sum and in fact eliminates any integrals from it since the integral of $|\psi_j(i_j)|^2$ is 1. In (21), a_j is the distance between the jth constituents on the surface of the sphere.

If we again assume, as we did in the case of resonance scattering, that the ka are small and introduce $k_j=M_j u_j^2/\hbar$, (21) can be written in the form

$$\sum_j \hbar i(\delta_{mj}+U_{1mj})u_{1j}u_{2j}^{-1}(\delta_{lj}-U_{2lj}^*)$$
$$+\hbar i(\delta_{mj}-U_{1mj})u_{1j}^{-1}u_{2j}(\delta_{lj}+U_{2lj}^*)$$
$$=2(E_1-E_2)\alpha_{1m}\alpha_{2l}^*. \tag{22}$$

This can be written with the notation explained above in the matrix form

$$hi(1+\mathfrak{U}_1)\mathfrak{u}_1\mathfrak{u}_2^{-1}(1-\mathfrak{U}_2^\dagger)$$
$$+hi(1-\mathfrak{U}_1)\mathfrak{u}_1^{-1}\mathfrak{u}_2(1+\mathfrak{U}_2^\dagger)$$
$$=2(E_1-E_2)\mathfrak{A}_{12}, \tag{23}$$

where

$$(\mathfrak{A}_{12})_{ml}=\alpha_{1m}\alpha_{2l}^* \quad \text{and} \quad \mathbf{1}_{ml}=\delta_{ml}; \tag{23a}$$

(23) is the generalization of (8) for resonance reactions. It will serve as a basis for most of the remainder of this work. It embodies the second approximation for the change of the wave function of the compound nucleus with energy. In the first approximation, this wave function was independent of energy. The relation of the second approximation to the first one was explained more fully through Eqs. (5) to (6a).

Setting $E_1=E_2$, it follows that $\mathfrak{u}_1=\mathfrak{u}_2$, $\mathfrak{U}_1=\mathfrak{U}_2$; and we have

$$(1+\mathfrak{U}_1)(1-\mathfrak{U}_1^\dagger)+(1-\mathfrak{U}_1)(1+\mathfrak{U}_1^\dagger)$$
$$=2-2\mathfrak{U}_1\mathfrak{U}_1^\dagger=0, \tag{24}$$

i.e., that the $\mathfrak{U}$ are unitary. This can be derived in the same way even if the assumptions in the introduction are not made. It corresponds to the conservation law for the number of systems[6] and has the same role which $U=1$ had in the case of resonance scattering.

If (18), (19) represent a solution, this also holds for their conjugate imaginary. In this conjugate imaginary, the second term in the bracket of (18) represents the incoming waves. Hence the conjugate imaginary is equal to

$$\varphi_l^*=-\sum_j U_{lj}^*\varphi_j. \tag{25}$$

Since the incoming waves of the right side of (25) are the same as on the left side, the two sides must be equal. It then follows, first, that the outgoing wave corresponding to the mth constituent is also the same

$$(4\pi)^{-\frac{1}{2}}(r_m u_m)^{-1}\exp(ik_m r_m)\delta_{ml}$$
$$=-\sum_j U_{lj}^*(4\pi)^{-\frac{1}{2}}(r_m u_m)^{-1}U_{jm}\exp(ik_m r_m)$$

or that

$$\delta_{ml}=\sum_j U_{lj}^*U_{jm}; \quad \mathfrak{U}^*\mathfrak{U}=1. \tag{26}$$

This, together with $\mathfrak{U}\mathfrak{U}^\dagger=\mathfrak{U}^\dagger\mathfrak{U}=1$ of (24b)

shows that $\mathfrak{U}^*=\mathfrak{U}$ or $\mathfrak{U}=\mathfrak{U}'$, i.e., that $\mathfrak{U}$ is symmetric. This is again known[6] to hold independently of the assumptions made in the introduction.

It follows from (25) also that the wave functions of both sides of (25) are the same in the region of the configuration space where all particles are close together. This gives, with (19)

$$\alpha_l^*\Psi^*=-\sum_j U_{lj}^*\alpha_j\Psi. \tag{27}$$

It follows from (27), first, that Ψ^* is, apart from a constant factor, equal to Ψ so that it can be made real by multiplication with a constant of unit absolute value. Since such a constant was left free when Ψ was defined, we can assume that Ψ is real and write instead of (27)

$$\alpha_l^*=-\sum_j U_{lj}^*\alpha_j;\quad \alpha_l=-\sum_j U_{lj}\alpha_j^*. \tag{27a}$$

In the matrix-vector notation explained above this reads

$$\boldsymbol{\alpha}_1^*=-\mathfrak{U}_1^*\boldsymbol{\alpha}_1;\quad \boldsymbol{\alpha}_1=-\mathfrak{U}_1\boldsymbol{\alpha}_1^*; \tag{27b}$$

(27b) is the generalization of (9) for resonance reactions and will play a role similar to that of (9).

The above derivations were abbreviated but it is believed that further details would make them more cumbersome without elucidating the situation very much more.

IV. SOLUTION OF EQUATIONS (23), (27b)

We introduce a quantity similar to the S of (10) by the equations

$$\mathfrak{U}_1=\frac{1+i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1}{1-i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1};\quad \mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1=i\frac{1-\mathfrak{U}_1}{1+\mathfrak{U}_1}. \tag{28}$$

The $\mathfrak{S}$ are now matrices just as the $\mathfrak{U}$ are and have the same number of rows and columns as the latter. Each row and column corresponds to a particular mode of disintegration of the compound nucleus. The $\mathfrak{u}$ are diagonal matrices, the diagonal elements, which are functions of energy, are the square roots of the velocities with which the particles collide. The $\mathfrak{S}$ are symmetric because the $\mathfrak{U}$ are, and they are hermitian because the $\mathfrak{U}$ are unitary. They are, therefore, real, symmetric matrices

$$\mathfrak{S}_1=\mathfrak{S}_1^*=\mathfrak{S}_1'=\mathfrak{S}_1^\dagger. \tag{28a}$$

Introducing (28) into (23) and multiplying the resulting equation with $\mathfrak{u}_1^{-1}(1-i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1)$ from the left, $(1+i\mathfrak{u}_2\mathfrak{S}_2\mathfrak{u}_2)\mathfrak{u}_2^{-1}$ from the right, one obtains

$$-\mathfrak{S}_2+\mathfrak{S}_1=(2\hbar)^{-1}(E_1-E_2)\mathfrak{u}_1^{-1}(1-i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1)\mathfrak{A}_{12}(1+i\mathfrak{u}_2\mathfrak{S}_2\mathfrak{u}_2)\mathfrak{u}_2^{-1}=2\hbar^{-1}(E_1-E_2)\mathfrak{B}_{12}. \tag{29}$$

The last part of (29) is the definition of $\mathfrak{B}_{12}$. One obtains for the matrix elements of $\mathfrak{B}_{12}$ in a straightforward way with the help of (23a)

$$(\mathfrak{B}_{12})_{ml}=\beta_{1m}\beta_{2l}^*, \tag{27a}$$

where

$$\beta_{1m}=\tfrac{1}{2}\sum u_{1m}^{-1}(1-i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1)_{mk}\alpha_{1k}$$

so that the vector $\mathfrak{\beta}$ is defined by

$$\mathfrak{\beta}_1=\tfrac{1}{2}\mathfrak{u}_1^{-1}(1-i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1)\boldsymbol{\alpha}_1. \tag{29b}$$

It follows from (29) that $\mathfrak{B}_{12}$ must be real and symmetric. It is a consequence of the former property (i.e., that all $\beta_{1m}\beta_{2l}^*$ are real) that the phase factor of all the β_{1m} is the same, independent of m and the energy E_1 to which they refer. It follows from the symmetry of $\mathfrak{B}$ that $\beta_{1m}\beta_{2l}^*=\beta_{1l}\beta_{2m}^*$ or $\beta_{1m}/\beta_{1l}=\beta_{2m}^*/\beta_{2l}^*$. This is equivalent with the statement that all vectors $\mathfrak{\beta}$ are multiples of a common vector

$$\mathfrak{\beta}_1=b_1\omega\mathfrak{\beta}\ (b_1 \text{ real}), \tag{30}$$

where b_1 is a real number, depending on energy, while the number ω and the vector $\mathfrak{\beta}$ are independent of energy. ω has the absolute value 1, the $\mathfrak{\beta}$ is real and we can assume it to be normalized

$$(\mathfrak{\beta},\mathfrak{\beta})=\sum\beta_j^2=1\ (\mathfrak{\beta}\text{ real}). \tag{30a}$$

Equation (27b) also can be written in terms of $\mathfrak{S}$

$$(1+i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1)\boldsymbol{\alpha}_1^*=-(1-i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1)\boldsymbol{\alpha}_1; \tag{31}$$

comparing this with (29b) one obtains $\mathfrak{u}_1\mathfrak{\beta}_1^*=-\mathfrak{u}_1\mathfrak{\beta}_1$ or with (30) that $\omega=\pm i$. One can assume $\omega=i$ since the sign of the vector $\mathfrak{\beta}$ remained undetermined so far. Uniting these equations, we can write

$$\boldsymbol{\alpha}_1=2ib_1(1-i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1)^{-1}\mathfrak{u}_1\mathfrak{\beta} \tag{32}$$

and[7]

$$\mathfrak{B}_{12}=b_1b_2\mathfrak{B};\ \mathfrak{B}_{ml}=\beta_m\beta_l;\ \mathfrak{B}=\mathfrak{B}^2=\mathfrak{B}^3=\cdots. \tag{32a}$$

[7] The mathematicians call matrices which are equal to their powers idempotents. $\mathfrak{A}$ and, later, $\mathfrak{B}$ are idempotents of rank 1. $\mathfrak{A}$ (and $\mathfrak{B}$) transform every vector into a vector of a definite direction which is, in the case of $\mathfrak{A}$, the direc-

This last result enables one to determine the $\mathfrak{S}$. One sees from (29) that $\mathfrak{S}_1$ must have the form

$$\mathfrak{S}_1=\mathfrak{S}_\infty-c_1\mathfrak{B}, \tag{32b}$$

where c_1 is a so far unknown real function of E_1 while the real symmetric matrix $\mathfrak{S}_\infty$ is independent from E_1.

Introducing now (32b) into (29), one obtains

$$-c_1+c_2=2\hbar^{-1}(E_1-E_2)b_1b_2. \tag{33}$$

If one writes down this equation for the pairs E_2, E_3 and E_3, E_1 instead of E_1, E_2 and adds the three resulting equations, the c's drop out and one obtains

$$[(E_1-E_2)b_2+(E_3-E_1)b_3]b_1=(E_3-E_2)b_2b_3.$$

This shows in the same way as was explained after (13) that $1/b_1$ is a linear function of the energy E_1 so that one can write

$$b_1=\frac{\frac{1}{2}\hbar b}{(E_1-E_0)} \quad (b, E_0 \text{ real}). \tag{34}$$

This gives with (33)

$$-c_1+c_2=\tfrac{1}{2}\hbar b^2\frac{(E_1-E_2)}{(E_1-E_0)(E_2-E_0)}$$

$$=\tfrac{1}{2}\hbar b^2\left(-\frac{1}{E_1-E_0}+\frac{1}{E_2-E_0}\right); \tag{35}$$

from this we have

$$c_1=\frac{\frac{1}{2}\hbar b^2}{E_1-E_0}+c_\infty. \tag{35a}$$

One can redefine S_∞ of (32b) by decreasing it by $c_\infty\mathfrak{B}$. As a result, the last term in (32b) will also have to be decreased by $c_\infty\mathfrak{B}$ or c_1 by c_∞. One sees that $c_\infty=0$ can be written without loss of generality. As a result, we have

$$\mathfrak{S}_1=\mathfrak{S}_\infty-\frac{\frac{1}{2}\hbar b^2}{E_1-E_0}\mathfrak{B} \tag{36}$$

so that the matrix $\mathfrak{S}_\infty$ which was introduced in (32b) merely as an energy independent matrix, becomes equal to $\mathfrak{S}_1$ for $E_1=\infty$. Equation (32) and (34) now give

$$\alpha_1=\frac{i\hbar b}{E_1-E_0}(1-i\mathfrak{u}_1\mathfrak{S}_1\mathfrak{u}_1)^{-1}\mathfrak{u}_1\beta. \tag{37}$$

In these, E_0 and b are arbitrary real numbers, β an arbitrary real vector of length 1, $\mathfrak{S}_\infty$ an arbitrary real symmetric matrix and

$$\mathfrak{B}_{jl}=\beta_j\beta_l;\ \mathfrak{B}=\mathfrak{B}^2=\mathfrak{B}^3=\cdots. \tag{36a}$$

All these quantities are independent of energy. For $\mathfrak{U}$ one has

$$\mathfrak{U}=(1+i\mathfrak{u}_1\mathfrak{S}_\infty\mathfrak{u}_1-\tfrac{1}{2}i\hbar b^2(E_1-E_0)^{-1}\mathfrak{u}_1\mathfrak{B}\mathfrak{u}_1)$$
$$\times(1-i\mathfrak{u}_1\mathfrak{S}_\infty\mathfrak{u}_1+\tfrac{1}{2}i\hbar b^2(E_1-E_0)^{-1}\mathfrak{u}_1\mathfrak{B}\mathfrak{u}_1)^{-1}$$
$$=[(1+i\mathfrak{u}_1\mathfrak{S}_\infty\mathfrak{u}_1)(E_1-E_0)-\tfrac{1}{2}i\hbar b^2\mathfrak{u}_1\mathfrak{B}\mathfrak{u}_1]$$
$$\times[(1-i\mathfrak{u}_1\mathfrak{S}_\infty\mathfrak{u}_1)(E_1-E_0)$$
$$+\tfrac{1}{2}i\hbar b^2\mathfrak{u}_1\mathfrak{B}\mathfrak{u}_1]^{-1}. \tag{38}$$

One can easily verify that the $\mathfrak{U}$ given by (38), together with the α given by (37) and (36), actually satisfy (24) and (27b). The reader will recognize that the derivation of this section, and indeed the result (38), is very closely patterned on the derivation of the similar formula (15) for resonance scattering. The only difference, as far as the result is concerned, is first that $\mathfrak{S}_\infty$ of (38) is an arbitrary real symmetric matrix rather than an arbitrary real number and, second, that the b^2 of (15) is replaced by $b^2\mathfrak{B}$ where $\mathfrak{B}$ is the matrix defined in (36a). In spite of these apparently minor differences, the calculation of the scattering and reaction cross sections from (38) (or (36)) in their general form is less immediate than the calculation of the former was from (15). Indeed, even the transformation of (38) into a form more suggestive of the resonance formula

$$\mathfrak{U}=\frac{1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}}{1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}}-\frac{i\hbar b^2}{E-E_0+\frac{1}{2}i\lambda}(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}$$
$$\times\mathfrak{u}\mathfrak{B}\mathfrak{u}(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}, \tag{39}$$

$$\lambda=\hbar b^2\sum_{jl}\beta_j u_j(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}_{jl}u_l\beta_l, \tag{39a}$$

$$\alpha=\frac{i\hbar b}{E-E_0+\frac{1}{2}i\lambda}(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\beta, \tag{39b}$$

tion of the vector α. Hence, for every vector $\mathbf{v}$, the $\mathfrak{A}\mathbf{v}$ is a constant times α. The length of $\mathfrak{A}\mathbf{v}$ (and of $\mathfrak{B}\mathbf{v}$) depends only on the component of $\mathbf{v}$ in a definite direction which is the direction of α in case of the matrix A. It is evident from this that $\mathfrak{A}\mathfrak{R}\mathfrak{A}$ is, for any matrix $\mathfrak{R}$, a multiple of $\mathfrak{A}$ since it has both the above properties of $\mathfrak{A}$. These remarks are designed to give a better understanding to some of the calculations carried out in the text and the appendix.

is quite cumbersome and will be given only in the appendix.

V. PARTICULAR CASES OF THE COLLISION MATRIX (38) FOR ZERO ANGULAR MOMENTUM

Although (38) is the most general $\mathfrak{U}$ compatible with (23) and (27b), it can be argued that it does not satisfy all conditions set forth in the introduction. It is seen that it goes over, for very large $E-E_0$, into $(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})(1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}$ while, according to (37), α goes to zero at the same time. This last fact shows that any reaction or scattering, if present, does not go through the intermediate state if $E-E_0$ is large since in this case the coefficient of Ψ in (19) tends to zero. One can add, therefore, to (23), (27b) the further condition that either $\mathfrak{U}=1$ for $E-E_0$ large, or that $\mathfrak{U}$ become a diagonal matrix in this case. In the former case, which will be taken up first, one stipulates that both reaction and scattering go through the intermediate state Ψ; in the latter case one assumes only that any actual reaction has to go through the intermediate state. It follows from the first, more stringent assumption that $(1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}=1$, i.e., $\mathfrak{S}_\infty=0$. The matrix $\mathfrak{U}-1$ then becomes

$$\mathfrak{U}-1=-\frac{ihb^2}{E-E_0+\frac{1}{2}i\lambda}\mathfrak{u}\mathfrak{P}\mathfrak{u}, \tag{40}$$

$$\lambda=hb^2\sum_j v_j\beta_j^2. \tag{40a}$$

The cross section for the transition from the state j to the state l is

$$\sigma_{jl}=\frac{\pi}{k_j^2}|(\mathfrak{U}-1)_{jl}|^2=\frac{\pi}{k_j^2}\frac{h^2b^4(u_j\beta_j\beta_l u_l)^2}{(E-E_0)^2+\frac{1}{4}\lambda^2},$$

$$\sigma_{jl}=\frac{\pi}{k_j^2}\frac{\Gamma_j\Gamma_l}{\frac{1}{4}\Gamma^2+(E-E_0)^2}, \tag{41}$$

where

$$\Gamma_j=hb^2\beta_j^2v_j,\quad \Gamma=\lambda=\sum_j\Gamma_j. \tag{41a}$$

One sees that the more rigorous restriction leads to the ordinary resonance formula just as the similar restriction specialized (16) to (16a). It is worth while to note that Γ_j is again proportional to the velocity with which the constituents of the jth type of particle collide before or separate after the reaction.

It may be remembered that the $1/v$ law of the slow neutron capture is a consequence of this: the $1/k^2$ factor gives proportionality with v^{-2} which multiplied with the Γ_{neutron} gives v^{-1}. Under the more stringent assumption which we consider now, (41) is valid also for scattering and one sees that the scattering cross section should go to zero for large $E-E_0$, i.e., (41) gives only what is usually considered to be the anomalous scattering cross section, but does not give the normal scattering.

The second restriction, the consequences of which one may be interested in, is that the first term of (39) be a diagonal matrix. In this case, $\mathfrak{S}_\infty$ also must be diagonal, its diagonal elements—which must be real—will be denoted by s_j. One obtains from (39)

$$(\mathfrak{U}-1)_{jl}=\frac{2is_jv_j}{1-is_jv_j}\delta_{jl}-\frac{ihb^2}{E-E_0+\frac{1}{2}i\lambda}\times(1-is_jv_j)^{-1}u_j\beta_j\beta_lu_l(1-is_lv_l)^{-1}, \tag{42}$$

$$\lambda=hb^2\sum_j\frac{\beta_j^2v_j}{1-is_jv_j}. \tag{42a}$$

It appears natural to separate real and imaginary parts of λ

$$\lambda=\Gamma+2i\Delta, \tag{43}$$

$$\Gamma=hb^2\sum\frac{\beta_j^2v_j}{1+s_j^2v_j^2};\quad \Delta=\tfrac{1}{2}hb^2\sum\frac{\beta_j^2s_jv_j^2}{1+s_j^2v_j^2}. \tag{43a}$$

Let us first consider the case of nuclear reaction, i.e., $j\neq l$. The first term of (42) then vanishes and we have for the effective cross section of the reaction yielding the particles l from a collision of particles j with the relative velocity v_j and total energy E:

$$\sigma_{jl}=\frac{\pi}{k_j^2}|(\mathfrak{U}-1)_{jl}|^2=\frac{\pi}{k_j^2}\frac{\Gamma_j\Gamma_l}{\frac{1}{4}\Gamma^2+(E-E_0-\Delta)^2};\quad (j\neq l), \tag{44}$$

where now

$$\Gamma_j=\frac{hb^2\beta_j^2v_j}{1+s_j^2v_j^2};\quad \Gamma=\sum_j\Gamma_j. \tag{44a}$$

Equations (44) differ in two ways from Eqs. (41): first, there is an energy shift Δ present in the former, the shift being velocity dependent.

It will appear, however, in the last paragraph of this section that, at least for reactions involving only uncharged particles, Δ is very small even if compared with Γ if v_j corresponds to less than about a million volts. If, on the other hand, v_j corresponds to a high energy, the corresponding term in Δ becomes energy independent so that the occurrence of Δ in the denominator of (44) is without much practical significance. The second difference between (44) and (41) is the definition of Γ_j, the dependence of which on energy became more complicated. However, virtually the same remarks apply as before: if v_j corresponds to a low energy, $v_j s_j \ll 1$, at least for uncharged particles. If the energy corresponding to the jth type of reaction is large, v_j changes so little within the resonance region that $1+s_j^2 v_j^2$ becomes practically independent of energy. As a result, the second, more general, assumption leads in practice to the same energy dependence of the cross section of resonance reactions as the first assumption.

For the scattering cross section, on the other hand, one has after a brief calculation

$$\sigma_{jj}=\frac{\pi}{k_j^2}|(\mathfrak{U}-1)_{jj}|^2=\frac{\pi v_j^2}{k_j^2(1+s_j^2v_j^2)^2}\left|2s_j(1-is_jv_j)-\frac{\hbar b^2\beta_j^2}{E-E_0-\Delta+\frac{1}{2}i\Gamma}\right|^2. \tag{45}$$

Expanding the square, this becomes with (43) and (44a)

$$\sigma_{jj}=\frac{4\pi s_j^2v_j^2}{k_j^2(1+s_j^2v_j^2)}+\frac{\pi}{k_j^2}\frac{\Gamma_j^2}{\frac{1}{4}\Gamma^2+(E-E_0-\Delta)^2}-\frac{4\pi s_j\Gamma_jv_j}{k_j^2(1+s_j^2v_j^2)}\frac{E-E_0-\Delta+\frac{1}{2}s_j\Gamma v_j}{\frac{1}{4}\Gamma^2+(E-E_0-\Delta)^2}. \tag{46}$$

The terms in the first two lines are the squares of the two terms in (45) between the absolute sign.[8] They correspond to ordinary and resonance scattering, respectively. The last term of (46) is the cross product of the two terms between the absolute sign. It corresponds to the interference between ordinary and resonance scattering. One sees from (45) or (46) that the scattering cross section remains finite for $v=0$.

It may be worth while to give a more visualizable interpretation of the quantities occurring in the equations of this section than is provided by the equations themselves and also to give expressions for them in more conventional units.

If one expresses the energy E in electron volts, π/k^2 becomes $(0.65\times10^{-18}/E)$ cm^2 for a particle of the mass of the neutron.

The quantities β_j are dimensionless, as evident from (30a), since the sum of their squares is 1. They serve in our formalism only to subdivide the total level width Γ into partial widths (cf. (41a) and (44a)). In the usual n,γ processes, the β_{neutron} is practically 1 while $\beta_{\text{radiation}}$ is very small, in spite of the $\Gamma_{\text{radiation}}$ being much greater than Γ_{neutron}, because the ratio of the neutron velocity to light velocity is much greater than the ratio of the corresponding Γ. The interpretation of b has been given before: $1/b^2\beta_j^2$ is the average distance which a particle of velocity v_j could cover during the partial lifetime of the compound nucleus for the j type of disintegration. This partial lifetime is defined as the reciprocal disintegration constant of the compound nucleus for the j type of disintegration. It is usual to express Γ_j in electron volts and write for it $\Gamma_j=aE_j^{\frac{1}{2}}$, where E_j is again expressed in electron volts. The $1/b^2\beta_j^2$ then becomes $1/b^2\beta_j^2=3.2\times10^3\ (M_{\text{neutron}}/M_j)^{\frac{1}{2}}a^{-1}(e^2/mc^2)$. Since a is usually less than 10^{-3} for slow neutron processes, one sees that $1/b^2$ is, at least in the case of the n,γ process, more than a million times greater than the electronic radius or the nuclear radius. The $\Gamma_{\text{radiation}}$ is usually of the order of 0.1 ev. As a result, $\beta_{\text{radiation}}$ is, in the slow neutron processes, of the order of one tenth.

The s are the diagonal elements of the matrix $\mathfrak{S}_\infty$ which determines the normal scattering. The dimension of the s is a reciprocal velocity, which is also the dimension of the matrix elements of all S. If we express the normal cross section σ_j for a collision in cm^2, $s_j=\pm13.4\times10^{12}(M_j/M_{\text{neutron}})\sigma_j^{\frac{1}{2}}/c$. Since σ_j is usually a few times 10^{-24} cm^2, $|s_j|$ for particles is of the order of $25/c$. The $s_{\text{radiation}}=0$ because the normal nuclear scattering of γ rays is negligible. It follows that,

[8] H. A. Bethe, Rev. Mod. Phys. 9, §55, 71 (1937) already derived this formula (p. 152) except that the ambiguity of the sign in (45), caused by the ambiguity of the sign of s, has been overlooked.

unless the energy E_j is very high, $s_j v_j \ll 1$, and then from (43a) that $\Delta \ll \Gamma$.

VI. REGION OF VALIDITY OF RESONANCE EQUATIONS

The basis for the equations for resonance scattering was (7) and it will be remembered that this appeared as a second approximation, obtained by replacing φ_1 and φ_2^* in the right side of (4) by $\alpha_1\Psi$ and $\alpha_2^*\Psi$, respectively. Since, as pointed out after (4), φ_1 and φ_2^* cannot be accurately equal to these expressions, one may wonder whether their use for the right side of (4) is justifiable.

Let us assume that, as the energy increases, the phase of U goes through -1 at E_0. Then $U_0 = -1$ holds accurately and we can denote by Ψ the exact expression for φ for $E=E_0$ inside the sphere of radius a defined in Section II. For all other values of the energy, φ will not be exactly a multiple of Ψ but will be, inside the sphere of radius a,

$$\varphi_1 = \alpha_1\Psi + \gamma_1\Psi_1, \tag{47}$$

where Ψ_1 is orthogonal to Ψ and is assumed to be normalized

$$(\Psi_1, \Psi_1) = 1, \ (\Psi_1, \Psi) = 0. \tag{47a}$$

For $E=E_0$, the second term of (47) vanishes, i.e., $\gamma_0 = 0$; α_0 is real because of (9).

Equation (8) now reads for the barred quantities defined in (8a)

$$\frac{u_1}{u_2}(1+U_1)(1-U_2^*) + \frac{u_2}{u_1}(1-U_1)(1+U_2^*) = \frac{2}{ih}(E_1-E_2)(\alpha_1\alpha_2^* + \gamma_{12}), \tag{48}$$

where

$$\gamma_{12} = \gamma_1\gamma_2^*(\Psi_2, \Psi_1). \tag{48a}$$

If we express the U again by the S by means of (10) this becomes

$$S_1 - S_2 = \frac{E_1-E_2}{2\hbar u_1 u_2}(1-iv_1S_1) \times (\alpha_1\alpha_2^* + \gamma_{12})(1+iv_2S_2). \tag{48b}$$

Since $\gamma_0 = 0$, also $\gamma_{10} = 0$ and one sees from (10) that $S_0 = \infty$. Setting $E_2 = E_0$ in (48b) and equating the coefficients of S_0 on both sides of (48b) gives

$$\alpha_1 = \frac{2ihu_1}{\alpha_0 u_0(1-iv_1S_1)(E_1-E_0)}. \tag{49}$$

Re-introducing this and a similar expression for α_2 into (48b) yields

$$S_1 - S_2 = \frac{2\hbar}{\alpha_0^2 v_0}\left(-\frac{1}{E_1-E_0} + \frac{1}{E_2-E_0}\right) + \frac{E_1-E_2}{2\hbar u_1 u_2}(1-iv_1S_1)\gamma_{12}(1+iv_2S_2). \tag{50}$$

We can introduce $b^2 = 4/\alpha_0^2 v_0$ in this and obtain

$$S_1 + \frac{\frac{1}{2}\hbar b^2}{E_1-E_0} = S_2 + \frac{\frac{1}{2}\hbar b^2}{E_2-E_0} + \frac{E_1-E_2}{2\hbar}(1-iv_1S_1)\frac{\gamma_{12}}{u_1u_2}(1+iv_2S_2). \tag{50a}$$

It is seen from this equation that the left side would be independent of energy if the last term could be neglected. No approximations were used in Eqs. (47)–(50a). As could have been anticipated, the accuracy of (14) depends on the possibility to neglect γ_{12} or $\gamma_1\gamma_2^*$.

The fact that φ cannot be accurately a multiple of Ψ, i.e., that $\gamma_1 \neq 0$, can be best seen from the fact that the logarithmic derivative of the asymptotic expression (1) for φ is not independent of energy. Since the result obtained in Section II gives at least an approximate expression for φ outside the sphere, one can see what the deviation of the value and gradient of this approximate expression is from the value and gradient of $\alpha\Psi$ which are also given in that section. If the difference between the values of φ and $\alpha\Psi$, as calculated in Section II, has the value δ_1, and if the difference between their derivatives is δ_1', one can assume that γ_1^2 is of the order of magnitude

$$|\gamma_1|^2 \sim (4\pi a^3/3)(\delta_1^2 + a^2\delta_1'^2). \tag{51}$$

This assumes that the discrepancy between the outside and inside solutions obtained in Section II does not get magnified toward the inside of the sphere. It is very likely that such a magnification would occur only in the neighborhood of another resonance.

When calculating δ_1, it is well to remember that the quantities obtained in Section II are

barred quantities from which the unbarred quantities must be first obtained by the Eqs. (8a). One has, therefore,

$$\delta_1=\bar{\alpha}_1\exp(-ik_1a)\Psi(a) - \frac{\exp(-ik_1a)-\bar{U}_1\exp(-2ik_1a)\exp(ik_1a)}{(4\pi)^{\frac{1}{2}}u_1a}$$
$$=\exp(-ik_1a)\left(\bar{\alpha}_1\Psi(a)-\frac{1-\bar{U}_1}{(4\pi)^{\frac{1}{2}}u_1a}\right). \quad (52)$$

This is, because of (15), (15a) equal to

$$\frac{i\hbar bu_1\Psi(a)+(2iu_1S_\infty(E_1-E_0)-i\hbar b^2u_1)(4\pi)^{-\frac{1}{2}}a^{-1}}{(1-iv_1S_\infty)(E_1-E_0)+\frac{1}{2}i\hbar b^2v_1}.$$

Since, however, Ψ(a) is accurate for $E_1=E_0$, it must be true that $\Psi(a)=b(4\pi)^{-\frac{1}{2}}a^{-1}$ and we have

$$\delta_1=\frac{\exp(-ik_1a)}{(4\pi)^{\frac{1}{2}}au_1}\times\frac{2iv_1S_\infty(E_1-E_0)}{(1-iv_1S_\infty)(E_1-E_0)+\frac{1}{2}i\hbar b^2v_1}. \quad (52a)$$

It is worth noticing that the solution obtained in Section II gives no discontinuity in φ if $S_\infty=0$.

One can calculate δ_1' in a similar way and obtain

$$a\delta_1'=\frac{\exp(-ik_1a)}{(4\pi)^{\frac{1}{2}}au_1}\times\frac{(2ik_1a-2iv_1S_\infty)(E_1-E_0)}{(1-iv_1S_\infty)(E_1-E_0)+\frac{1}{2}i\hbar b^2v_1}. \quad (52b)$$

These equations enable us to estimate the last term of (50a). It appears to be an overestimate to write for this term

$$\frac{E_1-E_2}{2\hbar u_1u_2}(1-iv_1S_1)\gamma_1\gamma_2^*(1+iv_2S_2)$$
$$\leq\frac{|E_1-E_2|}{2\hbar u_1u_2}\frac{4\pi}{3}a^3|(1-iv_1S_1)(\delta_1+a\delta_1')\times(\delta_2+a\delta_2')(1+iv_2S_2)|$$
$$=\frac{|E_1-E_2|}{2\hbar u_1u_2}\frac{4\pi}{3}a^3\frac{|2ik_1a||2ik_2a|}{4\pi a^2u_1u_2}=\frac{2(E_1-E_2)a}{3\hbar v_1v_2}k_1k_2a^2. \quad (53)$$

Strictly speaking $|\delta_1|+a|\delta_1'|$ and $|\delta_2|+a|\delta_2'|$ should stand in the second line of (53) which gives the result obtained in (53) only if $k_1a>v_1S_\infty$. Otherwise, the last term of (50a) can be estimated to be

$$\frac{2(E_1-E_2)a}{3\hbar}S_\infty^2. \quad (53a)$$

In general, the correction term of (50a) will be of the order of the larger of (53) and (53a). One sees at any rate that $S_2+\frac{1}{2}\hbar b^2(E_2-E_0)^{-1}$ must converge to a finite value as E_2 converges to E_0 and this value can be denoted by S_∞ if we keep in mind that this may not be the value of S for infinite energy. One then sees that (14a) holds with an error the order of magnitude of which is given by the greater of two expressions

$$\frac{2(E_1-E_0)a}{3\hbar}\left(\frac{M^2a^2}{\hbar^2}\text{ or }S_\infty^2\right); \quad (54)$$

$hk=Mv$ has been used in the first expression. Actually, if one assumes a to be of the order of a nuclear radius, the two expressions of (54) become about equal, except if S_∞ is exceptionally small. The order of magnitude of S_∞ is taken to be $25/c$, following the estimates given at the end of the last section.

The expression of (54) has to be small compared with (14a) if our approximations are to be valid. This will be the case unless $E-E_0$ is very large. However, in order that (54) be larger than S_∞, the $E-E_0$ must be either larger than $\hbar^3S_\infty/M^2a^3$ which is, for $a=e^2/mc^2$, equal to $137^3\times1840^{-2}S_\infty cmc^2$. This is several million electron volts and larger than $\hbar/S_\infty a\sim137\,mc^2/S_\infty c$. This latter expression is also of the order of a couple of million electron volts unless S_∞ and the normal scattering cross section is unreasonably large. This shows that one must be far indeed from the resonance before the approximations break down.

The other region in which (54) is larger than (14a) lies around $E=E_0+\frac{1}{2}\hbar b^2/S_\infty$ where (14a) vanishes. The range within which (54) is larger than (14a) has the extension of $\Delta E=\hbar b^4a/S_\infty$ if the first expression of (54) is the larger one. ΔE is in this case $ab^2\sim10^{-6}$ times smaller than its distance from E_0. This case can be disregarded. On the other hand, if the second of the expressions of (54) holds, the width of the region in

which (14a) is invalid is $b^4a^3M^2/\hbar S_\infty^3$ and the ratio of this width to the distance of the region from E_0 becomes $M^2a^3b^2/\hbar^2S_\infty^2 \sim 180ab^2/(S_\infty c)^2$. Ordinarily, this is of the same order as ab^2 but can be much larger if S_∞ is very small, i.e., if the ordinary scattering cross section is anomalously low. However, even in this case, there is little danger that the region of invalidity of (14a) assumes any importance as long as ab^2 remains of the order of 10^{-6}.

The above discussion appears to show that the validity of the resonance formula extends over a region of the order of a million electron volts—that is a much wider region than expected. One must remember, however, that the discussion was based on the assumption that the discontinuity of the solution used in Section II gives a fair measure for the deviation of the actual wave function from the wave function used there, even inside the sphere of radius a on the surface of which the discontinuity occurs. This is certainly not true in the neighborhood of another resonance and, in fact, the validity of the formulae of the preceding sections will be limited in most cases by the occurrence of other resonances. The above calculated limitation would apply only if there were no resonances other than the one considered within the limits calculated. It is true also that the above consideration applies only to the developments of Section II, i.e., when one has to deal only with resonance scattering. However, a similar consideration can be carried out also on the basis of the calculations of Section IV and the result is substantially the same inasmuch as the calculated limits of the validity of the equations are only inconsequentially narrower. The real limit of validity will be, also in this case, the occurrence of other resonances unless there is a substantial probability for non-resonance reactions all over the resonance regions.

VII. HIGHER SPIN AND ANGULAR MOMENTA

This subject will not be taken up here in full generality but only two special cases considered. In the first of these, the relative angular momentum of the two colliding particles still remains zero but the particles themselves have spins J_1 and J_2, respectively. If the compound state has a spin J_0, the reaction will be possible only if the spins J_1 and J_2 are combined in such a way that their resultant be J_0. From the $(2J_1+1)(2J_2+1)$ states of the separated particles one can make $2J_0+1$ linear combinations with a total angular momentum J_0. Since the original $(2J_1+1)(2J_2+1)$ states are all equally probable, this holds also for their linear combinations which can be obtained, as in the present case, by an orthogonal transformation. Hence the *a priori* probability that the system be in a state which can lead to a reaction is

$$f=\frac{2J_0+1}{(2J_1+1)(2J_2+1)} \tag{55}$$

and the reaction cross sections of Section V must be multiplied by this factor. J_1 and J_2 are the spins of the colliding particles in the original system which is denoted by j throughout Section VI. In order to calculate the scattering cross section, one must average over the scattering cross sections for the different total angular momenta of the whole system. This gives

$$\sigma=(2J_1+1)^{-1}(2J_2+1)^{-1}\sum_{J=|J_1-J_2|}^{J_1+J_2}(2J+1)\sigma_J. \tag{56}$$

In this, all the σ_J will be normal scattering cross sections, only for the term corresponding to $J=J_0$ will (45) or (46) have to be applied.

In the second case to be considered, the spins of the colliding particles vanish but they have an orbital momentum in the collision. This momentum is equal to the spin of the compound state. The equation for the resonance scattering can be derived in this case very much in the same way as it was derived in Section II for angular momentum zero. The only difference is that the asymptotic form of the wave function is, instead of (1), in case of angular momentum $1\hbar$ (this is the only case which will be taken up)

$$=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\frac{x}{ru}\left[\left(\frac{i}{r}+\frac{1}{kr^2}\right)e^{-ikr}+U\left(\frac{i}{r}-\frac{1}{kr^2}\right)e^{ikr}\right]\psi(i). \tag{57}$$

The $U=e^{2i\delta}$ again determines the scattering by means of the phase shift. The equation analogous

to (8) or (48) now is

$$\frac{u_1}{u_2}(1-\bar{U}_1)(1+\bar{U}_2^*)+\frac{u_2}{u_1}(1+\bar{U}_1)(1-\bar{U}_2)+\frac{i\hbar}{Ma}\left(\frac{u_1}{u_2^3}-\frac{u_2}{u_1^3}\right)(1-\bar{U}_1)(1-\bar{U}_2^*)=\frac{2}{i\hbar}(E_1-E_2)(\bar{\alpha}_1\bar{\alpha}_2^*+\bar{\gamma}_{12}). \quad (58)$$

The $\bar{U}_1=U_1\exp(2ik_1a)$; $\bar{\alpha}_1=\alpha_1\exp(ik_1a)$; $\bar{\gamma}_{12}$ $\gamma_{12}\exp[i(k_1-k_2)a]$ but the bar will be left off these quantities henceforth. The important difference between (58) and (8) or (48) is the last term on the left side which becomes infinite if a, that is the radius of the sphere in configuration space over which we integrate, goes to zero. The reason for this apparently anomalous behavior is that the irregular solution of the equation $\Delta\psi=-k^2\psi$ is not square integrable for angular momentum $1\hbar$ while both solutions are integrable for angular momentum 0. Similar terms appear in the solutions for all higher angular momenta. The γ_{12} is introduced for (58) in the same way as for (48): it vanishes if either E_1 or E_2 are equal to a definite energy E_0 for which $U_0=-1$. It will be assumed here that there is such an energy. Equation (9), which remains valid in this case, then shows that α_0 is real.

One can introduce again by (10) the real quantity $vS=i(1-U)/(1+U)$ instead of the U. In terms of these, (58) reads

$$v_1^2S_1-v_2^2S_2-(\hbar/Ma)(v_1^2-v_2^2)S_1S_2=(2\hbar)^{-1}(E_1-E_2)(1-iv_1S_1)u_1\times(\alpha_1\alpha_2^*+\gamma_{12})u_2(1+iv_2S_2). \quad (59)$$

If one substitutes E_0 for E_2, the γ_{12} will vanish and α_0 becomes real. Since S_0 is infinite, its coefficients on both sides of (59) must be equal. This gives for

$$\alpha_1=\frac{2i\hbar u_0}{E_0\alpha_0u_1}\frac{E_0+(\hbar/Ma)(E_1-E_0)S_1}{(E_1-E_0)(1-iv_1S_1)}. \quad (60)$$

The velocities v were expressed by the corresponding energies in (60). One can now substitute this α_1, and a similar expression for α_2, into (59). Again expressing the velocities in terms of the energies

$$E_1S_1-E_2S_2-(\hbar/Ma)(E_1-E_2)S_1S_2=\frac{2(E_1-E_2)\hbar}{E_0v_0\alpha_0^2}\frac{[E_0+(\hbar/Ma)(E_1-E_0)S_1][E_0+(\hbar/Ma)(E_2-E_0)S_2]}{(E_1-E_0)(E_2-E_0)}+(M/4\hbar)(E_1-E_2)(1-iv_1S_1)u_1\gamma_{12}u_2(1+iv_2S_2). \quad (60a)$$

Neglecting γ_{12}, for the time being, this equation can be written after division by the square brackets and multiplication by E_0 as

$$\frac{E_1S_1}{E_0+(\hbar/Ma)(E_1-E_0)S_1}-\frac{E_2S_2}{E_0+(\hbar/Ma)(E_2-E_0)S_2}=\frac{2\hbar}{v_0\alpha_0^2}\left(\frac{1}{E_0-E_1}-\frac{1}{E_0-E_2}\right). \quad (60b)$$

This shows that the difference of the first terms of both sides is a constant B independent of the energy E_1. Hence one can express S_1 in terms of $Bv_0\alpha_0^2$ and E_0. In order to simplify the formulae one can introduce $F=E_0+2\hbar/Bv_0\alpha_0^2$ and $E_0'=F(1-Ma/\hbar B)^{-1}$ and obtain

$$S_1=\frac{Ma}{\hbar}\frac{E_0E_0'}{F}\frac{F-E_1}{(E_0-E_1)(E_0'-E_1)} \quad (61)$$

and

$$\alpha_1=\frac{2iE_1}{u_1F}\frac{[(E_0-F)(F-E_0')E_0E_0'a]^{\frac{1}{2}}}{(E_0-E_1)(E_0'-E_1)(1-iv_1S_1)}; \quad (61a)$$

(61) appears to correspond to two resonances: one at E_0 and another at E_0'. The S becomes infinite for both these values of energy and the cross section therefore appears to assume its maximum possible value of $12\pi/k^2$. The E_0, E_0', F are arbitrary constants, except that the quantity under the square root sign of (61a) must be positive.

The S given by (61) is in most cases smaller than the S calculated in Section II. It is more important, therefore, to make the correction indicated in that section (10c), i.e., to subtract

$Ma/\hbar$ from it. This then gives

$$S_1=\bar{S}_1-\frac{Ma}{\hbar}=\frac{Ma}{\hbar}\frac{E_1(FE_0+FE_0'-FE_1-E_0E_0')}{F(E_0-E_1)(E_0'-E_1)}$$

$$=\frac{Ma(F-E_0)E_0'}{\hbar F(E_0'-E_1)}\frac{E_1}{E_0-E_1}+\frac{Ma}{\hbar}\frac{E_1}{E_0'-E_1}. \quad (61b)$$

If the domain in which (61) is valid is much smaller than $E_0'-E_0$—and the discussion of the validity of (61) will necessitate restriction of E_1 to such a domain—$E_0'-E_1$ can be considered to be a constant. Hence S_1 can be written in a form very similar to (14a)

$$S_1=\frac{-\frac{1}{2}\hbar c^2E_1}{E_1-E_0}+CE_1. \quad (61c)$$

The only difference is the factor E_1 which occurs in both terms of S_1. The first of these has the effect that, if one writes the cross section in the form of a resonance formula, the width Γ becomes proportional to the third rather than the first power of the velocity. The effect of the factor E_1 in the second term is to render the total scattering zero for $E_1=0$. One has, in fact, for $E\sim 0$

$$\sigma(E)\sim 12\pi a^2E^2\left(\frac{1}{E_0}+\frac{1}{E_0'}-\frac{1}{F}\right)^2 \quad (62)$$

so that the cross section due to the wave with angular momentum $1\hbar$ is proportional to E^2 at $E\sim 0$. Of course, there is always a cross section present from the spherical wave with zero angular momentum.

Above $E=0$, the cross section increases and reaches the theoretical maximum at E_0. The half-width of the absorption line can be obtained by calculating the distance at which vS becomes 1. The full half-width Γ_0 is twice this distance and is equal to

$$\Gamma_0=2ak\frac{F-E_0}{F}\frac{E_0'}{E_0'-E_0}E_0, \quad (63)$$

if all the differences occurring in (63) are large compared with Γ_0. This will be true in general if ak is small, i.e., the energy E_0 not excessively large. In the neighborhood of E_0 the cross section will follow the usual resonance form. As the energy increases further, the cross section drops and becomes zero at F. It then increases again and (61) would indicate that it reaches the theoretical limit again at E_0'. It will be shown, however, that (61) ceases to be valid somewhere midway between E_0 and E_0'.

The above discussion applies if $0<E_0<F<E_0'$. In addition to this, the following arrangements are possible, $0<E_0'<F<E_0$, in which case (61) ceases to be valid at low energies. The same holds true in general if $E_0'<0<E_0<F$ and if $F<E_0'<0<E_0$. This last case appears somewhat unlikely. It is possible, finally, that $E_0<0$ which is usually called a virtual resonance. The discussion of all these cases is very similar to the one above.

Before going into the discussion of the validity of (61), it appears worth while to calculate the ratio of the square integral $|\alpha|^2$ of the wave function inside the sphere with radius a, to the value which this square integral would assume if the wave function were as large inside this sphere as it is on its surface. Evidently, this ratio is very large if we have a real resonance. The first quantity is $|\alpha|^2$ and is given by (61a). The second one is the volume of the sphere multiplied by the square of the coefficient of $\psi(i)$ in (57) for $r=a$. In this last expression x^2/r^2 can be replaced by $\frac{1}{3}$. One has, therefore, for this ratio

$$\frac{3|\alpha_1|^2v_1}{a^3|i(1+\bar{U}_1)/a+(1-\bar{U}_1)/ka^2|^2}$$

$$=\frac{3(E_0-F)(F-E_0')E_0E_0'}{4(FE_1-FE_0-FE_0'+E_0E_0')^2}. \quad (64)$$

For a real resonance, this should be large at least for $E_1=E_0$. This means that

$$\frac{E_0}{E_0'}\frac{E_0'-F}{F-E_0}\gg 1 \quad (64a)$$

is certainly a necessary condition for a real resonance. Hence either E_0' must be very small or F must be very close to E_0.

It will be seen below that (61) may be valid over a considerable region even if neither of these conditions is fulfilled. The reason for this is that the centrifugal potential *outside* a acts as a barrier and assures a long lifetime for the compound nucleus even if there is no proper reson-

ance inside a. In this case, the lifetime of the compound nucleus is essentially given by this barrier and the width of the level will be of the order $2akE_0$. If either of the above conditions for a real resonance is satisfied, the lifetime of the compound nucleus will be considerably increased and the width of the level a good deal smaller than $2akE_0$—as is seen from (63).

As a last point, we discuss the validity of (61) and (61a), i.e., try to take the γ_{12} of (58) into account. The definition of γ_{12} is the same as in the preceding section (48a) and the calculation will be similar also. It hinges on the calculation of γ_1 and γ_2 which are defined by (47) and estimated in (51). In the latter formula, we shall take into account only the first term of the bracket, involving the discontinuity δ_1 of the value of the wave function at a. This is

$$\delta_1 e^{ika} = \alpha_1 \Psi(a, i) - \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \frac{\cos\vartheta}{u_1} \times \left[\frac{i}{a}(1+U_1) + \frac{1}{ka^2}(1-U_1)\right]\psi(i). \quad (65)$$

In this, $\cos\vartheta$ has been written for x/r and the α_1 and U_1 are the barred quantities, given by (61), (61a). It follows that the exponentials occurring in (57) give a common factor just as in (52). This factor was brought to the left side of (65). The variables of Ψ were replaced by i, the internal coordinates of the reacting particles, and by the distance between these which is a where the discontinuity occurs.

Since we assume that there is no discontinuity in the wave function at $E_1 = E_0$, (65) must vanish at that point. This condition determines $\Psi(a, i)$ and greatly simplifies the expression for

$$\delta_1 e^{ika} = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \frac{2i\cos\vartheta}{u_1 a} \frac{E_1}{(E_0' - E_1)(1 - iv_1 S_1)}. \quad (65a)$$

We can now estimate $(1-iv_1S_1)\gamma_{12}(1+iv_2S_2)$ easily to be

$$|(1-iv_1S_1)\gamma_{12}(1+iv_2S_2)| = \frac{4\pi}{3}a^3 |(1-iv_1S_1)\delta_1\delta_2^*(1+iv_2S_2)| \sim \frac{4a}{u_1u_2}\frac{E_1E_2}{(E_0'-E_0)(E_0'-E_2)}. \quad (66)$$

However, unless we have a real resonance as discussed above, this estimate is less safe than the corresponding estimate given in the preceding section because there is reason to believe that the error at the surface of the sphere becomes greatly magnified inside the sphere. This is made plausible already by the form of the asymptotic form of the wave function (57) which increases as r^{-2} toward the inside of the sphere.

The above estimate can be substituted into (60a). At the same time $S_1+\epsilon_1$ and $S_2+\epsilon_2$ will be written for S_1 and S_2, respectively, where S_1 and S_2 are the values of S given by (61), and ϵ_1 and ϵ_2 indicate the amount by which this expression is in error. Neglecting then terms in which the product of ϵ_1 and ϵ_2 occurs, (60a) gives

$$\frac{E_1-E_0'}{E_2-E_0'}E_2\epsilon_1 - \frac{E_2-E_0'}{E_1-E_0'}E_1\epsilon_2 \sim \frac{Ma}{\hbar}\frac{(E_1-E_2)E_1E_2}{(E_1-E_0')(E_2-E_0')}. \quad (66a)$$

Since for $E=E_0$ the equations are accurately satisfied, ϵ_2 will vanish for $E_2=E_0$. Hence

$$\epsilon_1 \sim \frac{Ma}{\hbar}\frac{(E_1-E_0)E_1}{(E_1-E_0')^2}. \quad (67)$$

A comparison of (67) and (61) shows that (61) is certainly grossly inaccurate if E_1 is as close or even closer to E_0' as it is to E_0. It also appears that (61) is inaccurate if E_1 is in the neighborhood of F. This is, however, only apparent and a similar phenomenon to that observed after (54): it only shows that the position of zero scattering may be slightly displaced from the position indicated in (61). However, the first limitation is real and shows that (61) is not valid in the neighborhood of the second resonance and, in fact, becomes invalid at best halfway in between E_0 and E_0'.

On the other hand, for $E_1 \sim 0$ the ϵ_1 becomes so small that the cross section given in (62) remains valid unless F is very close to E_0. This shows that Eqs. (61) have a validity quite comparable to that of the similar equations for zero relative angular momentum of the colliding particles.

APPENDIX

In order to prove Eqs. (39) of the text it is necessary to prove first that every expression $\mathfrak{B}\mathfrak{C}\mathfrak{B}$, where $\mathfrak{C}$ is an arbitrary matrix, is a multiple of $\mathfrak{B}$. This is a consequence of (36a), or, as the mathematicians express it, that $\mathfrak{B}$ is an idempotent of rank 1. We have, in fact

$$(\mathfrak{B}\mathfrak{C}\mathfrak{B})_{kl}=\sum_{jm}\beta_k\beta_j C_{jm}\beta_m\beta_l=(\sum_{jm}\beta_j C_{jm}\beta_m)B_{kl}. \tag{A1}$$

We now can go over to verifying (39). Equation (28) and (32b) give for $\mathfrak{U}$

$$\begin{aligned}\mathfrak{U}&=\frac{1+i\mathfrak{u}\mathfrak{S}\mathfrak{u}}{1-i\mathfrak{u}\mathfrak{S}\mathfrak{u}}=(1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}-ic\mathfrak{u}\mathfrak{B}\mathfrak{u})(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}+ic\mathfrak{u}\mathfrak{B}\mathfrak{u})^{-1}\\&=[1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}-ic\mathfrak{u}\mathfrak{B}\mathfrak{u}][(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})\\&\quad\times(1+ic(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u})]^{-1}\\&=[1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}-ic\mathfrak{u}\mathfrak{B}\mathfrak{u}][1+ic(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u}]^{-1}\\&\quad\times(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}.\end{aligned} \tag{A2}$$

For the first factor of this we shall try to write

$$\begin{aligned}1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}-ic\mathfrak{u}\mathfrak{B}\mathfrak{u}=&[1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}-\mu(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u}]\\&\times[1+ic(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u}].\end{aligned} \tag{A3}$$

If this proves to be possible, the product of the first two factors of (A2) will be equal to the first factor on the right side of (A3). We can substitute, by (A1), for the product of the last two terms on the right side of (A3) $-ic\mu\gamma(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u}$ with

$$\gamma=\sum_{jm}\beta_j(\mathfrak{u}(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u})_{jm}\beta_m. \tag{A4}$$

The right side of (A3) then gives

$$\begin{aligned}1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}+(-\mu+2ic-ic\mu\gamma)&(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u}\\&+ic(-1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u}.\end{aligned}$$

This is equal to the left side of (A3) if

$$-\mu+2ic-ic\mu\gamma=0 \tag{A5}$$

or, with (35a) and $c_\infty=0$

$$\mu=\frac{2ic}{1+ic\gamma}=\frac{i\hbar b^2}{E-E_0+\frac{1}{2}i\hbar b^2\gamma}. \tag{A6}$$

With this μ, (A3) becomes an identity. Equation (A2) therefore gives

$$\begin{aligned}\mathfrak{U}&=(1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}-\mu(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u})(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\\\mathfrak{U}&=\frac{1+i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}}{1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u}}-\mu(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}\mathfrak{u}\mathfrak{B}\mathfrak{u}(1-i\mathfrak{u}\mathfrak{S}_\infty\mathfrak{u})^{-1}.\end{aligned} \tag{A7}$$

This is indeed equivalent to (39). The λ of (39a) is $\hbar b^2\gamma$.

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The Temperature Dependence of Secondary Electron Emission from Oxide-Coated Cathodes*

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The secondary electron emission from alkaline-earth oxide-coated cathodes has been investigated under both continuous and pulsed bombardment. Experiments have been performed with three types of apparatus. Yield *vs.* energy data reveal values of δ of 4–7 at room temperature, with a more or less flat maximum at approximately 1000 volts primary energy. The yield increases with temperature in an exponential manner, and plots of log $\Delta\delta$ (i.e., $\delta_{K^\circ}-\delta_{300^\circ K}$) *vs.* $1/T$ give straight lines. Values of Q_1 between 0.9–1.5 ev are generally indicated, and from extrapolation of these curves, yields exceeding 100 at 850°C are deduced. The secondary emission depends upon the degree of activation, and increases with enhancement of the thermionic emission characteristics. Short time effects such as growth or decay of secondary current after the onset of primary bombardment or persistence after the cessation of bombardment have not been observed, and values of yield obtained by pulsed methods are in accord with those obtained under d.c. conditions. Tail phenomena reported by J. B. Johnson and interpreted as "enhanced thermionic emission" from oxide-coated cathodes become manifest only under experimental conditions characterized by certain space-charge effects, and have been effectively simulated by bombarding a tantalum target adjacent to an electron-emitting tungsten filament. Various measurements of the energy distribution of secondary electrons as a function of primary voltage and temperature have been obtained. It was observed that the average energy of the secondary electrons decreases with temperature at a rate which more than compensates for the increase in the number of secondaries emitted per incident primary. The mechanism of the observed dependence of yield upon temperature is not well understood. Various alternative explanations are discussed and, in the light of the present state of our knowledge, regarded as untenable.

I. INTRODUCTION

MEASUREMENTS of the secondary electron emission from BaSr oxide-coated cathodes at room temperature have been reported by several investigators.[1] The first suc-

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[1] K. Sixtus, Ann. d. Physik **3**, 1017 (1929); M. Zeigler,