THE DIFFRACTION RING PATTERN IN THE SHADOW OF A CIRCULAR OBJECT.

By Mason E. Hufford.

DURING a series of experiments[1] in producing diffraction patterns by use of circular openings and steel spheres, it was observed that the concentric ring system produced by interference in the shadow of a disc or sphere was clearly brought out.

Lommel[2] has deduced formulas for calculating the radii of these rings and has compared the calculated and measured values for a few rings in microscopic diffraction patterns.

Since the number of rings here shown is much larger than that mentioned or shown by others,[3] an opportunity is offered to compute the radii by Lommel's formulas for many more rings and to compare computed and measured values over a much wider range.

In Lommel's expression,

$$M^2 = C^2 + S^2,$$

which represents the intensity of light at points in the geometric shadow, the integrals C and S involve quantities,

$$y = \frac{2\pi(a + b)r^2}{\lambda ab} \quad \text{and} \quad z = \frac{2\pi rR}{\lambda b},$$

in which a is the distance of a point source of monochromatic light of wave-length λ from the diffracting object of radius r, b is the distance of the object from the screen where the shadow is observed, and R is the distance from the center of the geometric shadow to the point in the shadow where the intensity of light is observed.

In order to determine the presence of points of maximum and minimum intensity of light in the shadow, the intensity M^2 is differentiated in the usual way with respect to a factor z. The result shows that maxima and minima of intensity will occur when,

$$V_0 = 0$$

where,

$$V_0 = J_0(z) - \left(\frac{z}{y}\right)^2 J_2(z) + \left(\frac{z}{y}\right)^4 J_4(z) - \cdots$$

[1] Phys. Rev., N. S., Vol. 3, No. 4, April, 1914.

[2] Lommel, Abhandlungen der Bayerische Academie der Wissenschaften, B. 15, 1886.

[3] Croft, Phil. Mag., S. 5, V. 38, 1894. Arkadiew, Phys. Zeit, 14, 1913.

and J_0 is the Bessel function. If y is large V_0 becomes equal to J_0 and the values of z for $J_0 = 0$ are approximately the values of z for $V_0 = 0$.

The values of y and z in $V_0 = 0$ plotted give a series of curves as shown in Fig. 1. The dotted lines are the graphs of $J_0 = 0$. The line $y = z$ marks the boundary of the geometrical shadow. All parts of the curves above this line apply to the shadow region. The parts of the curves which give minima are ruled heavy while the parts giving maxima are ruled light.

The possibility of using the values of z for which $J_0 = 0$ instead of those for which $V_0 = 0$ suggests itself.

The first ten of the values given in the following table of root values of the Bessel function were taken from tables. Beyond the tenth the values were calculated by McMahon's formula.[1] They are here given to the twenty-fifth value.

Z	Value of z by Formula.	Z	Value of z by Formula.	Z	Value of z by Formula.
Z_1	2.4048255577	Z_{10}	30.6346064684	Z_{19}	58.90712206
Z_2	5.5200781103	Z_{11}	33.7758991042	Z_{20}	62.04861462
Z_3	8.6537279129	Z_{12}	36.9171861194	Z_{21}	65.19011752
Z_4	11.7915344391	Z_{13}	40.05852068	Z_{22}	68.33162936
Z_5	14.9309177086	Z_{14}	43.19989382	Z_{23}	71.47314895
Z_6	18.0710639679	Z_{15}	46.34129753	Z_{24}	74.61450030
Z_7	21.2116366299	Z_{16}	49.49272626	Z_{25}	77.75602563
Z_8	24.3524715308	Z_{17}	52.63417501		
Z_9	27.4934791320	Z_{18}	55.76564161		

It is clear that if the value of y were infinite the values of z in $V_0 = 0$ would coincide with the values of z in $J_0 = 0$. Since the value of y is finite and rather small, the values of z corresponding to the value of y as they are taken from the graphs will differ more and more from the values of z in $J_0 = 0$ as they are taken from larger and larger graphs. It is clear that for the largest ring in any diffraction pattern, the value of z in $J_0 = 0$ will lack most of being the value of z for $V_0 = 0$. It is sufficient therefore to determine whether the value of z in $J_0 = 0$ may be substituted for the value of z in $V_0 = 0$ for the largest ring in any pattern.

This may be done if the error introduced is not greater than the experimental error.

To show how V_0 depends upon z we have:

$$\frac{dV_0}{dz} = -J_1(z)\left(\frac{z}{y}\right)^2 + J_3(z)\left(\frac{z}{y}\right)^4 - J_1(z)\left(\frac{z}{y}\right)^2 + \cdots,$$

[1] Annals of Mathematics, V. 9, 1894–5, p. 25.

since $J_0 = 0$ and,

$$J_n' = J_{n-1} - \frac{n}{z} J_n.$$

The Bessel functions with higher indices are decreasing quantities and also within the geometric shadow the value of y is always greater than that of z. Therefore the fractions will decrease rapidly and the expression is converging.

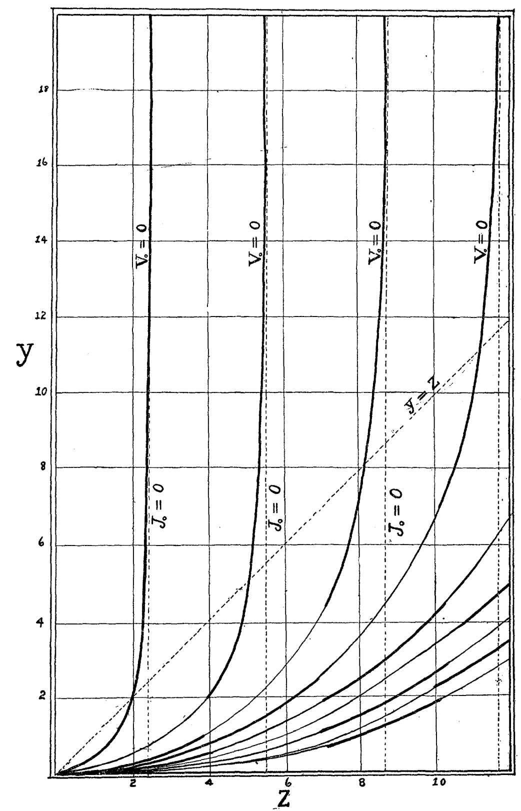


Fig. 1.

The accompanying photographs are the shadows of steel spheres, .317 cm., .6345 cm. and .9517 cm. in radius. These were obtained by apparatus similar to that described in a previous article.[1] The light from an arc was filtered through double thickness violet colored glass. From photographs of the spectrum of this filtered light it was observed that the most effective light which passed through the glass was that of wave-length, 3,880 A.U.

[1] See Phys. Rev., N. S., April, 1914, p. 243, Fig. 2.

By means of suitable lenses this light was focused on an opening .027 cm. in diameter. The light then traversed a box 3,255 cm. in length. At a distance of 1,550 cm. from the opening the spheres were suspended by means of very fine wires. The exposures were about six hours in length.

Substituting the values, $a = 1{,}550$, and $b = 1{,}705$ in the formula,

$$y = \frac{2\pi(a+b)r^2}{\lambda ab},$$

y for the smallest sphere is 20.02 and for the next larger one, $y = 80.23$.

The photographic negatives show that in the shadow of the small sphere, as shown in Photograph A, there are six rings representing minima of light intensity. In the shadow of the next larger sphere, as shown in Photograph D, there are about twenty-five rings representing minima of intensity. Photograph C is the shadow of the largest sphere used and Photographs B and E are enlargements.

The sixth and twenty-fifth roots of the Bessel function are, 18.0710639679 and 77.7560256302 respectively. These values apply to the sixth ring in the smallest pattern, Photograph A and to the twenty-fifth ring in the larger pattern, Photograph D. Whether these and smaller values may be used for calculating the radii of the rings depends upon how V_0 changes with respect to z in the immediate vicinity of these values.

If z is taken as 17 for the smallest sphere and as 75 for the next in size, then dV_0/dz for the former is .268 and for the latter it is, .2718. If the actual values of z could be found they would probably be larger than these and therefore the rates of variation would be less.

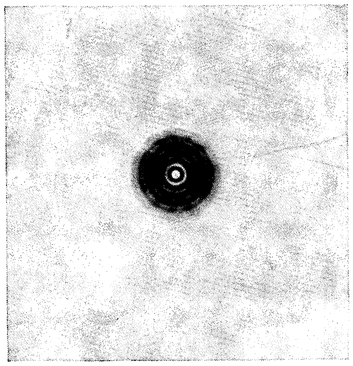
For calculating the first of these rates, the values of J_1, J_3, J_5, etc., were taken from tables in Gray and Matthew's Treatise on Bessel Functions. For calculating the second of the rates, the corresponding values of J_1, J_3, etc., were found by formulas given in Gray and Matthews, pp. 40 and 13.

These values are given below:

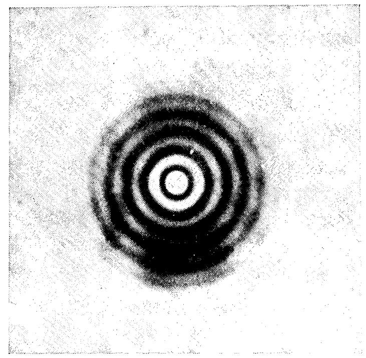
$J_1(75)$	-0.085137365	$J_5(75)$	-0.078166306	$J_9(75)$	-0.0540197166
$J_2(75)$	-0.03691648	$J_6(75)$	-0.0573188674	$J_{10}(75)$	-0.0834193863
$J_3(75)$	$+0.083168445$	$J_7(75)$	$+0.0689952873$	$J_{11}(75)$	$+0.0317745466$
$J_4(75)$	$+0.046896693$	$J_8(75)$	$+0.0701979877$		

Six terms of the expansion, dV_0/dz were used in calculating the rates of variation.

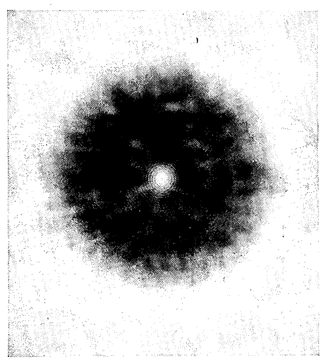
PHYSICAL REVIEW, Vol. VII., SECOND SERIES.
May, 1916.



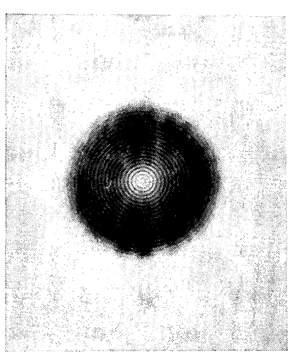
A.



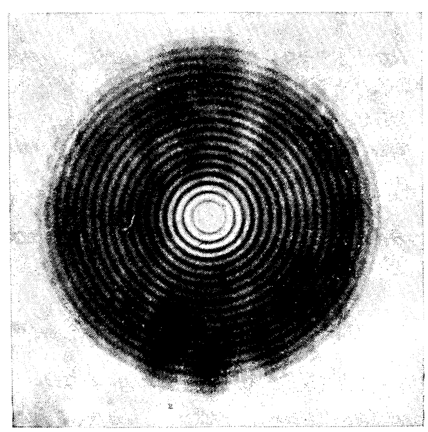
B.



C.



D.



E.

MASON E. HUFFORD.

The results of several trials showed that there was an error of about one per cent. in setting the cross hair of the measuring instrument on the transparent rings of the negatives.

Since the rather large rates of variation of V_0 with z apply only to the last rings of the patterns, it was thought sufficiently accurate to use the root values of z for calculating the radii of the diffraction rings.

The calculated and measured values are given below with the difference between them expressed as a per cent. of the calculated value.

$$a = 1{,}550 \text{ cm.} \qquad b = 1{,}705 \text{ cm.}$$

$$\lambda = 3{,}880 \text{ A.U.} \qquad D = .027 \text{ cm.}$$

Sphere No. 1. *Radius* .317 *cm.* $y = 20.02$.

No.	Calculated Cm.	Measured Cm.	Diff. in Per Cent.
1	.079	.080	+1.2
2	.183	.184	+0.5
3	.287	.287	+0.1
4	.391	.386	−1.4
5	.485	.483	−0.6
6	.600	.575	−4.0

Sphere No. 2. *Radius* .6345 *cm.* $y = 80.23$.

No.	Calculated Cm.	Measured Cm.	Diff. in Per Cent.	No.	Calculated Cm.	Measured Cm.	Diff. in Per Cent.
1	.039	.042	+7.0	13	.663	.652	−1.6
2	.091	.092	+1.5	14	.715	.704	−1.5
3	.143	.144	+0.5	15	.767	.755	−1.6
4	.195	.194	−0.3	16	.819	.802	−2.1
5	.247	.243	−1.7	17	.871	.858	−1.4
6	.299	.292	−2.4	18	.923	.910	−1.8
7	.351	.344	−2.0	19	.975	.958	−1.9
8	.403	.396	−1.8	20	1.027	1.013	−1.3
9	.455	.436	−4.2	21	1.079	1.062	−1.6
10	.507	.499	−1.8	22	1.132	1.113	−1.7
11	.559	.549	−1.7	23	1.184	1.162	−1.8
12	.611	.601	−1.6	24	1.236	1.208	−1.4

The above calculated values of the rings were found from the formula:

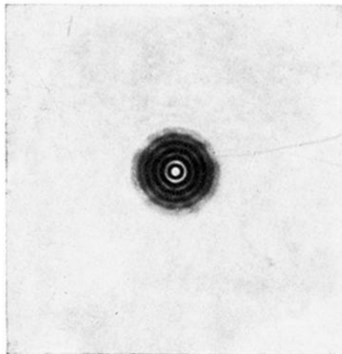
$$R = \frac{2\pi r}{\lambda b z}.$$

The measured values are the averages of four independent measurements of each radius. These were made by means of a comparator.

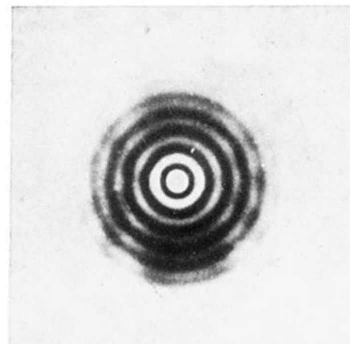
While it was possible to count twenty-five rings on the larger negative without the aid of a microscope it was found that there was so much lack of detail near the edge of the shadow that the cross hair could not be set on the outermost ring with any degree of accuracy. Twenty-four of the twenty-five rings were therefore measured.

In conclusion I wish to express my thanks to Prof. A. L. Foley, who has made many valuable suggestions throughout the progress of this work. I wish also to thank Prof. R. D. Carmichael, formerly of Indiana University, for valuable hints in the mathematical part of the work. I am indebted to Indiana University for placing the apparatus at my disposal and to the Chicago University Library for the loan of useful books.

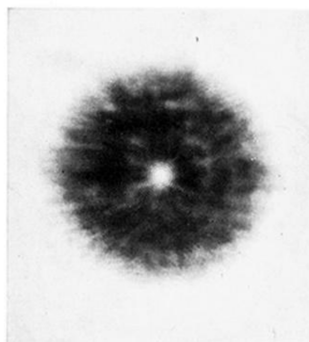
LABORATORY OF PHYSICS,
INDIANA UNIVERSITY,
December, 1915.



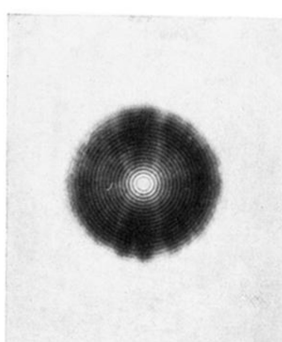
A.



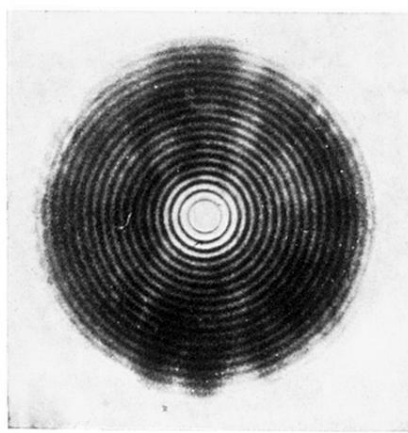
B.



C.



D.



E.