Redundant Zeros in the Discrete Energy Spectra in Heisenberg's Theory of Characteristic Matrix

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An important recent development in Heisenberg's theory of characteristic matrix is the determination of discrete energy values of closed stationary states.[1] This is made possible by the process of analytic continuation from real values of the momentum variables to imaginary values in the complex plane. Those imaginary values for which the amplitude of the outward scattered wave vanishes correspond to the closed stationary states and the corresponding negative energy values can be identified with the energy values of these states. It has been pointed out by Pauli that this procedure gives correct results for the scattering by a potential hole and the scattering by an attractive Coulomb field. Using this method, the writer has studied the case of an attractive exponential field, which presents a certain new feature.

We consider the spherically symmetrical state in the field

$$V(r) = -V_0 e^{-r/a},$$

where V_0 and a are positive constants. Let $u(r)/r$ be the wave function and E the energy of the scattered particle, which is positive. If we write

$$V_0 = \frac{\hbar^2}{2m} U_0,\ E = \frac{\hbar^2 k^2}{2m},$$

the wave equation takes the form

$$(d^2u/dr^2) + (k^2 + U_0 e^{-r/a})u = 0.$$

The solutions of this equation are

$$u = J\pm 2aki(2a(U_0 x)^{\frac{1}{2}})$$

where the J's are Bessel functions and $x = e^{-r/a}$. The condition that the function u should vanish at the origin requires it to be of the form

$$u = c[J - 2aki(2a(U_0)^{\frac{1}{2}})J_{2aki}(2a(U_0 x)^{\frac{1}{2}}) \\ - J_{2aki}(2a(U_0)^{\frac{1}{2}}J - 2aki(2a(U_0 x)^{\frac{1}{2}})],$$

where c is a constant. The asymptotic expression for large values of r is

$$u = c[J_{-2aki}(2a(U_0)^{\frac{1}{2}})(a(U_0)^{\frac{1}{2}})^{2aki}e^{-ik\gamma}/\Gamma(2aki+1) \\ - J_{2aki}(2a(U_0)^{\frac{1}{2}})(a(U_0)^{\frac{1}{2}})^{-2aki}e^{ik\gamma}/\Gamma(-2aki+1)].$$

From this asymptotic expression it can be seen that, as usual, there is no closed state when k is real. When k becomes imaginary, however, the above general consideration leads to the following conditions for closed states:

$$J2aki(2a\sqrt{U_0}) = 0, \qquad \text{(i)}$$
$$1/\Gamma(-2aki+1) = 0. \qquad \text{(ii)}$$

The first condition gives the discrete spectrum obtained by Bethe.[2] The second condition gives

$$-2aki + 1 = -n \qquad (n = 0, 1, 2, \cdots)$$

or

$$E = -\hbar^2(n+1)^2/8ma^2.$$

These eigenvalues do not correspond to any actual closed states. They may therefore be regarded as redundant zeros.

This special problem is of some interest in connection with the general formulation of the method for determining closed stationary states, as in Heisenberg's new theory there is no longer a wave equation to determine the wave function completely and consequently some other criterion is necessary for discarding such redundant zeros as are given by condition (ii).

1 Møller, Kgl. Danske Vid. Sels., Math.-Fys. Medd. 23, No. 1 (1945).
2 Bethe and Bacher, Rev. Mod. Phys. 8, 111 (1936).

Physical Interaction of Electrons with Liquid Dielectric Media. The Properties of Metal-Ammonia Solutions

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The author wishes to outline briefly the primitive theoretical considerations which preceded recent experimental discoveries relating to the properties of metal-ammonia solutions.[1] The basis of the treatment is the model, first proposed by Kraus,[2] of an electrolyte-like character for the solute metal in ammonia solutions, the negative ion constituent being the "solvated" electron. At extremely low concentrations, inter-ionic forces may be neglected. It is proposed that under these conditions individual electrons are "self-trapped" in physical cavities created in the solvent medium. If surface tension effects are neglected, the potential energy is due only to electrical polarization of the medium surrounding the cavity. The dielectric constant D of ammonia is sufficiently large ($\sim$22) so that without serious error the term $1/D$ may be neglected in comparison with unity in the well-known Born expression for the polarization energy. Hence as a first approximation, for a charge Ze symmetrically distributed in a spherical cavity of radius r_0, the potential energy

$$V = -Z^2e^2/2r_0,\ r < r_0;\ V = 0,\ r > r_0.$$

The attractive force, tending to shrink the cavity, is opposed by an especially simple quantum-effect repulsion. In essence, the magnitude of the de Broglie wave-length must correspond to the cavity diameter, leading to a zero-point kinetic energy. As a first approximation, the wave function is considered to have a node at the cavity boundary. The problem then reduces to that of a particle in a spherical box.[3] Solution of the Schrödinger equation yields for the lowest s state the kinetic energy $T = h^2/8mr_0^2$. Hence the total energy $W = T + V = (h^2/8mr_0^2) - (e^2/2r_0)$. Setting dW/dr_0 equal to zero, the equilibrium values of W and r_0 are found to be, respectively, $-(me^4/2h^2)$ (i.e., some -0.38 electron volt) and $h^2/2me^2$ (i.e., some 9.9×10^{-8} cm). The corresponding experimental values[1] are approximately -0.8 volt and 7×10^{-8} cm. Consideration of the finite "depth" of the potential well, with resultant penetration of the wave function outside the cavity (requiring a self-consistent field method) should improve the agreement