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The Magnetic Moments of Light Nuclei

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The sum of the magnetic moments of two nuclei which can be obtained from each other by interchanging neutrons and protons is shown to be related in a simple way to the probabilities of occurrence of each of the states of given spin and orbital angular momentum that can be combined to form the ground state. This result, which is independent of any nuclear model, has immediate application to the nuclei with equal numbers of neutrons and protons, in particular, to H^2, Li^6, B^{10}, and N^{14}. From the observed moments it is found that the ground state of B^{10} is a combination of about 50 percent ^3S and 50 percent ^3D function. The ground state of Li^6 contains at least 70 percent but no more than 90 percent of the ^3S term. It may contain as much as 30 percent of the ^3P term, or 15 percent of the ^1P term, or, finally, 10 percent of the ^3D term. The ground state of N^{14} contains at least 52 percent but no more than 84 percent of the ^3D function. It may contain as much as 48 percent ^1P function, but the ^3P term cannot contribute more than 24 percent or the ^3S more than 16 percent. All of these results are based on the assumption that the only terms contributing appreciably to the ground states of the nuclei are those found to be near the ground state in the Hartree approximation. It will be possible to obtain similar information concerning the ground states of other light nuclei if a technique for measuring the magnetic moments of radioactive nuclei is developed.

1. INTRODUCTION

THE magnetic moment constitutes a valuable source of information concerning the distribution of angular momentum between the orbital motion and spin in the ground state of a nucleus. In general, it is possible to guess from the observed moment which of the possible terms is predominant in the wave function of the ground state. However, the relative contribution of this term and the other possible terms usually cannot be obtained quantitatively because the relation between the magnetic moment and wave function involves cross products between the various terms. In addition the moment depends to some extent on the detailed functional properties of each term.

As has already been pointed out,[1] this difficulty can be overcome to a considerable extent by combining a measurement of the magnetic moment of one nucleus with a measurement of the magnetic moment of the nucleus which is obtained from the original one by replacing all of the protons by neutrons and all of the neutrons by protons. The latter nucleus will henceforth be referred to as the nucleus *conjugate* to the original nucleus. The sum of the moments of these two nuclei depends only on the relative probability of occurrence of each possible term in the ground state, as will be shown in the next section. This result depends on the assumption of equality of neutron-neutron and proton-proton forces and, in consequence, is not applicable to heavy nuclei for which the Coulomb interaction between protons becomes important.

One of the pair of conjugate nuclei will usually be radioactive, in which case application of the stated result will have to await development of a technique for the measurement of the magnetic moments of active nuclei. It can be hoped that such techniques will be forthcoming shortly.

There is one case in which the sums of the moments of conjugate nuclei have been obtained; that is, for the nuclei containing an equal number of neutrons and protons. These nuclei will be referred to as *self-conjugate*. For the self-conjugate nuclei, the sum of the moments is obviously just twice the moment of the nucleus. Magnetic moments have been measured for all of the stable self-conjugate nuclei (for the alpha-particle nuclei they are, of course, zero). The conclusions that can be drawn from them will be discussed in Section 3.

2. THE SUM OF THE MAGNETIC MOMENTS OF CONJUGATE NUCLEI

In the approximation that the Coulomb forces between protons can be neglected, that is, for light nuclei, it is generally assumed that the forces between neutrons and those between protons are the same. Then the only difference

* This work is in no way related to the author's work for the Ordnance Department.

[1] R. G. Sachs and J. Schwinger, Phys. Rev. 61, 732A (1942).

between each of a pair of conjugate nuclei is the number of neutrons and protons, so the role played by the neutrons in the one nucleus is the same as that played by the protons in the conjugate nucleus.[2] More explicitly, the wave function of the one nucleus can be obtained from that of the conjugate nucleus by identifying those variables in the one wave function which refer to the neutrons as the variables referring to protons in the other wave function, and by treating the proton variables in a similar manner.

Because of this symmetry between neutrons and protons, the sum of the magnetic moments of the pair of conjugate nuclei is the same as the sum of the moments of two hypothetical nuclei, one consisting entirely of protons and the other of neutrons.[3] The wave functions of these hypothetical nuclei have the same form as the wave functions of the original nuclei. The magnetic moment of a system consisting of one type of particle depends in a simple way on the total orbital angular momentum and the total spin, as is well known for the electronic magnetic moment of atoms. Thus the moments of each of the hypothetical nuclei can be expressed in terms of the relative probability of occurrence of each allowed combination of orbital and spin angular momentum. The sum of the moments of the two hypothetical nuclei can then be expressed in similar terms. Thus the same statement can be made for the sum of the moments of the original nuclei, since this is equal to the sum of the moments of the hypothetical nuclei.

In order to demonstrate this result quantitatively, we write the wave function of one nucleus as

$$\Psi_{JM}(r_1, r_2, \cdots r_n; r_{n+1}, \cdots r_{n+m}; s_1, \cdots s_n; s_{n+1}, \cdots s_{n+m}),$$

where $r_1 \cdots r_n$, $s_1 \cdots s_n$ refer to the space coordinates and the z components of the spins of the n neutrons, respectively, and the other r's and s's refer to the m protons. J is the total angular momentum of the nucleus and M is the projection of this angular momentum on the z axis. The wave function of the conjugate nucleus is the same but with the variables numbered from 1 to n now referring to protons and the other variables to neutrons. This wave function can be expressed as a sum of functions each of which refers to a state of given total orbital angular momentum, L, and given total spin, S:

$$\Psi_{JM} = \sum_{LS} c_{LS} \Phi_{JM}{}^{LS}. \tag{1}$$

The coefficients satisfy the normalization condition

$$\sum_{LS} |c_{LS}|^2 = 1. \tag{2}$$

The values of L and S which appear in these sums are those for which $L+S$, $L+S-1$, $L+S-2$, $\cdots$, or $|L-S|$ equals J.

The projection on the z axis of the magnetic moment of the nucleus consisting of n neutrons and m protons will be designated by g_1M and that of the conjugate nucleus by g_2M. If the g-factors in units of the nuclear magneton for neutron and proton are g_n and g_p, respectively, the magnetic moment in the same units of the first nucleus is

$$g_1M = \langle L^z{}_{n+1} + \cdots + L^z{}_{n+m} + g_n(s_1 + \cdots + s_n) + g_p(s_{n+1} + \cdots + s_{n+m})\rangle_{JM}, \tag{3a}$$

where $L^z{}_j$ is the projection on the z axis of the orbital angular momentum of the jth particle. The bracket $\langle\ \rangle_{JM}$ is meant to indicate the average over the wave function Ψ_{JM}. The magnetic moment of the conjugate nucleus is

$$g_2M = \langle L^z{}_1 + \cdots + L^z{}_n + g_p(s_1 + \cdots + s_n) + g_n(s_{n+1} + \cdots + s_{n+m})\rangle_{JM}, \tag{3b}$$

and the sum of the moments is

$$GM = g_1M + g_2M = \langle L^z + (g_n+g_p)S^z\rangle_{JM}, \tag{4}$$

where L^z and S^z are the projections on the z axis of the total orbital and total spin angular momentum, respectively. Since $L^z + S^z = J^z$, the z component of the total angular momentum, this expression can be rewritten as

$$GM = \tfrac{1}{2}(1+g_n+g_p)M + \tfrac{1}{2}(1-g_n-g_p)\langle L^z - S^z\rangle_{JM}. \tag{5}$$

The average values of L^z and S^z, and therefore the average of their difference, can be calculated

[2] The difference between the masses of neutron and proton is neglected.

[3] A similar statement can be made for any of the properties of the nuclei which is the sum of operators referring individually to each of the particles in the nucleus.

TABLE I. Observed g-factors, values of $g-g_0$, and values of $(g_0-g)/0.0945$ for odd-odd, self-conjugate nuclei.

Nucleus:	H^2	Li^6	B^{10}	N^{14}
g	0.856	0.821	0.598	0.403
$g-g_0$	−0.022	−0.057	−0.280	−0.475
$\frac{g_0-g}{0.0945}$	0.23	0.60	2.96	5.03

by means of the usual vector sum rule. This gives

$$\langle L^2-S^2\rangle_{JM}=\frac{\langle(\mathbf{L}-\mathbf{S})\cdot\mathbf{J}\rangle_{JM}}{J(J+1)}\cdot M$$
$$=\frac{\langle|\mathbf{L}|^2-|\mathbf{S}|^2\rangle_{JM}}{J(J+1)}\cdot M.$$

Inserting this value into Eq. (5), the formula for G reduces to

$$G=\tfrac{1}{2}(1+g_n+g_p)+\tfrac{1}{2}(1-g_n-g_p)\langle|\mathbf{L}|^2-|\mathbf{S}|^2\rangle_{JM}/J(J+1). \quad (6)$$

Thus it is seen that G is made up of two terms, one that is the same for all nuclei and another that is proportional to the average value of the difference of the squares of the orbital and spin angular momenta. The value of G then depends on the characteristics of the nuclear wave function only through the latter term. Since this term depends only on the total orbital and spin angular momenta, the average involved is determined by the relative probabilities of occurrence of states of given orbital angular momentum and spin in the ground state. These probabilities are given by $|c_{LS}|^2$, where the c_{LS} are defined by Eq. (1). In terms of the $|c_{LS}|^2$ the average under consideration is

$$\langle|\mathbf{L}|^2-|\mathbf{S}|^2\rangle_{JM}=\sum_{LS}|c_{LS}|^2[L(L+1)-S(S+1)].$$

Therefore Eq. (6) takes the form

$$G=\tfrac{1}{2}(1+g_n+g_p)+\tfrac{1}{2}(1-g_n-g_p)\times\sum_{LS}|c_{LS}|^2\frac{L(L+1)-S(S+1)}{J(J+1)}, \quad (7)$$

where the sum is over all states that can be combined to form a state of total angular momentum J. This expression shows how a measurement of the quantity G can be combined with the accepted values of g_n and g_p to obtain information concerning the probability coefficients $|c_{LS}|^2$. It should be kept in mind that these coefficients also satisfy the normalization condition Eq. (2), so one of them may be expressed in terms of the others if it is convenient.

In order to investigate the use of the relation Eq. (7) in detail we now turn to those nuclei for some of which the necessary data are available, namely the self-conjugate nuclei.

3. APPLICATION TO SELF-CONJUGATE NUCLEI

It has been pointed out in the Introduction that for self-conjugate nuclei, i.e., those with an equal number of neutrons and protons, the G-factor for the sum of the moments reduces to twice the g-factor for the single nucleus. Therefore the magnetic moments of these nuclei are expressible in terms of the probabilities $|c_{LS}|^2$. If g is the gyromagnetic ratio of a self-conjugate nucleus, we find from Eq. (7) that

$$g=\tfrac{1}{4}(1+g_n+g_p)+\tfrac{1}{4}(1-g_n-g_p)\times\sum_{LS}|c_{LS}|^2\frac{L(L+1)-S(S+1)}{J(J+1)}. \quad (8)$$

The result expressed by Eq. (8) is particularly useful because some of the odd-odd, self-conjugate nuclei, namely, H^2, Li^6, B^{10}, and N^{14}, are stable and their magnetic moments have been measured. Since the heavier odd-odd, self-conjugate nuclei are unstable, their moments have not been measured, but, of course, Eq. (8) applies equally well to them as long as the Coulomb energy can be neglected. The alpha-particle nuclei are also self-conjugate but, since their moments are all expected to vanish, no useful information can be obtained from them.

The observed g-factors for the nuclei under consideration are given[4] in Table I. The total angular momentum of each of these nuclei is $J=1$. On the basis of Russell-Saunders coupling, it has been surmised[5] that the ground state of each nucleus is a ^3S state. In this case we would have $g=g_0=\frac{1}{2}(g_n+g_p)=0.878$ for all of the nuclei if we take[4] $g_n=-3.822$ and $g_p=5.579$. The difference $g-g_0$ is given in the third row of Table I. From these differences it seems clear that

[4] These figures are taken from the summary of results given by S. Millman and P. Kusch, Phys. Rev. 60, 91 (1941).

[5] M. E. Rose and H. A. Bethe, Phys. Rev. 51, 205 (1937).

the deviation from Russell-Saunders coupling becomes increasingly important for the heavier nuclei. This fact has already been pointed out by Rose.[6]

According to Eqs. (8) and (2), the difference $g-g_0$ is

$$g-g_0=-0.0945\sum{}'_{LS}|c_{LS}|^2k_{LS}, \qquad (9)$$

with $k_{LS}=L(L+1)-S(S+1)+2$. The $\sum{}'_{LS}$ indicates the sum over all states except the ^3S state. The values of $(g_0-g)/0.0945=\sum{}'|c_{LS}|^2k_{LS}$ for each of the nuclei under consideration are given in the fourth row of Table I.

A large number of possible terms are included in the expansion of the wave function of the ground states for all of these nuclei except H^2. In order to make use of the results obtained thus far, it is desirable to indicate which of the terms are probably most important. In the Hartree approximation, it is found[7] that certain of the terms arise from one configuration and it seems likely that these would contribute most strongly to the wave function of the ground state. The resulting terms corresponding to unit total angular momentum are tabulated in Table II for each of the nuclei under consideration.* For the $^1P_1$, $^3P_1$, and $^3D_1$ terms the values of k_{LS} are 4, 2, and 6, respectively. Using the results given in the last row of Table I, we find the following linear relations between the $|c_{LS}|^2$ for the various nuclei:

$$H^2:\quad 6|c_{21}|^2=0.23, \qquad (10a)$$

$$Li^6:\quad 4|c_{10}|^2+2|c_{11}|^2+6|c_{21}|^2=0.60, \qquad (10b)$$

$$B^{10}:\quad 6|c_{21}|^2=2.96, \qquad (10c)$$

$$N^{14}:\quad 4|c_{10}|^2+2|c_{11}|^2+6|c_{21}|^2=5.03. \qquad (10d)$$

TABLE II. Low-lying terms with $J=1$ in the odd-odd, self-conjugate nuclei.

Nucleus:	H^2	Li^6	B^{10}	N^{14}
Terms:	$^3S_1$	$^3S_1$	$^3S_1$	$^3S_1$
	$^3D_1$	$^3D_1$	$^3D_1$	$^3D_1$
		$^1P_1$		$^1P_1$
		$^3P_1$		$^3P_1$

In addition the coefficients must satisfy Eq. (2) with the coefficient for the ^3S state, $|c_{01}|^2$, included in the sum.

Equation (10a) does not lead to a new result since it is on the basis of this relation, $|c_{21}|^2=0.038$, that the magnetic moment of the neutron was obtained.[4]

Equation (10c) gives, for B^{10}, $|c_{21}|^2=0.49$, so there appears to be a mixing of the ^3D state with the ^3S state to such an extent that the probability of each is about $\frac{1}{2}$. It should be borne in mind that this result is valid only if none of the other terms, such as the ^1P, ^3P, ^5P, ^5D, etc., contribute appreciably to the ground state, as is suggested by the Hartree model.

Rose[6] assumed that the ^1P and ^3P states would not make an important contribution to the ground states of Li^6 or N^{14}, but there seems to be no *a priori* reason for making such an assumption. It is true that, if the tensor spin-orbit interaction is responsible for the mixing of terms, only the D state would occur in first order if the S state were dominant. However, it turns out that in N^{14}, for example, the D state occurs with a much higher probability than the S state. It follows that the P state may also occur with a high probability since the tensor force will mix P and D states.

On this basis we assume that all three of the coefficients in Eqs. (10b) and (10d) are to be treated on an equal footing. Considering first Eq. (10b), it is seen that the largest possible values of each of the coefficients is obtained by setting the other two equal to zero. Thus we find for Li^6

$$|c_{10}|^2\leq 0.15, \quad |c_{11}|^2\leq 0.30, \quad |c_{21}|^2\leq 0.10. \qquad (11)$$

Since Eq. (10b) can also be written in the form

$$|c_{10}|^2+|c_{11}|^2+|c_{21}|^2=0.10+\tfrac{1}{3}|c_{10}|^2+\tfrac{2}{3}|c_{11}|^2,$$

it follows that the sum of the squares of the three coefficients must be greater than or equal to 0.10 or, in virtue of Eq. (2), the contribution of the ^3S state, $|c_{01}|^2$, cannot be greater than 90 percent. Equation (10b) can also be written in the form

$$|c_{10}|^2+|c_{11}|^2+|c_{21}|^2=0.30-|c_{10}|^2-2|c_{21}|^2,$$

which shows in a similar way that the contribution of the ^3S state cannot be less than 70 percent. Therefore we see that the ground state of Li^6 is predominately a ^3S state, with a mixture

[6] M. E. Rose, Phys. Rev. **56**, 1064 (1939).

[7] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937). The terms for H^2 are taken from W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941).

* For Li^6 and N^{14} all terms that occur in the Hartree approximation are given in the table, but out of the many possible terms for B^{10} only the two lowest are given in order to avoid complicating the argument.

of at least 10 percent and at most 30 percent of the other three states. The individual contributions of the other three states are limited by the conditions Eq. (11).

For N^{14}, Eq. (10d) shows that there must be an appreciable contribution from the ^3D state. As a consequence it is convenient to rewrite the equation with the coefficient $|c_{21}|^2$ eliminated in favor of the ^3S coefficient, $|c_{01}|^2$, by means of the normalization condition Eq. (2). In this way Eq. (10d) becomes

$$2|c_{10}|^2+4|c_{11}|^2+6|c_{01}|^2=0.97. \quad (12)$$

The largest possible value of each of the coefficients is obtained, as for Li^6, by setting the other two equal to zero:

$$|c_{10}|^2\leq 0.48, \quad |c_{11}|^2\leq 0.24, \quad |c_{01}|^2\leq 0.16. \quad (13)$$

Also as for Li^6, it is easy to show that the sum of the squares of these three coefficients can be no less than 0.16 and no greater than 0.48. Therefore, it follows that the contribution of the ^3D state must be at least 52 percent and may be as large as 84 percent. Thus the ground state of this nucleus is either predominantly a ^3D state or it is an almost equal mixture of ^3D and ^1P states. The contribution of the ^3S state cannot be greater than 16 percent.

Again it must be emphasized that this discussion is subject to the condition that the only states which contribute to the ground state are those found to be near the ground state on the basis of the Hartree model.

4. CONCLUSION

In this discussion no attempt has been made to predict theoretically what the magnetic moments of nuclei *should* be on the basis of a particular nuclear model. Rather, an attempt has been made to show how the observed moments can be used to give information about the structure of the wave functions of the ground states. In the present state of our experimental knowledge of magnetic moments, the results obtained here are particularly applicable to the odd-odd nuclei consisting of equal numbers of protons and neutrons. The conclusions that can be drawn concerning the nuclei of this type which are stable are given in Table III. When a technique is developed for the measurement of the moments of radioactive nuclei, it will be possible to obtain similar information concerning the structure of the ground states of those light nuclei for which the moments of the conjugate pair are observed.

TABLE III. Percentage contributions of Russell-Saunders terms to the ground states of the stable, odd-odd, self-conjugate nuclei as found from the observed gyromagnetic ratios. (The values for H^2 are included for completeness.)

Term:	^3S	^3D	^1P	^3P
Nucleus				
H^2	(96%)	(4%)	—	—
Li^6	70–90%	0–10%	0–15%	0–30%
B^{10}	51%	49%	—	—
N^{14}	0–16%	52–84%	0–48%	0–24%

It should be remarked that the additivity of the magnetic moments of the neutron and proton has been assumed here. Since current ideas concerning nuclear forces make it seem quite possible that these moments are altered by the interaction with other particles in the nucleus, it is conceivable that the assumption of additivity is not valid. In this case marked deviations from the results obtained here would be expected.

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