# Relativistic Correction in Calculating the Magnetic Moment of the Deuteron

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This paper presents a study of the variation in the magnetic moment of protons and neutrons caused by their kinetic energy; it assumes that these particles satisfy the relativistic Dirac-Pauli equation, which is characteristic of a particle with spin $\frac{1}{2}$, and that the particles possess a supplementary magnetic moment (not depending on charge). An hypothesis, formulated by Margenau about the variation law for magnetic moments, and the effect of the relativistic correction in calculating the deuteron magnetic moment are discussed.

IT is important in the study of nuclear forces to know the magnetic moments of elementary particles which compose atomic nuclei. One of the most interesting problems is to establish exactly how the magnetic moments of proton and neutron determine the magnetic moment of the deuteron. In this connection it is interesting —as Margenau[1] has first pointed out—to consider the fact that the field of force in a deuteron impresses a movement upon proton and neutron, whose magnetic moments may be changed by this movement. This change appears as a dependence of the magnetic moments on the kinetic energy of the particle, and it is calculated by Margenau assuming that the field of force in the deuteron be a central one and that the neutron and the supplementary proton magnetic moment (that is the part of the magnetic moment not contained in Dirac's equation) change, on account of the field of force, in accordance with the law characteristic of the magnetic moment of a particle obeying Dirac's equation. Jt is also possible, and results in a more coherent method, to take account of such a supplementary magnetic moment by supposing that proton and neutron satisfy, not the ordinary Dirac equation, but the equation corrected by Pauli.[2] Using this corrected equation, which we will call the Dirac-Pauli equation, and without prejudicing the relativistic invariance, a magnetic moment is introduced which is not necessarily dependent on the particle charge. We have solved this equation for the case of a particle moving in a central field of force and subjected to a constant magnetic field $H$, a case closely related to the deuteron problem.

Use is made of the assumption that every particle having spin $\frac{1}{2}$ and rest mass $M_0$ satisfies the relativistic Dirac equation:

$$\left[\boldsymbol{\alpha}\cdot\left(\mathbf{p}-\frac{\epsilon}{c}\mathbf{A}\right)+\beta M_0 c\right]\Psi=-\left(\frac{\hbar}{i}\frac{1}{c}\frac{\partial}{\partial t}+\frac{1}{c}V\right)\Psi, \quad (1)$$

where $\boldsymbol{\alpha}$ is the operator for $v/c$, $\mathbf{p}$ the operator with components $(\hbar/i)(\partial/\partial x_i)$, $\mathbf{A}$ the vector potential, and $V(x_1, x_2, x_3)$ the potential energy of the static forces acting on the particle.

Pauli has shown that Dirac's equation may be modified without destroying its relativistic invariance—by adding to the operator that appears in the first member, the term

$$\lambda \sum_{(\mu<\nu)} \sigma^{\mu\nu}F^{\mu\nu} \quad (\mu, \nu=1, 2, 3, 4),$$

where $\lambda$ is a constant factor, $F^{\mu\nu}$ the antisymmetrical tensor of the electromagnetic field, and $\sigma^{\mu\nu}$ the operators defined by means of the relations:

$$\sigma^{mn}=i\beta\alpha^m\alpha^n, \quad \sigma^{4n}=i\beta\alpha^n, \quad \sigma^{\mu\mu}=0,$$

$$(m, n=1, 2, 3).$$

In our case, in which only a magnetic field is present, the additional Pauli term becomes:

$$i\lambda\beta \sum_{(m\neq n)} \alpha^m\alpha^n F^{mn}=\frac{\lambda}{c}(\mathbf{S}\times\mathbf{H}),$$

where $\mathbf{S}$ is the vector operator having components:

[1] H. Margenau, Phys. Rev. **57**, 383 (1940).
[2] W. Pauli, *Handbuch der Physik*, first series, Vol. 24, p. 232.

$$S_1=\begin{vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{vmatrix},\quad S_2=\begin{vmatrix}0&i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&i&0\end{vmatrix},$$

$$S_3=\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{vmatrix}.$$

Therefore the Dirac-Pauli equation, which a particle with spin $\frac{1}{2}$ obeys when it is in a field of force derivable from the scalar potential function $V$ and subjected to a magnetic field $\mathbf{H}$, must be written:

$$\left[\boldsymbol{\alpha}\cdot\left(\mathbf{p}-\frac{\epsilon}{c}\mathbf{A}\right)+\beta M_0c+\frac{\lambda}{c}\mathbf{S}\cdot\mathbf{H}\right]\Psi=-\frac{1}{c}\left(\frac{h}{i}\frac{\partial}{\partial t}+V\right)\Psi. \quad (2)$$

The particle obeying such an equation has, as one easily verifies, a magnetic moment $\lambda$ not directly associated with its charge; we will assume Eq. (2) as a characteristic equation describing proton and neutron ($\epsilon=0$). Since in our problem the magnetic field is constant and uniform and all forces acting on the particle are constant with time, Eq. (2) admits a solution of the type:

$$\Psi=u\exp\left(-\frac{i}{h}Wt\right),$$

where $u$ satisfies the equation

$$\left[\frac{-W+V}{c}+\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M_0c+\Lambda\right]u=0, \quad (3)$$

provided

$$\Lambda=\frac{\epsilon}{c}\boldsymbol{\alpha}\cdot\mathbf{A}+\frac{\lambda}{c}\mathbf{S}\cdot\mathbf{H}=\frac{\epsilon H_0}{c}(x\alpha_2-y\alpha_1)-\frac{\lambda}{c}S_3H_0. \quad (4)$$

We will solve Eq. (3) by the perturbation method, assuming $\Lambda$ to be a perturbing term. For the unperturbed system we have the ordinary Dirac equation:

$$\left[\frac{-W^0+V}{c}+\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M_0c\right]u^0=0,$$

whose solutions are well known.

The action of the perturbing term yields a change of the eigenvalue given by:

$$\Delta W^0=-c\iiint_\infty\sum_{k=1}^{4}u_k^{0*}\Lambda u_k^0d\tau$$
$$=-\frac{\epsilon H_0}{2}\iiint_\infty[-u_1^{0*}i(x+iy)u_4^0+u_2^{0*}i(x-iy)u_3^0-u_3^{0*}i(x+iy)u_2^0+u_4^{0*}i(x-iy)u_1^0]d\tau$$
$$+\lambda H_0\iiint_\infty[u_1^{0*}u_1^0-u_2^{0*}u_2^0-u_3^{0*}u_3^0+u_4^{0*}u_4^0]d\tau,$$

where the integrations are easily carried out with the use of well-known relations for the spherical functions.

Then, assuming (see Margenau's work):

$$\kappa=\iiint_\infty F_+^2d\tau \quad (5)$$

and following a procedure quite similar to Margenau's, the result is ($j=l+\frac{1}{2}$):

$$\Delta W^0=\frac{2m-1}{2l+1}H_0\left\{(l+1)\mu_0-\frac{2(l+1)^2}{2l+3}x\mu_0+\lambda+\frac{\lambda}{2l+3}\kappa\right\}. \quad (6)$$

The magnetic moment is obtained from (6) by putting $m=l+1$ and dividing $\Delta W^0$ by $H$. Thus

$$\mu=(j+\tfrac{1}{2})\mu_0+\lambda-\left[\frac{(j+\frac{1}{2})^2}{j+1}\mu_0-\frac{1}{2(j+1)}\lambda\right]\kappa. \quad (7)$$

The expression for $\kappa$ may be easily evaluated for the limiting cases of high and low energy particles. Thus we have

*Newton-approximation:*

$$\mu=(j+\tfrac{1}{2})\mu_0+\lambda; \quad (8)$$

*approximation for $v\ll c$:*

$$\mu=(j+\tfrac{1}{2})\mu_0+\lambda-\left[\frac{(j+\frac{1}{2})^2}{j+1}\mu_0+\frac{1}{2(j+1)}\right]\bar{T}; \quad (9)$$

*extreme-relativistic approximation:*

$$\mu=\frac{1}{2(j+1)}[(j+\tfrac{1}{2})\mu_0+(2j+3)\lambda], \quad (10)$$

where $\bar{T}$ is the average kinetic energy, written in units $M_0c^2$:

$$\bar{T}=\int\int\int_{\infty}\frac{E-V+M_0c^2}{M_0c^2}(g^2+f^2)d\tau.$$

Quite analogously we may carry through the calculations for the case $j=l-\frac{1}{2}$. But the results can immediately be obtained from the preceding formulas on merely replacing $l$ by $-(l+1)$.

For the applications, the case of a particle in an $s$-state, for which $l=0$, is important. From (9), putting $j=0$, we have for $v\ll c$:

$$\mu=\mu_0+\lambda+\tfrac{1}{3}\bar{T}(-2\mu_0+\lambda).$$

Then the change in magnetic moment due to the relativistic correction is

$$\Delta\mu=\tfrac{2}{3}\bar{T}\mu_0+\tfrac{1}{3}\bar{T}\lambda.$$

Observing furthermore that the proton's rest magnetic moment is

$$\mu_p=\mu_0+\lambda_p \quad \text{with} \quad \lambda_p=1.789\mu_0,$$

while that of the neutron is:

$$\mu_n=\lambda_n \quad \text{with} \quad \lambda_n=-1.935\mu_0,$$

we have, respectively, the following expressions for the relativistic corrections:

$$\Delta\mu_p\simeq-0.072\bar{T}\mu_0, \quad \Delta\mu_n\simeq-0.645\bar{T}\mu_0.$$

These corrections are not proportional—in contrast with Margenau's assumption—to the value obtained for the rest magnetic moments of the two particles.

In further investigation of this question we may observe that, if we had used the following equation instead of (2),

$$\left[\boldsymbol{\alpha}\cdot\left(\mathbf{p}-\frac{\epsilon}{c}\mathbf{A}\right)+\beta M_0c+\frac{\lambda}{c}\boldsymbol{\sigma}\cdot\mathbf{H}\right]\Psi=-\frac{1}{c}\left(\frac{h}{i}\frac{\partial}{\partial t}+V\right)\Psi, \quad (11)$$

with:

$$\sigma_1=\begin{vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{vmatrix}, \quad \sigma_2=\begin{vmatrix}0&i&0&0\\-i&0&0&0\\0&0&0&i\\0&0&-i&0\end{vmatrix},$$

$$\sigma_3=\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{vmatrix},$$

we should have obtained the following results:

*Newton-approximation:*

$$\mu=(j+\tfrac{1}{2})\mu_0+\lambda,$$

*approximation for $v\ll c$:*

$$\mu=(j+\tfrac{1}{2})\mu_0+\lambda-\left[\frac{(j+\frac{1}{2})^2}{j+1}\mu_0+\frac{j+\frac{1}{2}}{j+1}\right]\bar{T},$$

*extreme-relativistic approximation:*

$$\mu=\frac{1}{2(j+1)}[(j+\tfrac{1}{2})\mu_0+\lambda].$$

These results agree, as one may easily see, with Margenau's assumption that the magnetic moment of the elementary nuclear particles depends on their energy in accordance with the same law which describes the magnetic moment characteristic of the Dirac equation and which considers the moment as strictly bound up with the particle's charge. However, Eq. (11) is not invariant with respect to Lorentz transformations, because the ordinary vector $\sigma$ that appears here is the spatial part of a 4-vector; it is not—as Lorentz invariance requires—a part of an antisymmetrical tensor of the second rank.

It is very interesting to discuss the deuteron magnetic moment in order to know the variation of the sum $\mu_p+\mu_n$. That is, after our hypothesis

$$\Delta(\mu_p+\mu_n)=\tfrac{1}{3}\bar{T}(\lambda_p+\lambda_n-2\mu_0)\simeq-0.72\bar{T}\mu_0,$$

while after Margenau

$$\Delta(\mu_p+\mu_n)=-\tfrac{2}{3}\bar{T}(\lambda_p+\lambda_n+\mu_0)\simeq-0.57\bar{T}\mu_0.$$

We will assume—following Margenau—that proton and neutron have, in the field of force acting in the deuteron (assuming the potential hole model), an average kinetic energy $\bar{T}\simeq0.012$. Then we have, respectively, on the two hypotheses

$$\Delta(\mu_p+\mu_n)\simeq-0.007\mu_0, \quad \Delta(\mu_p+\mu_n)\simeq-0.006\mu_0.$$

In conclusion, though the two hypotheses—Margenau's and our own—imply quite different laws for the variation in the magnetic moment of an elementary particle, yet they give about the same result in calculating the whole magnetic moment of the deuteron. Surely—as Margenau has remarked—the problem here considered will take on added interest when the neutron magnetic moment is known with greater accuracy.