Some Properties of Very Intense Shock Waves

R. G. SACHS
Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland
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Conditions have been obtained for the existence of a steady shock wave of such an intensity that radiation pressure plays a role in determining the properties of the shock. These conditions are completely analogous to the Rankine-Hugoniot equations for ordinary shocks; they are obtained by consideration of the conservation of mass, momentum, and energy. The results are applied to hydrogen and other very light gases. The application to other media requires a much more complicated discussion of the equation of state and specific heat under extremely high pressures and temperatures. In the light gases, the thickness of the shock front is extremely large because the radiation free path, which is determined by Compton scattering, is very large. The velocity of sound in a medium under very high pressures and temperatures is also discussed, and it is found that this velocity continues to increase with increasing pressure, a condition that is necessary for the shock to be stable.

1. INTRODUCTION

THE conditions for the existence of a steady shock wave in matter are well known. These conditions, which may be called the Rankine[1]-Hugoniot[2] equations, relate the pressure, density, and material velocity behind the shock to those in front of the shock for a given velocity of the shock front. They are simple consequences of the conditions for conservation of mass, momentum, and energy across the shock front, as well as of the equation of state of the medium through which the shock is moving.

The Rankine-Hugoniot equations have served to treat problems concerned with any shocks that can be produced in the laboratory. However, it may be of interest, for example in the study of stellar explosions, to consider shocks which are so intense that radiation pressure plays an important role in determining their properties. It is the purpose of this discussion to indicate the changes in the Rankine-Hugoniot relations that are necessary to take account of the radiation pressure, and to derive some of the consequences of these new relations.

Since the derivation of the Rankine-Hugoniot relations will serve as a guide for our purpose, it will be recalled in the next section. The radiation problem will then be treated in subsequent sections.

[1] W. J. M. Rankine, Trans. Roy. Soc. London, **A160**, 277 (1870).
[2] H. Hugoniot, J. Ecole Poly. **58** (1889).

2. THE RANKINE-HUGONIOT EQUATIONS

In the derivation of these equations, a shock wave is assumed to be represented by a discontinuity in pressure, density, and temperature which moves through the medium with a velocity, V. This discontinuity exists as such only on the macroscopic scale. On the molecular scale, the changes in pressure, etc., must necessarily take place over a distance of several mean free paths in order to maintain equilibrium in the transition. It has recently been shown by Thomas[3] that this condition is satisfied for quite intense shock waves in gases.

Instead of considering the medium to be initially at rest and the shock moving with speed V, it is more convenient to observe the phenomenon from a coordinate system in which the shock is at rest and the material is being fed into the shock front with the speed V. This condition

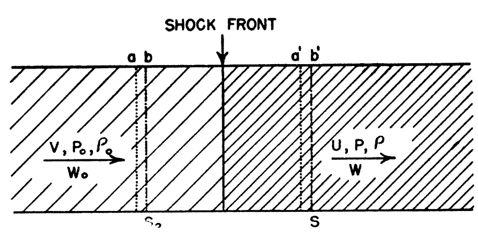

FIG. 1. Schematic representation of shock in coordinate system moving with the shock front. S_0 and S are planes fixed with respect to the shock front. The slab ab of material of thickness Vdt moves into the section S_0S in time dt, and the slab $a'b'$ of thickness Udt moves out of that section in the same time.

[3] L. H. Thomas, J. Chem. Phys. **12**, 449 (1944).

is indicated in Fig. 1. Material at pressure P_0 and density ρ_0 is being fed into the shock with velocity V (from left to right)—and material at pressure P and density ρ is leaving the front with a velocity $U(U<V)$.

Consider now the section S_0S of the material between the vertical dashed lines,[4] which are taken to be at rest in this coordinate system. Since the shock is assumed to be steady, the mass, momentum, and energy of the material in this section must be conserved during the infinitesimal time dt.[5] The mass that flows into S_0S in this time is the mass of the slab ab of material or $\rho_0 Vdt$, and the mass that flows out is the mass of the slab $a'b'$ or ρUdt, so conservation of mass requires that

$$\rho_0 V=\rho U. \tag{1}$$

Since the mass $\rho_0 Vdt$ has a velocity V, it imparts a momentum $\rho_0 V^2dt$ to the section. Similarly, a momentum $\rho U^2dt=\rho_0 VUdt$ leaves the section. Since the section is under a pressure P_0 on the left and P on the right, it also acquires a momentum toward the right of $(P_0-P)dt$, in this time. The condition that the momentum be conserved is, then:

$$\rho_0 V(V-U)=P-P_0. \tag{2}$$

Similarly, the slab ab carries in with it a kinetic energy $\frac{1}{2}\rho_0 V^3dt$, and the slab $a'b'$ carries out a kinetic energy $\frac{1}{2}\rho_0 VU^2dt$. In addition, the material that moves into the section in time dt must move a distance Vdt against a pressure P_0 while that moving out moves a distance Udt against a pressure P, so an amount of work $(P_0V-PU)dt$ is done on the section. Finally, the internal energy (E_0) of unit mass of the material at pressure P_0 and density ρ_0 is different from that (E) at pressure P and density ρ. Thus the section ab carries in an energy $\rho_0 VE_0dt$, and the section $a'b'$ carries out an energy $\rho_0 VEdt$. The condition that all of these changes in energy lead to no over-all change in the total energy of the section S_0S is:

$$\rho_0 V[\tfrac{1}{2}(V^2-U^2)+E_0-E]=PU-P_0V. \tag{3}$$

The Eqs. (1)–(3) are the Rankine-Hugoniot equations. Taken along with the equation of state, they provide the necessary relations for determining the physical conditions existing behind a shock front which is moving with a known velocity. For a perfect gas, the equation of state is

$$E=P/(\gamma-1)\rho, \tag{4}$$

where γ is the ratio of specific heats. Equation (4) may be used to eliminate E and E_0 from Eq. (3). Then the Eqs. (1)–(3) give three independent relations between the four unknown quantities P, ρ, U, V.

It is generally convenient to deal with the dimensionless quantities $p=P/P_0$, $x=\rho/\rho_0$, $u=U/V$, $v=V(\rho_0/P_0)^{\frac{1}{2}}$, and $\epsilon=E\rho_0/P_0$. In terms of these quantities, the Eqs. (1) to (4) become

$$ux=1, \tag{5}$$

$$v^2(1-u)=p-1, \tag{6}$$

$$\tfrac{1}{2}v^2(1-u^2)+\epsilon_0-\epsilon=pu-1, \tag{7}$$

$$\epsilon=p/(\gamma-1)x. \tag{8}$$

These equations, which now in virtue of (8) refer to a perfect gas, may easily be solved to give p, x, and u in terms of v. One consequence of them which we will use is that, as the pressure becomes infinitely great, the density ratio x approaches the finite value $(\gamma+1)/(\gamma-1)$.

3. THE STEADY SHOCK CONDITIONS FOR VERY INTENSE SHOCKS

If the shock pressure is very high, that is of the order of 10^5 atmospheres, the temperature is correspondingly high, so the energy density of radiation is comparable with the internal energy density of the material. Similarly, the radiation pressure is comparable with the material pressure. Under this condition, the Eqs. (5)–(8) can no longer be correct. It is necessary to introduce terms in them that will take account of the momentum and energy transferred by the radiation.

If the radiation energy density is W, the radiation pressure is

$$P_{\mathrm{rad}}=\tfrac{1}{3}W. \tag{9}$$

The radiation density and pressure at every point is determined by the temperature at that point

[4] If the shock front is not plane, the distances S_0S, ab, and $a'b'$ must be small compared to the radius of curvature.

[5] It is necessary that ab and $a'b'$ be large compared to a mean free path so dt must be large compared to the mean free path divided by V.

by the relation

$$W=a(kT)^4 \tag{10}$$

where $a=2.23\times10^{49}$ erg^{-3} cm^{-3}.

Turning now to Fig. 1, a radiation energy density W must be included on the right side of the shock. For generality we include an energy density W_0 on the left side (in front of the shock) although this quantity is negligibly small in the usual case of a shock advancing into a medium at nearly standard conditions.

In discussing ordinary shocks, it was pointed out that the changes represented as discontinuous in Fig. 1 must actually take place over a distance which is large compared to a molecular mean free path. An equivalent condition applies to the present case. The thickness of the shock front is large compared to some length λ' where λ' must obviously be at least as great as the mean free path, λ, for the scattering or absorption of radiation. However, λ' is not to be set equal to λ because the velocity of light, c, is so great compared to the shock velocity that an appreciable amount of radiation can be lost by diffusion even if the thickness of the shock front is large (but not large enough) compared to the mean free path. Thus for a shock front which is initially very sharp, the radiation will diffuse through the front and raise the temperature of the medium just ahead of the shock. The front is continually broadened by this process. The broadening eventually ceases to be important, however, because the time required for the diffusion of the radiation through a given distance is proportional to the square of the distance while the time required for the shock to negotiate the same distance is linear in the distance. Thus as the shock becomes thicker, a stage is reached in which only a negligible fraction of the radiation gets ahead of the shock.

A rough estimate of the thickness, λ', at which the broadening process virtually ceases is obtained by taking it to be that distance for which the average velocity of diffusion is equal to the velocity, V, of the shock. The average diffusion velocity over a distance λ' is roughly $c\lambda/\lambda'$. Setting this equal to V, the estimate

$$\lambda'=c\lambda/V$$

is obtained for the minimum thickness of a steady shock.

In the following considerations, it is assumed that the distance S_0S (Fig. 1) is large compared to λ'. On the other hand, the time interval dt which will be considered need only be so large that the thickness of each of the slabs ab and $a'b'$ in Fig. 1 is large compared to λ. The path length λ' does not enter in this case because each slab is cut from a region of uniform radiation density, and diffusion need not be considered in such a region.

The first of the conditions given in Section 2, that is, the conservation of mass, is not changed by the presence of radiation since shock waves so intense as to require a relativistic treatment will not be considered.[6] This condition is expressed in dimensionless form by Eq. (5).

The condition for conservation of momentum must be modified to include the effect of the radiation pressure. Since the thickness of the section S_0S of material is large compared to the radiation free path, the radiation pressure will transfer momentum to the section in the same degree as that transferred by the particle pressure. Thus Eq. (2) is to be changed by adding the corresponding radiation pressure to each of the particle pressure terms. Since the radiation pressure is $\frac{1}{3}W$, the equation to be substituted for Eq. (2) is:

$$\rho_0V(V-U)=P+\tfrac{1}{3}W-(P_0+\tfrac{1}{3}W_0). \tag{11}$$

In dimensionless form, this equation becomes:

$$v^2(1-u)=p+w-1-w_0, \tag{12}$$

where

$$w=\tfrac{1}{3}W/P_0. \tag{13}$$

Consideration of the conservation of energy leads to the result that Eq. (3) must be modified by the addition of a radiation pressure term and a radiation energy term, as would be expected. As a consequence of the condition that the thickness of the material ab or $a'b'$ flowing into or out of the section S_0S in time dt is large compared to the radiation free path, the slabs ab and $a'b'$ absorb the full radiation momentum P_{rad}. They thereby do an amount of work $\frac{1}{3}W_0Vdt$ and $-\frac{1}{3}WUdt$ on

[6] In the relativistic case, the conservation of mass would appear as part of the energy conservation equation, but an additional condition would be required for conservation of particles or for conservation of electric charge if pair production is taken into account.

the section S_0S. It follows that $\frac{1}{3}W$ and $\frac{1}{3}W_0$ are to be added to the corresponding particle pressures on the right side of Eq. (3).

In addition to this change in the pressure term, there is a corresponding change in the internal energy term in Eq. (3). It is to be remembered that that term was due to the internal energy carried into and out of the section S_0S by the material in ab and in $a'b'$, respectively. In order to obtain the corresponding contribution of the radiation, let us imagine that perfectly reflecting, double faced mirrors which move along with the material are placed at a, b, a', and b'. These have no effect on the radiation equilibrium, so between the mirrors the radiation density is W (or W_0).[7] The total radiation energy between the mirrors ab is carried into the section S_0S, and the radiation energy between $a'b'$ is carried out of the section in time dt. The consequent increase in the energy of the section is $W_0Vdt-WUdt$, so a term W_0V-WU must be added to the left side of Eq. (3). With both of the necessary changes, Eq. (3) is replaced by

$$\rho_0V[\tfrac{1}{2}(V^2-U^2)+E_0-E]+VW_0-UW \\ =(P+\tfrac{1}{3}W)U-(P_0+\tfrac{1}{3}W_0)V. \quad (14)$$

In dimensionless form, this is

$$\tfrac{1}{2}v^2(1-u^2)+\epsilon_0-\epsilon+4(w_0-uw)=pu-1. \quad (15)$$

Equations (5), (12), and (15) provide the necessary description of the shock phenomenon. Since these equations are useful primarily for very intense shock waves, some simplification of them can be secured by neglecting the quantities w_0 and 1 which are small compared with w, and the quantity ϵ_0 which is small compared to ϵ, if the shock wave is advancing into a medium at nearly standard conditions. The resulting equations are:

$$ux=1, \quad (16a)$$

$$v^2(1-u)=p+w, \quad (16b)$$

$$\tfrac{1}{2}v^2(1-u^2)-\epsilon=(p+4w)u. \quad (16c)$$

[7] If the mirrors (assumed to be penetrable by the material) move with a velocity different from the material, they disturb the radiation energy equilibrium because the reflected intensities on the two sides are different from the incident intensities and different from each other as a consequence of the Doppler effect. This point was called to the author's attention by Professor J. E. Mayer.

4. THE EQUATIONS OF STATE

The Eqs. (16) must be augmented by the equation of state for the material and the equation of state for the radiation. The latter is simply given by Eq. (10), or, in terms of the dimensionless w,

$$w=a(kT)^4/3P_0. \quad (17)$$

When the equation of state for the material is given, the temperature, T, in (17) can be expressed in terms of the material pressure and density. Then these conditions, along with Eqs. (16), are sufficient to determine all of the variables of the system as a function of one of them, which is the desired result. However, under the conditions to be considered, the equation of state of the material cannot always be expressed in simple form. The temperatures are so high that molecules and solids are dissociated into a gas of highly ionized ions and electrons. Many of the free electrons have sufficiently high thermal energy to be treated as a non-degenerate classical gas, but there are additional electrons in the excited ionic states which will make a somewhat complicated contribution to the specific heat, and therefore to the thermodynamic equation of state.

The problem is complicated further if we are dealing with a material which is initially liquid or solid because although the atomic structure is destroyed by the high temperature, the density is very high, so the material behaves more like a liquid with a very high vapor pressure than a gas. This liquid is highly ionized, so there are free electrons throughout. The higher atomic levels are split into bands because of the high density, and these bands may be only partially occupied, as if the liquid were a metallic conductor. The electrons in these partially occupied bands behave very much like free electrons.[8] These free and almost-free electrons contribute to the equation of state as if they constituted a perfect monatomic gas. The contribution to the equation of state made by the liquid consisting of atomic ions cannot be obtained without a detailed investigation of the material under consideration.

In order to avoid the difficulty of obtaining the

[8] The really free electrons have energies greater than the ionization potential or, speaking in terms of a metal, their energies are greater than the work function.

equation of state and yet still obtain a qualitative picture, the properties of a shock wave in a perfect monatomic gas containing Z free (non-degenerate) electrons per atom will be considered. The contribution of the bound electrons to the specific heat will be included by taking the value of γ, the ratio of specific heats, to be temperature dependent. It will introduce no serious complication to admit that Z is temperature dependent. Then these assumptions yield a reasonable approximation for a shock in a light gas. However, for the more interesting case of a shock in a dense medium, the approximations are poor and the results will indicate only the qualitative behavior of the shock.

The thermodynamic equation of state can be formally written as

$$E = P/(\gamma-1)\rho, \tag{18}$$

where now γ is assumed to be temperature dependent to take account of the contribution of bound electrons to the specific heat. In dimensionless form, this is

$$\epsilon = p/(\gamma-1)x. \tag{19}$$

The equation of state for a gas containing an effective number Z of free electrons per atom is given by

$$P/\rho = (Z+1)N_0kT/A, \tag{20}$$

where A is the atomic weight and N_0 the Avogadro number. It follows that

$$kT = Ap/\rho(Z+1)N_0. \tag{21}$$

Inserting this in Eq. (10) gives the relation between the radiation energy density and the particle pressure and density:

$$W = a[AP/\rho(Z+1)N_0]^4. \tag{22}$$

In terms of the dimensionless $w = \frac{1}{3}W/P_0$, this becomes

$$w = \alpha^4(p/x)^4, \tag{23}$$

with

$$\alpha = \left(\frac{a}{3P_0}\right)^{\frac{1}{4}} \frac{AP_0}{(Z+1)N_0\rho_0} = 5.23\times 10^{10}\frac{AP_0}{(Z+1)N_0\rho_0}, \tag{24}$$

where the numerical factor is in units of erg^{-1}. It is to be remembered that Z, and therefore α, may be temperature dependent.

Equations (19) and (23), when used in conjunction with Eqs. (16), completely determine the properties of an intense shock wave in a medium of the type under discussion. That is, the properties are determined in terms of the functions $\alpha(T)$ and $\gamma(T)$. It will be shown in the next section that the essential results can be obtained in such a way that the rather insensitive function $\alpha(T)$ can be inserted as a last step.

5. FORMAL SOLUTION OF THE EQUATIONS

The Eqs. (16), (19), and (23) can now be solved to give p, u, and w in terms of x. The relation between u and x is, according to Eq. (16a)

$$u = 1/x. \tag{25}$$

Substituting this in Eq. (16b):

$$v^2 = x(p+w)/(x-1). \tag{26}$$

Inserting Eq. (25), (26), and (19) in Eq. (16c) yields after straightforward algebraic operations:

$$p = (7-x)w/(x-X_0), \tag{27}$$

where

$$X_0 = (\gamma+1)/(\gamma-1). \tag{27a}$$

Using the expression Eq. (23) for w, the relation between p and x is found to be

$$p = \left(\frac{x}{\alpha}\right)^{4/3}\left(\frac{x-X_0}{7-x}\right)^{1/3}. \tag{28}$$

From Eq. (27) the expression for w is

$$w = \left(\frac{x}{\alpha}\cdot\frac{x-X_0}{7-x}\right)^{4/3}; \tag{29}$$

and from Eq. (26), that for v is

$$v = \left[\frac{(7-X_0)}{(7-x)}\frac{xp}{x-1}\right]^{\frac{1}{2}}, \tag{30}$$

where p can be obtained from (28).

It is to be remembered that the Rankine-Hugoniot equations lead to a limit for the density ratio of $X_0 = (\gamma+1)/(\gamma-1)$ for very high pressures. Now we find that, with radiation, the density ratio becomes greater than X_0 (where, it is to be remembered, the value of X_0 depends on the temperature) but reaches a limiting value of 7. It is seen that these equations do not apply for

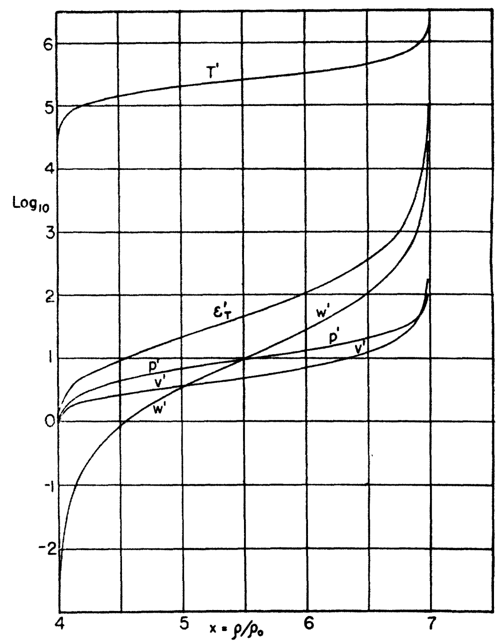

FIG. 2. Log curves of reduced pressure, velocity, etc., as a function of density ratio in shock. The scale for hydrogen is: p, w, $\epsilon_T = 9.2\times 10^3$ atmos., $T = 9.8$°K, $V = 2.7\times 10^4$ meters/sec.

$x \leq X_0$. To obtain the correct results for such values of x, it is necessary to retain the terms of the order of 1 which were neglected in arriving at Eqs. (16).

A quantity that is of interest is the total internal energy per unit volume, that is, the sum of the particle energy density and the radiation energy density. The energy per unit volume due to matter is $E\rho$, which, if measured in units of P_0, becomes $\epsilon x = p/(\gamma-1)$, according to Eq. (19). The radiation energy density in units of P_0 is $3w$, so the total energy density, ϵ_T, is given by

$$\epsilon_T = p/(\gamma-1)+3w. \tag{31}$$

This quantity is probably the best measure of the intensity of the shock wave, and, as such, will simply be called the intensity of the shock.

It will also be of interest to know the temperature behind the shock front. This can easily be obtained as a function of w from Eq. (17). The result is:

$$T = 1.41\times 10^5 w^{\frac{1}{4}}\ \text{°K}, \tag{32}$$

for $P_0 = 10^6$ ergs/cm^3 = 1 atmos.

It is convenient to express the equations obtained in this section in terms of the new dependent variables $p' = \alpha^{4/3}p$, $w' = \alpha^{4/3}w$, $\epsilon_T' = \alpha^{4/3}\epsilon_T$, $v' = \alpha^{\frac{1}{3}}v$, $T' = \alpha^{\frac{1}{3}}T$. The primed quantities are obtained from the unprimed by a change of scale, the scale factor in general depending on the temperature and therefore, on the independent variable x. In these terms, the equations become

$$p' = x^{4/3}\left(\frac{x-X_0}{7-x}\right)^{1/3}, \tag{33a}$$

$$w' = \left(x\frac{x-X_0}{7-x}\right)^{4/3}, \tag{33b}$$

$$\epsilon_T' = p'/(\gamma-1)+3w', \tag{33c}$$

$$v' = \left[\frac{7-X_0}{7-x}\frac{xp'}{x-1}\right]^{\frac{1}{2}}, \tag{33d}$$

$$T' = 1.4\times 10^5 w'^{\frac{1}{4}}. \tag{33e}$$

6. QUANTITATIVE RESULTS FOR HYDROGEN AND OTHER VERY LIGHT GASES

The Eqs. (33) could be solved to give the desired dependence of pressure, velocity, etc., on density if the functions $\alpha(T)$ and $X_0(T)$ were known. These functions depend in detail on the particular gas under consideration. The function $\alpha(T)$ can be treated to a fair approximation without great difficulty because it does not appear explicitly in Eqs. (33). $X_0(T)$, on the other hand, is an unpleasant function which will show rapid changes whenever kT is equal to the energy of an excited state of an ion in the gas. Therefore, the treatment of the problem for most gases would be tedious. One gas for which this is not true is hydrogen. It will be shown that the temperature is high enough so that the hydrogen is ionized over the entire region of interest. Therefore, the gas consists of (unexcitable) ions and free electrons. In this case $\gamma = 5/3$, so $X_0 = 4$.

With this value of X_0, the values of p', w', v', etc., have been computed from Eqs. (33), and their logarithms have been plotted in Fig. 2. To obtain the corresponding values of p, w, v, etc., it is necessary to determine α, and thereby the scale factors p/p', v/v', etc. Since in this case $Z=1$, Eq. (24) gives for α

$$\alpha = 2.61\times 10^{10}\, AP_0/N_0\rho_0 = 2.61\times 10^{10}\, kT_0,$$

where $T_0 = 300$°K. Thus

$$1/\alpha = 940. \tag{34}$$

The factor by which T' is to be multiplied to obtain T in °K is

$$1/\alpha^{\frac{1}{3}}=9.8, \tag{35a}$$

and that by which p', w', ϵ_T' are to be multiplied to obtain p, w, ϵ_T in atmospheres is

$$1/\alpha^{4/3}=9.2\times 10^3. \tag{35b}$$

The factor by which v' is to be multiplied to obtain the velocity of the shock in units of $(P_0/\rho_0)^{\frac{1}{2}}=280$ m/sec. is

$$1/\alpha^{\frac{2}{3}}=96. \tag{35c}$$

It can be seen in Fig. 2 that the radiation ceases to play an important role in the shock for $x\leq 4.01$. This then is the lowest value of x which is of interest for this discussion. At $x=4.01$, Eqs. (33) with $X_0=4$ give $T'=3.36\times 10^4$. Therefore, $T=3.3\times 10^5$ °K, which corresponds to a value of kT of 28 electron volts or over twice the ionization potential of hydrogen. It can be seen from Fig. 2 that the temperature increases rapidly as x increases beyond the value 4.01, so for almost all values of x in the region between $x=4$ and $x=7$, the thermal energy is far greater in magnitude than the potential energy, which can then be neglected. It follows that the assumption that the gas consists of free electrons and protons with its consequent values of $X_0=4$ and $1/\alpha=940$ is a reasonable approximation over most of the region of interest. The properties of the shock in hydrogen are therefore given by Fig. 2 in the following units:

$$\begin{array}{cl} p,\, w,\, \epsilon_T & 9.22\times 10^3 \text{ atmospheres,} \\ T & 9.8\text{°K,} \\ v & 2.7\times 10^4 \text{ meters/sec.} \end{array} \tag{36}$$

The curves in Fig. 2 can also be used to determine the properties of an intense shock in other light gases as long as the intensity of the shock is so great that kT is large compared to the total ionization potential of the gas. The ionization potential for the last electron of an atom of atomic number Z is Z^2 times greater than that of hydrogen, while the temperature for the same value of x is only $[(Z+1)/2]^{\frac{1}{3}}$ times greater than that for hydrogen. Therefore, the range of intensities over which our approximation applies narrows rapidly with increasing Z, and even for air, it is only to the very highest shock intensities considered that Fig. 2 can be applied. It should be pointed out that there is an upper limit to the intensity that can be considered on the basis of the equations used here since at $x=6.99$, the velocity of the shock in hydrogen is about 5×10^8 cm/sec. or 2 percent of the velocity of light. Above this intensity, relativistic effects would become appreciable, and our considerations would have to be modified as indicated in footnote 6.

In the range of intensities to which Fig. 2 applies for light gases other than hydrogen, the units are, of course, different from those given in (36). According to Eqs. (24) and (34), they may be obtained from Eqs. (35) by setting

$$1/\alpha=940(Z+1)/2.$$

7. CONDITION FOR THE FORMATION OF THE SHOCK

Up to this point, it has been assumed that a steady shock can exist, and some consequences of this assumption have been derived. The question naturally arises as to whether the steady shock will ever be formed under reasonable conditions. For ordinary shocks, the answer to this question is usually given by pointing out that the velocity of sound increases with increasing pressure of the medium through which the sound wave is traveling. Therefore, a pressure pulse that has a smooth rather than a discontinuous change in pressure will change form by the high pressure region moving faster than the low pressure region in front of it. The ultimate form of the pulse will then be that of a shock. It will now be shown that a similar process does occur in a medium at extremely high pressure.

For this purpose, it is necessary to write down the hydrodynamic equations for such a medium and thereby determine the velocity of sound. In doing so, it will be assumed that there are only negligible changes in pressure, temperature, etc., over distances of the order of $\lambda''=\lambda c/C$, where C is the velocity of sound. This assumption is made to avoid treating the effects of diffusion of radiation in obtaining the sound velocity. It will be shown that this "non-diffusion" velocity, C, increases with increasing pressure. Therefore, a large amplitude sound pulse which satisfies the non-diffusion condition stated above will become steeper at the front as it progresses. Ultimately, the thickness of the front will reduce to a distance of the order of λ'', whereupon diffusion of the radiation becomes important, and the simple hydrodynamic theory does not apply. The diffusion process sends more energy forward than backward since the pulse has, at this stage, a steeper gradient toward the front than toward the back. Thus, the added effect of diffusion is to sharpen the front of the pulse even further. This process of sharpening continues until the thickness is of the order of $\lambda' = \lambda c/V$, as shown in Section 3. It is to be noted that $\lambda' \ll \lambda''$ since $C \ll V$.

It appears then that a sufficiently broad, large amplitude sound pulse will eventually take the form of a shock of the type discussed in the foregoing sections. However, a pulse of thickness less than λ'' will broaden by diffusion until it becomes so broad that the non-diffusion, hydrodynamic effects take control. Then it will proceed to form a shock, but the peak of the shock would be expected to have a smaller intensity than the maximum in the original pulse.

We now consider the hydrodynamic theory under the non-diffusion conditions described above, in order to show that C increases with pressure. Methods analogous to those described in Section 3, show that the equation of continuity has the usual form:

$$\mathrm{div}\ \rho\mathbf{U} = -\partial\rho/\partial t. \tag{37}$$

The equation for conservation of momentum now takes the form

$$\rho d\mathbf{U}/dt = -\mathrm{grad}\ (P + \tfrac{1}{3}W), \tag{38}$$

and conservation of energy yields

$$\rho\frac{d}{dt}(E + \tfrac{1}{2}U^2) + \rho\frac{d(W/\rho)}{dt} = -\mathrm{div}\ [(P + \tfrac{1}{3}W)\mathbf{U}]. \tag{39}$$

The latter equation, with the help of the other two, may be written as

$$\frac{dE}{dt} + \frac{1}{\rho}\frac{dW}{dt} = \frac{(P+4W/3)}{\rho}\frac{d\rho}{dt}, \tag{40}$$

that is, just the condition for constant entropy.

From the form of the Eqs. (37) and (38), it is apparent that the velocity of sound is given by

$$C = \left[\frac{\partial(P + \frac{1}{3}W)}{\partial\rho}\right]_S^{\frac{1}{2}} \tag{41}$$

as would be expected. The constant entropy condition, which is indicated by the subscript S, is expressed by Eq. (40). Since the radiation energy density is given by Eq. (23) or

$$W = 3\beta(P/\rho)^4 \tag{42}$$

with

$$\beta = \frac{a}{3}\left[\frac{A}{(Z+1)N_0}\right]^4,$$

the velocity of sound becomes

$$C = \left[\left(\frac{\partial P}{\partial\rho}\right)\right]_S^{\frac{1}{2}}\left[1 + \frac{4}{3}W\left(\frac{1}{P} - \frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_S\right)\right]^{\frac{1}{2}}. \tag{43}$$

In the case that $W \ll P$, that is, for negligible radiation pressure, this reduces to the normal expression for the velocity of sound, as it should.

The derivative, $(\partial P/\partial\rho)_S$ may be evaluated with the help of Eq. (40), the thermodynamic equation of state, Eq. (18), and Eq. (42). Straightforward manipulation leads to

$$\left(\frac{\partial P}{\partial\rho}\right)_S = \frac{\gamma P + 16(\gamma-1)W/3}{P + 4(\gamma-1)W}\cdot\frac{P}{\rho}, \tag{44}$$

which reduces to the usual expression when $W \ll P$. For $W \gg P$, this becomes

$$(\partial P/\partial\rho)_S = 4P/3\rho,$$

and Eq. (43) becomes

$$C = \tfrac{2}{3}(W/\rho)^{\frac{1}{2}}. \tag{45}$$

Under all conditions, the coefficient of P/ρ in Eq. (44) is larger than one. Consequently, the coefficient of W in Eq. (43) is positive. Therefore, the velocity of sound, C, is a steadily increasing function of the pressure, which is the required condition for the formation of a shock, as expressed in the beginning of the section.

The equation for the velocity of sound obtained in this section could also have been derived by applying Eqs. (1), (11), and (14) to the case of a very weak shock in a medium at high pressure (P_0, W_0 large).

8. CONCLUSION

It has been shown that the conditions prevailing in very intense steady shocks can be obtained by a direct generalization of the Rankine-Hugoniot equations. These conditions yield simple quantitative results for a shock in a very light gas such as hydrogen. For other media, the equation of state and dependence of specific heat on temperature under extremely high temperature conditions are required. Because of this added complication, they have not been treated here.

The results obtained for the light gases are applicable only to plane shock waves of very long duration since the radius of curvature[4] and the length of the shock must be large compared to the thickness of the shock front. In turn, the thickness of the shock front is large compared to λ' and therefore to the free path for radiation, as shown in Section 3. Since the radiation free path is determined by Compton scattering for the gas of free electrons and unexcitable ions under consideration, it will be of the order of[9] 10^4 cm, which is undesirably large.

For the more practical case of a heavy gas, the greater part of the radiation frequency distribution will fall in the neighborhood of the absorption lines of the excited ions, so the radiation free path will be very short. However, the medium will still be nearly transparent to the high frequency tail of the distribution, so there will be a continual degradation of the energy in the shock front due to the loss of radiation of high energy. In a solid or liquid, the free path for almost the entire range of frequencies will be short, i.e., of the order of a millimeter, due to Compton scattering, since the density of electrons is very high. In this case, the medium is appreciably transparent only to the extremely high energy x-rays. Since the frequency of occurrence of such energetic quanta is extremely small, their escape should have only a small effect on the properties of the shock.

One further point is worth mentioning before concluding. In all of these considerations, it has been assumed that the shock wave is passing through a stable medium. However, at the temperatures prevailing in very intense shock waves, very ordinary media may undergo nuclear reactions. The existence of these reactions would lead to an added internal energy term in Eq. (14) and subsequent equations. If the reaction were sufficiently exothermic and the reaction rate were high enough,[10] a detonation wave rather than a shock wave would pass through the medium.

[9] See W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1936), p. 157.

[10] *High enough* means that an appreciable fraction of the material must undergo a nuclear transformation within the duration of the shock wave. Reaction rates are usually quite slow even at the temperatures considered here because of the Coulomb potential barrier, so the duration of the shock would need to be quite long.