

Conduction and Dispersion of Ionized Gases at High Frequencies

H. MARGENAU*

*Radiation Laboratory,** Massachusetts Institute of Technology, Cambridge, Massachusetts*

(Received January 7, 1946)

The distribution in energy of electrons in a high frequency electromagnetic field is derived by kinetic theory methods. By use of the distribution law, the current density and hence the (complex) conductivity are calculated as functions of electron density, pressure, and frequency of the field. The real part of the conductivity has a maximum for gas pressures, or frequencies, such that the mean free time of an electron is approximately equal to the period of the field. From the conductivity, the dielectric constant of the medium, its index of refraction, and its extinction coefficient are deduced. The results are applicable in microwave researches and in ionosphere problems.

IN several phases of microwave work and in the analysis of ionosphere problems¹ it is necessary to know the conductivity of an ionized gas as a function of electron density, pressure of the gas, and frequency of the waves. Two limiting cases of the conductivity problem are well understood: for low frequencies and high pressures, the current density set up by an alternating field strength $E \cos \omega t$ is given by Langevin's mobility formula

$$I = \frac{4}{3} \frac{e^2 E \lambda n}{(2\pi m k T)^{\frac{1}{2}}} \cos \omega t; \quad (1)$$

it is in phase with the field strength. For high frequencies and low pressures, on the other hand, the current is in quadrature with the field and is represented by the formula characteristic of free electrons,

$$I = (e^2 E n / m \omega) \sin \omega t. \quad (2)$$

The symbols here have their usual significance; λ is the mean free path of the electrons and n their volume concentration.

Under conditions intermediate between the two extremes described by Eqs. (1) and (2) it becomes necessary to consider the effect of electron collisions upon the forced oscillations which the field imposes on them. This is usually done² by including in the equation of motion of the electron a friction term of the form $g(dx/dt)$

which has the result of converting Eq. (2) into

$$I = \frac{e^2 E m \omega n}{m^2 \omega^2 + g^2} \sin \omega t. \quad (2')$$

When this formula is compared with experiment, g is frequently treated as an empirical parameter, and the use of (2') has led to considerable success.² Hulburt³ and others have used a suggestion of Lorentz to the effect that $g = 2m$ times the collision frequency of the electrons, and Childs⁴ has discussed some of the shortcomings of this procedure. Aside from its artificiality, it may be criticized primarily for ignoring the distribution-in-energy of the electrons and hence for failing to provide a proper average over electron speeds.

Avoiding these approximations and using kinetic theory methods, we describe in the present note a compact way of obtaining formulas for the conductivity and dispersion of an ionized gas at all pressures and frequencies.

I. DISTRIBUTION FUNCTIONS

We assume that, owing to agencies other than the electromagnetic waves dispersed, a concentration of n electrons per cc is maintained in the gas. At present it will be assumed that they have two sources of energy only, elastic collisions and the external electric field. The case in which the waves are themselves capable of ionizing or exciting the molecules will be treated in a later paper; it cannot be handled with the same precision as the one where the electrons make only elastic collisions.

* Now at Yale University.

** This paper is based on work done for the Office of Scientific Research and Development under contract OEMsr-262.

¹ See Lee H. R. Mimno, *Rev. Mod. Phys.* **9**, 1 (1937).

² See K. K. Darrow, *Bell System Tech. J.* **11**, 576 (1932); **12**, 91 (1933).

³ E. O. Hulburt, *Phys. Rev.* **29**, 706 (1927).

⁴ E. C. Childs, *Phil. Mag.* **13**, 873 (1932).

Of the n electrons per cc, $f(v_x, v_y, v_z)dv_xdv_ydv_z$ have velocity components about v_x, v_y, v_z . The function f satisfies the Boltzmann "transfer equation" which reads

$$\frac{e}{m} \mathcal{E} \frac{\partial f}{\partial v_x} + \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t} \right)_e, \quad (3)$$

provided the field \mathcal{E} is applied in the X direction. Here $\partial f/\partial t$ represents the local time rate of change caused in f by variations of \mathcal{E} , and $(\partial f/\partial t)_e$ that due to encounters with gas molecules. Collisions between electrons will be neglected, and a term involving the magnetic field strength need not be included in Eq. (3) because the velocity of the electrons is small when compared with that of light. Now let

$$\mathcal{E} = E \cos \omega t, \quad (4)$$

and put

$$\gamma = eE/m. \quad (5)$$

The distribution function $f(\mathbf{v})$ will be expanded as usual in spherical harmonics of the components of $\gamma\mathbf{v}$, and all harmonics beyond the first will be neglected. Thus

$$f(\mathbf{v}) = f_0(v) + \gamma v_x [f_1(v) \cos \omega t + g_1(v) \sin \omega t], \quad (6)$$

and the functions f_0, f_1 , and g_1 depend only on the magnitude of \mathbf{v} . It may be shown⁵ that

$$\left(\frac{\partial f_0}{\partial t} \right)_e = \frac{1}{v^2} \frac{m}{M} \frac{\partial}{\partial v} \left(\frac{v^4 f_0}{\lambda} \right) + \frac{kT}{Mv^2} \frac{\partial}{\partial v} \left(\frac{v^3}{\lambda} \frac{\partial f_0}{\partial v} \right), \quad (7)$$

$$\left(\frac{\partial (v_x f_1)}{\partial t} \right)_e = -\frac{v_x v}{\lambda} f_1, \quad (8)$$

$$\left(\frac{\partial (v_x g_1)}{\partial t} \right)_e = -\frac{v_x v}{\lambda} g_1, \quad (9)$$

where λ is the mean free path of the electrons and M the mass of a gas molecule. On inserting these expressions in (3) and equating coefficients of γv_x we have

$$\frac{\cos \omega t}{v} \frac{\partial f_0}{\partial v} - \sin \omega t \cdot \omega f_1 + \cos \omega t \cdot \omega g_1 = -\cos \omega t \cdot \frac{v}{\lambda} f_1 - \sin \omega t \cdot \frac{v}{\lambda} g_1. \quad (10)$$

Terms even in v_x yield, after averaging over directions,

$$\frac{\gamma^2}{3v^2} \cos^2 \omega t \cdot \frac{\partial}{\partial v} (v^3 f_1) + \frac{\gamma^2}{3v^2} \cos \omega t \sin \omega t \cdot \frac{\partial}{\partial v} (v^3 g_1) = \left(\frac{\partial f_0}{\partial t} \right)_e,$$

and this, when averaged over a period of the wave becomes, with the use of (7),

$$\frac{\gamma^2}{6} \frac{\partial}{\partial v} (v^3 f_1) = \frac{m}{M} \frac{\partial}{\partial v} \left(\frac{v^4 f_0}{\lambda} \right) + \frac{kT}{M} \frac{\partial}{\partial v} \left(\frac{v^3}{\lambda} \frac{\partial f_0}{\partial v} \right). \quad (11)$$

Equation (10), on the other hand, resolves itself on equating coefficients of sine and cosine terms into the following two:

$$g_1 = \frac{\omega \lambda}{v} f_1, \quad (12)$$

$$\frac{1}{v} \frac{\partial f_0}{\partial v} = -\frac{v}{\lambda} f_1 - \omega g_1, \quad (13)$$

so that

$$f_1 = -\frac{\lambda}{v^2 + \omega^2 \lambda^2} \frac{\partial f_0}{\partial v}. \quad (14)$$

This may be put into Eq. (11), which becomes after integration (the constant of integration being 0),

$$-\frac{\gamma^2}{3} \frac{\lambda v^4}{v^2 + \omega^2 \lambda^2} \frac{\partial f_0}{\partial v} = \frac{m v^4 f_0}{M \lambda} + \frac{2kT v^4}{M \lambda} \frac{\partial f_0}{\partial v},$$

hence

$$\log f_0 = - \int_0^v \frac{(m/2)d(v^2)}{kT + [M\gamma^2\lambda^2/6(v^2 + \omega^2\lambda^2)]}. \quad (15)$$

If γ is so small that kT outweighs the second term in the denominator, f is Maxwellian. If the second term is dominant, we obtain a distribution function similar to Druyvesteyn's,⁵ though differing from it by the presence of the term in ω :

$$f_0 = A \exp \left[-\frac{3m(v^4 + 2\omega^2\lambda^2 v^2)}{2M\gamma^2\lambda^2} \right]. \quad (16)$$

In arriving at this result λ has been treated as constant.

A more convenient way of expressing f_0 as well as the subsequent results is by introducing two energy parameters

$$\epsilon_1 = \frac{1}{2}m(\omega\lambda)^2 \quad \text{and} \quad \epsilon_2 = eE\lambda \quad (17)$$

⁵ S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1939), pp. 348 et seq. P. M. Morse, W. P. Allis and E. S. Lamar, *Phys. Rev.* **48**, 412 (1935).

characteristic of the distribution. In terms of them and of $\epsilon = \frac{1}{2}mv^2$, Eq. (16) reads

$$f_0 = A \exp \left[-\frac{6m_2}{M\epsilon_2}(\epsilon^2 + 2\epsilon_1\epsilon) \right]. \quad (16')$$

It is clear that for sufficiently large ϵ_1 the distribution differs appreciably from Druyvesteyn's formula even in the limiting case ($\epsilon_2 \rightarrow \infty$) here under consideration. On integrating (15) without

$$\begin{aligned} n &= 4\pi \int_0^\infty f_0 v^2 dv = 2\pi \left(\frac{2kT}{m} \right)^{\frac{1}{2}} A \int_0^\infty e^{-x} \left(1 + \frac{x}{x_1 + \alpha} \right)^\alpha x^{\frac{1}{2}} dx \\ &= \left(\frac{2\pi kT}{m} \right)^{\frac{1}{2}} A \left\{ 1 + \frac{3}{2} \frac{\alpha}{\alpha + x_1} + \sum_{j=2}^\infty \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2j-3)}{(\alpha+x_1)^j} \cdot \frac{4j^2-1}{2^j} \right\}, \end{aligned}$$

where

$$x_1 \equiv \frac{\epsilon_1}{kT} = \frac{m(\omega\lambda)^2}{2kT}; \quad x \equiv \frac{\epsilon}{kT}. \quad (20)$$

In this report our interest is confined chiefly to the condition

$$x_1 \gg \alpha, \quad (21)$$

which is satisfied for most high frequency electromagnetic waves with intensity below the level of ionization. If we consider as an example 3-cm waves passing through He gas at a pressure of 20 mm, x_1 is about 100 provided T is room temperature. Condition (21) then requires that $E \ll 7$, E being measured in volts/cm. For shorter waves and lower pressures E is allowed to be greater.

Wishing to approximate the parenthesis in

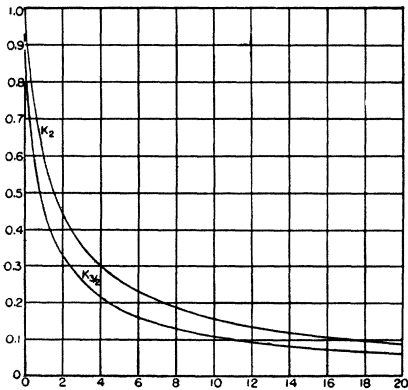


FIG. 1. K_2 and $K_{3/2}$ as functions of $X \equiv m\omega^2\lambda^2/2kT$.

approximation one obtains the accurate distribution law

$$f_0 = A e^{-\epsilon/kT} \left(1 + \frac{\epsilon/kT}{(\epsilon_1/kT) + \alpha} \right)^\alpha, \quad (18)$$

with

$$\alpha \equiv \frac{M}{12m} \left(\frac{\epsilon_2}{kT} \right)^2. \quad (19)$$

To determine the constant A we use the relation

Eq. (18) by an exponential, we expand

$$\begin{aligned} \log \left(1 + \frac{x}{x_1 + \alpha} \right)^\alpha &= \alpha \log e^{\alpha x / (x_1 + \alpha)} \\ &= \alpha \sum_{j=2}^\infty \left(-\frac{1}{j} \right) \left(\frac{x}{x_1 + \alpha} \right)^j \end{aligned}$$

and thus obtain

$$\begin{aligned} f_0 &= A \exp \left[-\frac{x_1}{x_1 + \alpha} x - \frac{1}{2} \frac{\alpha x^2}{(x_1 + \alpha)^2} \right. \\ &\quad \left. + \frac{1}{3} \frac{\alpha x^3}{(x_1 + \alpha)^3} - \cdots \right]. \quad (22) \end{aligned}$$

In the direct-current case, the first term in the exponent is zero, and the second represents the Druyvesteyn distribution. For alternating currents, and under condition (21), the first term alone is important and leads to a Maxwellian distribution

$$f_0 = A \exp \left[-x_1 / (x_1 + \alpha) \right] x, \quad (23)$$

corresponding to a temperature

$$T' = T(1 + \alpha/x_1). \quad (24)$$

In the present instance this is essentially T .

Because of the great convenience afforded by the use of the Maxwell formula which, perhaps somewhat unexpectedly, presents itself as a valid approximation in the a.c. case even for values of the field strength that calls for the use of the Druyvesteyn formula under d.c. conditions—it seems important to know whether condition (21)

is necessary for this approximation to hold. We first note that, if both x_1 and α are small, Eq. (22) to be sure becomes nugatory, but Eq. (15) clearly shows that f_0 is Maxwellian. Finally, to investigate the situation for larger values of α , numerical computations have been made. The results indicate that for practical purposes (18) and (23) may be regarded as identical in the part of the velocity range where f_0 is sufficiently large to matter, so long as x_1 is not smaller than α . Zeroth and first moments of the energy, when computed with (23), differ by less than 3 percent from their correct values based on (18). Condition (21) may therefore be replaced for most purposes by

$$x_1 \geq \alpha. \tag{21'}$$

II. CONDUCTIVITY OF THE MEDIUM WHEN DISTRIBUTION FUNCTION IS MAXWELLIAN

The current density through the gas is

$$I = ne\bar{v}_x = e\gamma \int v_x^2 (f_1 \cos \omega t + g_1 \sin \omega t) v^2 dv \sin \theta d\theta d\phi$$

because of (6). Hence in view of Eq. (12),

$$I = \frac{4\pi}{3} e\gamma \int_0^\infty f_1 \left(\cos \omega t + \frac{\lambda\omega}{v} \sin \omega t \right) v^4 dv. \tag{25}$$

In accordance with our assumption (21) Eq. (23) leads to

$$f_0 = Ae^{-\epsilon l k T}, \quad A = n(m/2\pi k T)^{3/2}, \tag{26}$$

so that (14) takes the form

$$f_1 = \frac{\lambda m v}{v^2 + \omega^2 \lambda^2} f_0. \tag{27}$$

Using the definition

$$\int_0^\infty \frac{x^\nu e^{-x}}{x_1 + x} dx \equiv K_\nu(x_1), \tag{28}$$

we obtain (cf. Eq. (20) for the meaning of x_1) from (25), (26), and (27)

$$I = \frac{4}{3} \frac{e^2 E \lambda n}{(2\pi m k T)^{3/2}} [K_2(x_1) \cos \omega t + x_1^{1/2} K_{3/2}(x_1) \sin \omega t]. \tag{29}$$

This formula is equivalent to the following one for the complex conductivity:

$$\sigma_{\text{complex}} = \frac{4}{3} \frac{e^2 \lambda n}{(2\pi m k T)^{3/2}} [K_2(x_1) - ix_1^{1/2} K_{3/2}(x_1)], \tag{30}$$

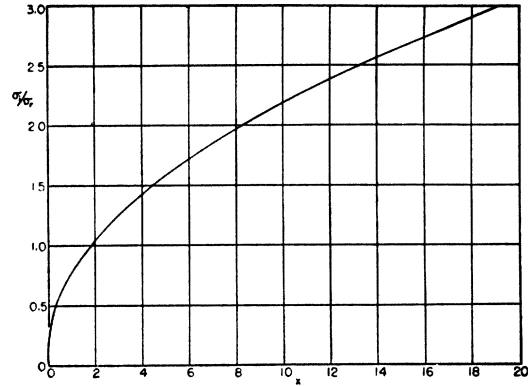


FIG. 2. Ratio of imaginary ("out of phase") to real ("in phase") part of the conductivity as a function of X . When this ratio is much larger than 1, formula (2) is applicable.

which is defined by the relation $I = \sigma_{\text{complex}} E e^{i\omega t}$. To exhibit the analytic nature of this formula we record that it may also be written

$$\sigma_{\text{complex}} = \frac{8}{3} \frac{e^2 \lambda n}{(2\pi m k T)^{3/2}} \int_0^\infty \frac{t^4 \exp[-t^2] dt}{t + i\lambda(m/2kT)^{1/2} \omega}. \tag{30'}$$

The functions K_2 and $K_{3/2}$ can be expressed in terms of the exponential integral $Ei(-x_1)$ and the error function $\text{erf} [(x_1)^{1/2}]$ (both defined as in Jahnke-Emde, *Funktionentafeln*), respectively:

$$K_2(x_1) = 1 - x_1 - x_1^2 \exp[x_1] Ei(-x_1),$$

$$K_{3/2}(x_1) = \left(\frac{1}{2} - x_1\right) \pi^{1/2} + \pi x_1^{1/2} \exp[x_1] \{1 - \text{erf} [(x_1)^{1/2}]\}.$$

Graphs of these are shown in Figs. 1 and 2. We note that

$$\lim_{x_1 \rightarrow 0} K_2 = 1, \quad \lim_{x_1 \rightarrow 0} K_{3/2} = \frac{1}{2} \pi^{1/2},$$

$$\lim_{x_1 \rightarrow \infty} K_2 = \frac{2}{x_1}, \quad \lim_{x_1 \rightarrow \infty} K_{3/2} = \frac{3\pi^{1/2}}{4x_1}.$$

Thus Eq. (29) has the following two limiting forms:

$$\lim_{x_1 \rightarrow 0} I = \frac{4}{3} \frac{e^2 E \lambda}{(2\pi m k T)^{3/2}} n \cos \omega t + \frac{\omega e^2 E \lambda^2}{3kT} n \sin \omega t, \tag{31}$$

$$\lim_{x_1 \rightarrow \infty} I = \frac{16}{3} \frac{e^2 E}{m\omega^2 \lambda} \left(\frac{kT}{2\pi m}\right)^{1/2} n \cos \omega t + \frac{e^2 E}{m\omega} n \sin \omega t. \tag{32}$$

The first term on the right of (31) represents the

Langevin mobility formula, Eq. (1); the second term in (32) is the familiar expression for the current (in quadrature with the applied e.m.f.) due to entirely free electrons, Eq. (2). In problems involving the passage of cm waves through ionized gases, the limiting form (32) is usually applicable.

This applies *a fortiori* to the positive ions. Because of their large mass, x_1 is very large, and (29) describes their contribution to the conductivity, which is seen to be very small and can be neglected for most purposes.

The real and imaginary parts of σ_{complex} are functions of λ , and hence of the pressure of the gas. The imaginary part, being proportional to $x_1 K_{3/2}(x_1)$, is a monotonically increasing function of x_1 : for a given frequency ω it starts with the value $e^2 n / m \omega$ at low pressures and goes to zero as $\omega e^2 \lambda^2 / 3 k T$ at high pressures. The real part, however, which is proportional to $x_1^{\frac{1}{2}} K_2(x_1)$, has the value zero at both ends of the range of x_1 and passes through a maximum.

On differentiating, it is seen that this point is fixed by the condition

$$K_2(x_1) + 2x_1 K_2'(x_1) = 0,$$

which leads to $x_1 = 2.1$. Thus maximum conductivity occurs when

$$\omega \lambda = 6.8 \times 10^{13} (kT)^{\frac{1}{2}}. \quad (33)$$

This equation places the maximum of conductivity nearly at that value of the pressure for which the mean free time τ_1 of an electron between impacts equals τ_2 , the "period" of the light

wave. For if we take the period as the time per radian, $\tau_2 = 1/\omega$, while $\tau_1 = \lambda/v = \lambda(m/3kT)^{\frac{1}{2}}$. Equality of τ_1 and τ_2 then implies $x_1 = 3/2$ or

$$\omega \lambda = 5.75 \times 10^{13} (kT)^{\frac{1}{2}}, \quad (34)$$

a result not very different from (33).

In some experiments⁶ high frequency waves are sent through gases maintaining an independent electrical discharge, and their dispersion and attenuation are measured. If the energy distribution of the electrons is Maxwellian, formula (29) may be shown to be applicable, and T appearing therein (also implicitly in x_1) refers to the temperature of the electrons sustaining the discharge.

III. EXACT THEORY OF CONDUCTIVITY

Starting with Eq. (18) and using the abbreviations (19) and (20), we obtain with the use of (14) and (15)

$$f_1 = 2A \left(\frac{m}{2kT} \right)^{\frac{1}{2}} \lambda \frac{(x+x_1+\alpha)^{\alpha-1}}{(x_1+\alpha)^\alpha} e^{-x} x^{\frac{1}{2}}. \quad (35)$$

When this is inserted in Eq. (25) there results, after an appropriate change of variable,

$$I = \frac{8\pi e^2 E \lambda}{3 m^2} A k T \left\{ \int_0^\infty \frac{(x+x_1+\alpha)^{\alpha-1}}{(x_1+\alpha)^\alpha} e^{-x} x^2 dx \cdot \cos \omega t \right. \\ \left. + x_1^{\frac{1}{2}} \int_0^\infty \frac{(x+x_1+\alpha)^{\alpha-1}}{(x_1+\alpha)^\alpha} e^{-x} x^{\frac{1}{2}} dx \cdot \sin \omega t \right\}. \quad (36)$$

The remaining integrals, as well as A , may be expressed in terms of confluent hypergeometric functions⁷ in the form

$$I = \frac{e^2 E \lambda n}{(2\pi m k T)^{\frac{1}{2}}} \left(W_{[(\alpha-1/2)/2], [(\alpha+3/2)/2]}(x_1+\alpha) \right)^{-1} \left\{ \frac{8}{3} (x_1+\alpha)^{\frac{1}{2}} W_{[(\alpha-3)/2], [(\alpha+2)/2]}(x_1+\alpha) \cdot \cos \omega t \right. \\ \left. + (\pi x_1)^{\frac{1}{2}} W_{[(\alpha-5/2)/2], [(\alpha+3/2)/2]}(x_1+\alpha) \cdot \sin \omega t \right\}. \quad (37)$$

For small integral values of α the integrands in Eq. (36) can be expanded easily and the integrals become sums of Γ -functions. When this procedure is possible it is far less laborious than the use of Eq. (37).

IV. DIELECTRIC CONSTANT, INDEX OF REFRACTION

For ready reference we compile here a few facts that are well known and indicate how the fore-

going results may be used in the computation of the optical properties of an ionized medium.

Ampere's law, for an isotropic medium without space charge and with permeability unity, has the form

$$\nabla \times \mathbf{H} = \frac{1}{c} \dot{\mathbf{D}} + \frac{4\pi}{c} \mathbf{I}.$$

⁶ A. Szekely, Ann. d. Physik [5], 3, 112 (1929). E. V. Appleton and F. W. Chapman, Proc. Roy. Soc. London 44, 246 (1932).

⁷ Whittaker and Watson, p. 340.

If we introduce the "ordinary" dielectric constant $\epsilon_0 = D/\mathcal{E}$ and the conductivity $\sigma = I/\mathcal{E}$, this equation when written for field strengths proportional to $e^{i\omega t}$ reads

$$\nabla \times \mathbf{H} = \frac{i\omega}{c} \epsilon_0 \mathcal{E} + \frac{4\pi\sigma}{c} \mathcal{E}.$$

If σ is real, the current density in phase with \mathcal{E} is $\sigma\mathcal{E}$, that out of phase is $(\omega/4\pi)\epsilon_0\mathcal{E}$. But if σ is complex, that is $\sigma = \sigma_r - i\sigma_i$, then ϵ_0 , the measure of the current in quadrature with \mathcal{E} , must be replaced by the new dielectric constant

$$\epsilon = \epsilon_0 - (4\pi\sigma_i/\omega), \tag{38}$$

while the dissipative current density is $\sigma_r\mathcal{E}$. For the present problem, σ_r and σ_i may be identified from Eqs. (30) [or (36), (37)].

The index of refraction, N , of the medium is defined by

$$\mathcal{E} = E e^{i\omega(t - (N/c)\mathbf{K}\cdot\mathbf{r})}, \tag{39}$$

\mathbf{K} being a unit vector in the direction of propagation. When Maxwell's equations are solved with insertion of this form of \mathcal{E} , it is seen that

$$N^2 = \epsilon - (4\pi i\sigma_r/\omega). \tag{40}$$

If, as usual, we introduce the "ordinary" index of refraction n and the extinction coefficient k , such that

$$N = n - ik, \tag{41}$$

we obtain from (40)

$$n^2 - k^2 = \epsilon = \epsilon_0 - (4\pi\sigma_i/\omega),$$

$$nk = 2\pi\sigma_r/\omega;$$

whence

$$n^2 = \frac{\epsilon}{2} \{ 1 \pm [1 + (4\pi\sigma_r/\epsilon\omega)^2]^{\frac{1}{2}} \}, \tag{42}$$

$$k^2 = \frac{\epsilon}{2} \{ -1 \pm [1 + (4\pi\sigma_r/\epsilon\omega)^2]^{\frac{1}{2}} \}.$$

Here the positive sign is to be taken when $\epsilon \geq 0$, the negative sign when $\epsilon \leq 0$, (see Eq. (35)) the reason being that n and k are real. For waves in the cm region ϵ becomes negative at electron concentrations of $10^{12}/\text{cm}^3$. Detailed applications of these results will be given in another communication.

The author wishes to express his indebtedness to Dr. C. G. Montgomery for suggesting this problem to him and for helpful comments.