comparison. In addition, the ratio of P to Φ_D/D^2 is plotted against y/D in Fig. 2 for both the case where a magnetic field is present (solid curve) and the case where there is no magnetic field (broken curve). The total charge per unit area on the anode is twice that which would be present on a condenser at the same potential as compared with four-thirds as much in case the magnetic field is absent.

In the preceding discussion the plane of cut-off has been supposed to coincide with the anode. Suppose, now, that the potential or charge distribution is desired for a case where the distance d of the anode from the cathode is less than the distance D of the cut-off plane. The potential of the anode being known, Φ_d as well as d is given. Hence, in the specified magnetic field, the quantity

$$M \equiv \Phi_d/\Omega^2 d^2$$

is a *known quantity*. By use of the cut-off relation $\Phi_D = \Omega^2 D^2$,

$$M \equiv \frac{\Phi_d}{\Phi_D}\frac{D^2}{d^2}.$$

Values of this quantity are given in the last column of Table II. From the value of M can be determined the ratio d/D and hence the potential and charge distribution everywhere between the electrodes. For example, if M is found to be 2.96, then $d/D = 0.1486$ from the first column in the table, or the distance between the electrodes is only 14.86 percent of the distance from the cathode to the (virtual) plane of cut-off.

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Space Charge in Cylindrical Magnetron

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The space charge equation for the cylindrical magnetron is solved, the current is obtained as a function of the magnetic field, and the effect of the magnetic field on the distribution of potential and charge is determined.

THE object of this paper is to apply the methods employed in discussing the space charge equation of the diode[1] to the cylindrical magnetron consisting of two coaxial cylindrical electrodes of radii a and b $(a<b)$ in a uniform magnetic field H parallel to their common axis. The inner electrode or cathode emits ions (electrons) with negligible initial velocities. These ions are accelerated toward the outer electrode or anode by a radial electric field because of a difference of potential between the electrodes. Saturation emission of ions from the inner electrode is assumed.

While Brillouin[2] has discussed the particular solution of the space charge equation corresponding to circular paths of the ions, very little progress seems to have been made in solving the space charge equation for the more important case of ions originating on the inner electrode.

With polar coordinates r, θ in a plane at right angles to the common axis of the electrodes, the fundamental equations required are, in Heaviside-Lorentz units,

$$\dot{r}^2 + r^2\dot{\theta}^2 = -\frac{2e}{m}V(r), \tag{1}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = -\frac{e}{mc}Hr\dot{r}, \tag{2}$$

[1] L. Page and N. I. Adams, Jr., Phys. Rev. **68**, 126 (1945).
[2] L. Brillouin, Phys. Rev. **60**, 385 (1941); Elec. Comm. **20**, 112 (1941).

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)=-\rho(r), \tag{3}$$

$$r\dot{r}\rho(r)=\frac{j_l}{2\pi}, \tag{4}$$

where $V(r)$ is the excess of the potential at distance r from the axis over that of the cathode, $\rho(r)$ is the space charge per unit volume, and j_l is the constant current per unit length of the electrodes.

Put $\Phi\equiv-2(e/m)V\geqslant 0$, $\Omega\equiv eH/mc$. Then, from (1) and (2),

$$\dot{r}^2=\Phi-\tfrac{1}{4}\Omega^2a^2\left(\frac{r}{a}-\frac{a}{r}\right)^2\equiv U(r), \tag{5}$$

and, when ρ is eliminated from (3) by means of (4), it is found that U must satisfy the differential equation

$$\frac{d}{dr}\left(r\frac{dU}{dr}\right)+\Omega^2a\left(\frac{r}{a}+\frac{a^3}{r^3}\right)=\frac{(e/m)(j_l/\pi)}{U^{\frac{1}{2}}} \tag{6}$$

subject to the boundary conditions that U and its derivative with respect to r vanish at the cathode $r=a$. Evidently U vanishes again at cut-off.

Next put

$$p\equiv(r/a)^{\frac{2}{3}},\quad U\equiv Kp\psi^2(p),$$

where K is a constant coefficient, getting

$$p\frac{d}{dp}\left(p\frac{d\psi^2}{dp}\right)+2p\frac{d\psi^2}{dp}+\psi^2+\frac{9}{4}\frac{\Omega^2a^2}{K}\left(p^2+\frac{1}{p^4}\right)=\frac{9}{4\pi}\frac{e}{m}\frac{j_la}{K^{\frac{3}{2}}}\frac{1}{\psi}.$$

We take $K^{\frac{3}{2}}\equiv(9/4\pi)(e/m)j_la$, so as to make the coefficient of $1/\psi$ on the right-hand side equal to unity. Then

$$\psi=\frac{1}{p^{\frac{1}{2}}}\frac{U^{\frac{1}{2}}/a}{\left(\dfrac{9}{4\pi}\dfrac{e}{m}\dfrac{j_l}{a^2}\right)^{\frac{1}{3}}},\quad\frac{\Omega^2a^2}{K}=\frac{\Omega^2}{\left(\dfrac{9}{4\pi}\dfrac{e}{m}\dfrac{j_l}{a^2}\right)^{\frac{2}{3}}}\equiv s^2, \tag{7}$$

and the equation for ψ becomes

$$\psi\left[p\frac{d}{dp}\left(p\frac{d\psi^2}{dp}\right)+2p\frac{d\psi^2}{dp}+\psi^2+\frac{9}{4}s^2\left(p^2+\frac{1}{p^4}\right)\right]=1. \tag{8}$$

If we make $s=0$ to pass to the case of the simple cylindrical diode in the absence of a magnetic field, it is easily seen that the function ψ of this paper becomes the cube root of the function g (equivalent to Langmuir's β^2) of the earlier paper.[1]

Following the method employed in that paper we shall seek a solution of (8) in the form of a "near formula" which converges rapidly in the neighborhood of $p=1$ and which satisfies the boundary conditions at the surface of the cathode. Then we shall look for a complete solution in the form of a "far formula" which converges rapidly for large values of p and the two arbitrary constants of which we can determine so as to make the two formulas fit in the region where both converge satisfactorily. Of course p is limited to values less than cut-off. We shall aim for an accuracy of one percent or better in our expansions in series.

NEAR FORMULA

To get the near formula satisfying the boundary conditions at the cathode, we change the independent variable in (8) to x, where $x^3\equiv1-1/p$. Then x is zero at the cathode, increasing as r

increases. In the new variable the differential equation takes the form

$$\psi\left[\frac{(1-x^3)^4}{9x^6}\left\{x\frac{d}{dx}\left(x\frac{d\psi^2}{dx}\right)-3x\frac{d\psi^2}{dx}\right\}+\frac{(1-x^3)^3}{3x^3}x\frac{d\psi^2}{dx}+(1-x^3)^2\psi^2+\frac{9}{4}s^2\{(1-x^3)^6+1\}\right]=(1-x^3)^2. \qquad (9)$$

The solution satisfying the boundary conditions is of the form

$$\psi=(9/4)^{\frac{1}{3}}x^2[\Phi_0-s^2x^2\Phi_1-s^4x^4\Phi_2-s^6x^6\Phi_3-\cdots],$$

where Φ_0, Φ_1, Φ_2, $\Phi_3\cdots$ are series in integral powers of $z\equiv x^3$. Evidently

$$\psi^2=(9/4)^{\frac{2}{3}}x^4[\chi_0-s^2x^2\chi_1-s^4x^4\chi_2-s^6x^6\chi_3-\cdots],$$

where

$$\chi_0=\Phi_0^2,\quad \chi_1=2\Phi_0\Phi_1,\quad \chi_2=2\Phi_0\Phi_2-\Phi_1^2,\quad \chi_3=2\Phi_0\Phi_3-2\Phi_1\Phi_2,\quad \cdots.$$

Substituting in (9) we find for Φ_0, Φ_1, Φ_2, Φ_3,

$$\Phi_0\left[\tfrac{1}{4}(4+4z+z^2)\chi_0+\tfrac{3}{4}(1-z)(5-2z)z\frac{d\chi_0}{dz}+\frac{9}{4}(1-z)^2z\frac{d}{dz}\left(z\frac{d\chi_0}{dz}\right)\right]=1,$$

$$\Phi_1(1-z)^2+\Phi_0^2\left[(1-z)^2\left\{\tfrac{1}{4}(18-18z+9z^2)\chi_1+\tfrac{3}{4}(1-z)(9-6z)z\frac{d\chi_1}{dz}+\frac{9}{4}(1-z)^2z\frac{d}{dz}\left(z\frac{d\chi_1}{dz}\right)\right\}-\left(\frac{9}{4}\right)^{4/3}\{(1-z)^6+1\}\right]=0,$$

$$\Phi_1^2+\Phi_0\Phi_2+\Phi_0^3\left[\tfrac{1}{4}(40-56z+25z^2)\chi_2+\tfrac{3}{4}(1-z)(13-10z)z\frac{d\chi_2}{dz}+\frac{9}{4}(1-z)^2z\frac{d}{dz}\left(z\frac{d\chi_2}{dz}\right)\right]=0,$$

$$\Phi_1^3+2\Phi_0\Phi_1\Phi_2+\Phi_0^2\Phi_3+\Phi_0^4\left[\tfrac{1}{4}(70-110z+49z^2)\chi_3+\tfrac{3}{4}(1-z)(17-14z)z\frac{d\chi_3}{dz}+\frac{9}{4}(1-z)^2z\frac{d}{dz}\left(z\frac{d\chi_3}{dz}\right)\right]=0\ ,$$

and continuing in this manner we obtain the near formula

$$\begin{aligned}\psi=1.3104x^2[&\{1-0.06667z-0.04139z^2-0.02719z^3-0.01916z^4-0.01420z^5-0.01091z^6\\&-0.00862z^7-0.00697z^8-0.00572z^9-0.00476z^{10}-0.00402z^{11}-0.00342z^{12}\cdots\}\\&-0.5897s^2x^2\{1+0.0238z+0.7452z^2+0.7781z^3+0.9224z^4+1.0675z^5+1.2136z^6+1.361z^7\\&+1.508z^8+1.656z^9+1.805z^{10}+1.955z^{11}+2.108z^{12}\cdots\}-0.1490s^4x^4\{1+0z+1.622z^2\\&+1.791z^3+2.794z^4+3.894z^5+5.272z^6\cdots\}-0.0749s^6x^6\{1-0.076z+2.528z^2+2.833z^3\cdots\}\\&-0.0472s^8x^8\{1\cdots\}-0.0334s^{10}x^{10}\{1\cdots\}-0.0263s^{12}x^{12}\{1\cdots\}-\cdots]. \qquad (10)\end{aligned}$$

The leading coefficients in this expression (i.e., the coefficients of s^2x^2 and its powers) check with the expression for U in the theory of the plane magnetron,[3] as should be the case, since, when $z\to 0$, the formula (10) becomes applicable to the plane magnetron. Of course, accurate computations cannot be made from (10) for the case where the magnetic field is so strong that cut-off occurs for a value of z very small compared with unity. That case, however, is provided for, to a sufficient degree of accuracy, by the theory of the plane magnetron.[3] Hence we limit the use of the near formula (10) to cases where cut-off occurs at a distance from the axis some ten or more times the radius a of the cathode.

FAR FORMULA

Now we must look for a complete solution of (8) valid for large values of p. We shall neglect the term in $1/p^4$ as compared with the term in p^2 in the coefficient of s. This introduces an error of only one-seventh of one percent even when p is as small as three. Physically this approximation amounts

[3] L. Page and N. I. Adams, Jr., Phys. Rev. 69, 492 (1946).

to neglect of the magnetic flux through a cross section of the cathode as compared with the magnetic flux through a circle of radius r.

Following the method used with the cylindrical diode[1] we find that solutions exist in power series of

$$Cp^{-1\pm i/2^{\frac{1}{2}}}=\frac{1}{p}e^{\pm i\theta},\quad \theta\equiv\frac{1}{2^{\frac{1}{2}}}\log p+\alpha.$$

Substituting a power series in these variables in the differential equation, we obtain the solution

$$\begin{aligned}\psi=1&+\frac{2c}{p}\cos\theta-\frac{2c^2}{p^2}\left\{\frac{6}{11}\cos 2\theta+\frac{2(2)^{\frac{1}{2}}}{11}\sin 2\theta\right\}+\frac{2c^3}{p^3}\left\{\frac{1}{2}\cos 3\theta+\frac{7(2)^{\frac{1}{2}}}{24}\sin 3\theta+\frac{5}{44}\cos\theta\right.\\&\left.+\frac{29(2)^{\frac{1}{2}}}{88}\sin\theta\right\}-\cdots-\frac{9}{76}s^2p^2\left[1-\frac{2c}{p}\left\{\frac{3}{4}\cos\theta-\frac{(2)^{\frac{1}{2}}}{8}\sin\theta\right\}\right.\\&\left.+\frac{2c^2}{p^2}\left\{\frac{17}{11}\cos 2\theta-\frac{17(2)^{\frac{1}{2}}}{22}\sin 2\theta-1\right\}-\cdots\right]-\frac{81}{12{,}274}s^4p^4\left[1-\frac{2c}{p}\left\{\frac{1121}{512}\cos\theta\right.\right.\\&\left.\left.-\frac{175(2)^{\frac{1}{2}}}{512}\sin\theta\right\}+\cdots\right]-\frac{60{,}831}{82{,}088{,}512}s^6p^6\left[1-\cdots\right]-\cdots.\end{aligned}\tag{11}$$

This is a complete solution since it contains the two arbitrary constants c and α. The numerical coefficients were verified by substituting the solution back in the differential equation after it had been expressed in trigonometrical form.

Now when $s=0$ (i.e., no magnetic field present), ψ must reduce to the cube root of the function g calculated for the cylindrical diode in the earlier paper.[1] Let C_{10} and C_{20} be the coefficients of the pair of variables appearing in ψ in that case. If, then, C_1 and C_2 are the coefficients when $s\neq 0$, it is evident from the form of the near formula (10) that

$$C_1=C_{10}+C_{12}s^2+C_{14}s^4+\cdots,\quad C_2=C_{20}+C_{22}s^2+C_{24}s^4+\cdots.$$

Consequently the terms in (11) involving all coefficients C_{ij} other than C_{10} and C_{20} are of higher order in $1/p$ than any we have retained, and therefore will be neglected. Hence we obtain at once from the theory of the diode,[1]

$$c=-0.3260,\quad \theta=93^\circ.288\log_{10}p+22^\circ.83,\tag{12}$$

as the proper values of the constants required to satisfy the assumed boundary conditions at the cathode, giving for the final form of ψ,

$$\begin{aligned}\psi=1&-\{0.65200\cos\theta\}\frac{1}{p}-\{0.11594\cos 2\theta+0.05466\sin 2\theta\}\frac{1}{p^2}-\{0.03465\cos 3\theta+0.02858\sin 3\theta\\&+0.00787\cos\theta+0.03229\sin\theta\}\frac{1}{p^3}-\cdots-0.11842s^2p^2\left[1+\{0.48900\cos\theta-0.11526\sin\theta\}\frac{1}{p}\right.\\&\left.+\{0.32850\cos 2\theta-0.23229\sin 2\theta-0.21256\}\frac{1}{p^2}+\cdots\right]-0.006{,}599s^4p^4\left[1+\{1.4275\cos\theta\right.\\&\left.-0.3152\sin\theta\}\frac{1}{p}+\cdots\right]-0.000{,}741s^6p^6\left[1+\cdots\right]-\cdots.\end{aligned}\tag{13}$$

As a test we have calculated ψ from both the near formula (10) and the far formula (13) for $p=3$ ($z=\frac{2}{3}$, $r=5.196a$) through terms in s^4. From the first we get

$$\psi=0.9215-1.016s^2-0.34s^4-\cdots,$$

although little confidence can be placed in the coefficient of s^4 since the coefficients involved in this term in (10) have been computed only through z^6 as against coefficients computed through z^{12} in the first two terms. On the other hand, from (13) we obtain

$$\psi = 0.9214 - 1.023s^2 - 0.49s^4 - \cdots.$$

Evidently the magnetic field must be such as to make s somewhat less than one-half in order to make the two calculations of ψ agree to within one percent. This is a rather severe test, however, particularly of the near formula, which was designed for small values of z and which we have applied to a value two-thirds of its upper limit unity.

The far formula (13) will give values of ψ accurate to better than one percent for $p \geqslant 4 (r/a \geqslant 8)$ provided only that the magnetic field is small enough so that sp is not much greater than unity. This means that the formula cannot be used close to cut-off.

CUT-OFF FORMULA

In order to apply the far formula (11) in the neighborhood of cut-off, where $\psi=0$, we need far more of the leading terms (i.e., those in s^2p^2 and its powers) than have been computed so far. Therefore we turn back to (8) and calculate to twenty-one terms the particular solution corresponding to $c=0$ in (11). In this calculation we have included the contribution to the first four terms of the series of the previously neglected term in $1/p^4$ in (8), mainly for the purpose of showing how utterly negligible is the contribution of this term for all except the very smallest values of p. The terms in c in (11) can be added to this particular solution if needed, but they are negligible if cut-off occurs at large p.

In order to avoid many zeros between the decimal point and the first significant figure of the coefficients, it will be convenient to put

$$q \equiv \frac{9}{38} s^2p^2 = 0.23684 s^2p^2.$$

Then we find for the desired particular solution

$$\begin{aligned}\psi = 1 - 0.50000q\left\{1+\frac{1}{p^6}\right\} - 0.11765q^2\left\{1+\frac{0}{p^6}+\frac{1.030}{p^{12}}\right\} - 0.05578q^3\left\{1-\frac{2.241}{p^6}+\frac{0.891}{p^{12}}+\frac{1.064}{p^{18}}\right\}\\ -0.03324q^4 - 0.02227q^5 - 0.01602q^6 - 0.01209q^7 - 0.00944q^8 - 0.00757q^9 - 0.00620q^{10}\\ -0.00516q^{11} - 0.00435q^{12} - 0.00371q^{13} - 0.00319q^{14} - 0.00277q^{15} - 0.00242q^{16}\\ -0.00213q^{17} - 0.00189q^{18} - 0.00168q^{19} - 0.00150q^{20} - \cdots. \quad (14)\end{aligned}$$

It should be noted that, although ψ is a function of the two independent parameters p and s for small or moderate p, it becomes a function of the single variable $(sp)^2$ for sufficiently large p.

From this series can be obtained the value of q at cut-off for large r/a. A few trials show that ψ vanishes for a value of q a little greater than unity. In order to secure greater accuracy, a correction is made to the computed value of $\psi(q)$ for values of q in this neighborhood by adding to (14) a geometrical series with ratio equal to that of the last few terms. The justification for this procedure lies in the fact that the last few terms in (14) in such cases have a nearly constant ratio, and, furthermore, the correction is small. The results are given in Table I.

TABLE I. Values of $\psi(q)$ near cut-off for large p.

q	$1-\psi(q)$ (uncorrected)	$1-\psi(q)$ (corrected)
1.03	0.872	0.907
1.05	0.921	0.988
1.06	0.948	1.047
1.08	1.007	1.257

The uncorrected values of $\psi(q)$ obtained from the twenty-one terms of (14) show that q is certainly less than 1.08 at cut-off, and the corrected values indicate that its value lies between 1.05 and 1.06, and closer to the first than the second. Therefore we infer that, for large p, cut-off occurs at $q=1.05$ with an error not greater than one percent.

CALCULATIONS AND GRAPHS

We are now in a position to plot curves from the formulas derived. We can cover all cases except that in which cut-off occurs close to the cathode. First we calculate ψ as a function of s for various values of p, although, in order to avoid large numbers of zeros after the decimal point, we shall use $q\equiv(9/38)s^2p^2$ as variable instead of s itself. In Table II are listed formulas for ψ for values of p covering the entire range from 1 up. To and including $p=2$ the near formula (10) is used, for $p=3$ and greater ψ is calculated from the far formula (13). In the region $2\leqslant p\leqslant 4$ the accuracy is least, as neither formula converges very well in this range. This is indicated in the table by giving fewer terms in the expressions for ψ in this region. For $p\geqslant 9$ terms from (14) may be added as cut-off is approached, the accuracy becoming greater the larger p. From the last series we have already found that cut-off comes at $q=1.05$ for very large p. By the same method we find that cut-off comes at $q=1.06$ for $p=25$.

The most important family of curves is that giving the current plotted against the magnetic field for a given Φ and p. From Eq. (7) defining s we have

$$q=\frac{9}{38}p^2\frac{\Omega^2}{\left(\dfrac{9}{4\pi}\dfrac{e}{m}\dfrac{j_l}{a^2}\right)^{\frac{2}{3}}},$$

and the square of Eq. (7) defining ψ, if we eliminate U by means of (5), becomes

$$\psi^2=\frac{\Phi-\frac{1}{4}\Omega^2a^2(r/a-a/r)^2}{a^2p\left(\dfrac{9}{4\pi}\dfrac{e}{m}\dfrac{j_l}{a^2}\right)^{\frac{2}{3}}}.$$

Eliminating j_l from these equations we find

$$\frac{\Omega r}{\Phi^{\frac{1}{2}}}=\frac{2.05480(q)^{\frac{1}{2}}}{[\psi^2+1.05556q(1-a^2/r^2)^2]^{\frac{1}{2}}},\tag{15}$$

TABLE II. Formulas for ψ for various values of $p\equiv(r/a)^{\frac{2}{3}}$.

p	r/a	$\psi(q)$ where $q\equiv(9/38)s^2p^2$
1.000	1.000	0
1.250	1.398	$0.4413-0.2547q-0.062q^2-0.029q^3-0.015q^4-\cdots$
1.333	1.540	$0.5098-0.3092q-0.079q^2-0.038q^3-0.02q^4-\cdots$
1.500	1.837	$0.6122-0.3814q-0.100q^2-0.05q^3-\cdots$
2.000	2.828	$0.7851-0.463q-0.11q^2-\cdots$
3.000	5.196	$0.9214-0.480q-0.11q^2-\cdots$
4.000	8.000	$0.9745-0.479q-0.12q^2-\cdots$
9.000	27.000	$1.0284-0.4822q-0.107q^2-0.056q^3-\cdots$
16.000	64.000	$1.0291-0.4867q-0.109q^2-0.056q^3-\cdots$
20.000	89.443	$1.0265-0.4885q-0.1098q^2-0.0558q^3-\cdots$
25.000	125.000	$1.0232-0.4904q-0.1110q^2-0.0558q^3-\cdots$
Very large	Very large	$1.0000-0.5000q-0.1177q^2-0.0558q^3-\cdots$

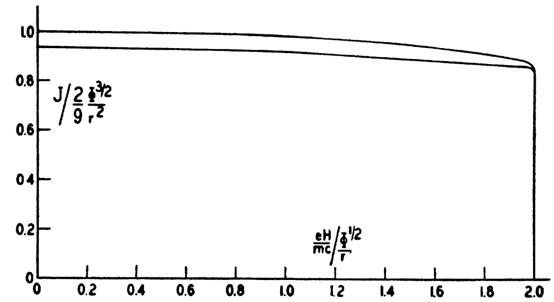

FIG. 1. Variation of relative current density with magnetic field, potential and distance. Upper curve r/a very large, lower curve $r/a=125$.

and eliminating Ω

$$\frac{J}{\dfrac{2}{9}\dfrac{\Phi^{\frac{3}{2}}}{r^2}}=\frac{1}{\left[\psi^2+1.05556q\left(1-\dfrac{a^2}{r^2}\right)^2\right]^{\frac{1}{2}}},\tag{16}$$

where we have put $j_l\equiv 2\pi rj$ and then substituted J for $(e/m)j$. Evidently j is the current density at distance r from the axis.

TABLE III. Current as function of magnetic field for r/a very large.

q	ψ	$\dfrac{eH}{mc}\Big/\dfrac{\Phi^{\frac{1}{2}}}{r}$	$J\Big/\dfrac{2}{9}\dfrac{\Phi^{\frac{3}{2}}}{r^2}$
0.00	1.000	0.000	1.000
0.10	0.949	0.648	0.992
0.20	0.895	0.914	0.983
0.30	0.838	1.115	0.973
0.40	0.776	1.284	0.964
0.50	0.710	1.430	0.953
0.60	0.638	1.560	0.942
0.70	0.557	1.678	0.930
0.80	0.464	1.785	0.917
0.90	0.349	1.883	0.901
1.00	0.177	1.971	0.883
1.05	0.000	2.000	0.857

TABLE IV. Current as function of magnetic field for $r/a=125$.

q	ψ	$\dfrac{eH}{mc}\Big/\dfrac{\Phi^{\frac{1}{2}}}{r}$	$J\Big/\dfrac{2}{9}\dfrac{\Phi^{\frac{3}{2}}}{r^2}$
0.00	1.023	0.000	0.934
0.10	0.973	0.633	0.926
0.20	0.920	0.893	0.919
0.30	0.864	1.091	0.912
0.40	0.804	1.257	0.904
0.50	0.740	1.401	0.897
0.60	0.670	1.530	0.889
0.70	0.591	1.649	0.882
0.80	0.499	1.758	0.875
0.90	0.386	1.860	0.868
1.00	0.217	1.957	0.864
1.06	0.000	2.000	0.845

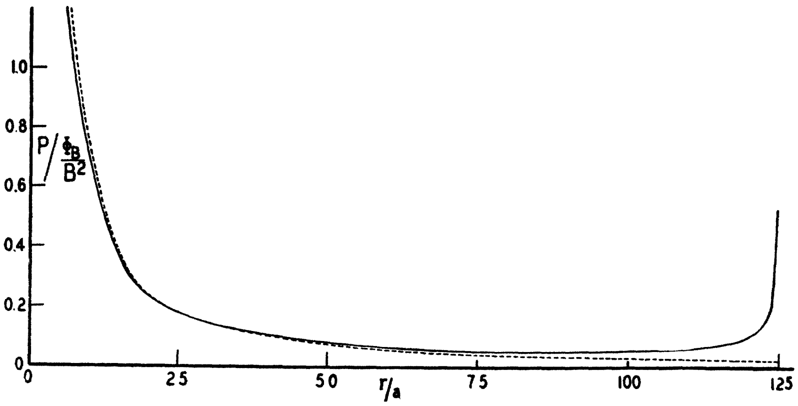

FIG. 2. Charge density with magnetic field (solid curve) and without magnetic field (broken curve) as a function of r/a.

We have plotted two curves of the family under consideration in Fig. 1. The upper one is for p very large, and the lower one for $p=25$ ($r/a=125$). In both cases we have neglected $(a/r)^2$ as compared with unity in (15) and (16). The calculated values for the two curves are given in Tables III and IV. Only for $q \geqslant 1$ has it been necessary to add a small correction to (14) to account for the effect of additional terms beyond the one in q^{20}. It will be noted that both curves are very nearly horizontal, the second even more so than the first. This is quite in accord with the experimental measurements of Hull,[4] who used a ratio of anode to cathode radius not very different from 125.

Next we consider the distribution of potential and of space charge for the case where the anode $r=b$ coincides with the surface $r=B$ of cut-off, limiting ourselves to the case where $a/b \ll 1$. Then (5) gives $\Phi_B=\frac{1}{4}\Omega^2 B^2$ at cut-off. Since, in general,

$$\Phi=\tfrac{1}{4}\Omega^2 r^2\left[\left(1-\frac{a^2}{r^2}\right)^2+\frac{18}{19}\frac{\psi^2}{q}\right],$$

it follows that the distribution of potential is given by

$$\frac{\Phi}{\Phi_B}=\left(\frac{r}{a}\right)^2\left(\frac{a}{B}\right)^2\left[\left(1-\frac{a^2}{r^2}\right)^2+0.94737\frac{\psi^2}{q}\right]. \quad (17)$$

Furthermore, if we put $\mathrm{P}\equiv(e/m)\rho$, it follows that $\mathrm{P}=J/U^{\frac{1}{2}}$, and the distribution of space charge is given by

$$\frac{\mathrm{P}}{\Phi_B/B^2}=\frac{4}{19q\psi}=\frac{0.21053}{q\psi}. \quad (18)$$

In Table V are given, in the third and fourth columns, respectively, the values of Φ/Φ_B and $\mathrm{P}/(\Phi_B/B^2)$ corresponding to the values of p as given in the first column or of r/a as given in the second, for $b/a=125$. The values these two quantities would assume if no magnetic field were present, are given in the fifth and sixth columns. It is to be noted that the potential distribution is not greatly affected by the magnetic field. The charge density, on the other hand, rises very abruptly in the immediate vicinity of cut-off when a magnetic field is present, becoming infinite at the surface of cut-off. In Fig. 2 the graph of $\mathrm{P}/(\Phi_B/B^2)$ plotted against r/a is shown by the solid curve. The broken curve represents the same quantity when no magnetic field is present.

TABLE V. Distribution of potential and of space charge when anode coincides with cut-off surface.

p	r/a	$[\Phi/\Phi_B]_H$	$[\mathrm{P}/(\Phi_B/B^2)]_H$	$[\Phi/\Phi_B]_0$	$[\mathrm{P}/(\Phi_B/B^2)]_0$
1.000	1.000	0.000	∞	0.000	∞
1.250	1.398	0.009	180.3	0.009	192.3
1.333	1.540	0.012	137.2	0.013	146.3
1.500	1.837	0.020	90.4	0.022	96.4
2.000	2.828	0.044	39.7	0.047	42.3
3.000	5.196	0.091	15.09	0.097	16.00
4.000	8.000	0.136	8.07	0.145	8.51
9.000	27.000	0.343	1.596	0.364	1.592
16.000	64.000	0.620	0.613	0.647	0.504
20.000	89.443	0.781	0.505	0.805	0.323
22.000	103.189	0.863	0.533	0.883	0.267
24.000	117.576	0.946	0.804	0.961	0.225
25.000	125.000	1.000	∞	1.000	0.207

[4] A. W. Hull, Phys. Rev. 18, 31 (1921).