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Space Charge in Plane Magnetron

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The space charge equation for the plane magnetron is solved, the current is obtained as a function of the magnetic field, and the effect of the magnetic field on the distribution of potential and charge is discussed.

BEFORE attempting to apply to the cylindrical magnetron the method employed by the authors to solve the space charge equation of the cylindrical diode,[1] it was felt desirable to develop the theory of the limitation of current by space charge in the plane magnetron. Although the space charge equation of the plane magnetron can be solved in closed form and is of sufficient importance to have merited attention, the authors of this paper have been unable to find its solution in the literature. Following the present paper is an investigation of the more important space charge equation of the cylindrical magnetron.

The plane magnetron consists of two infinite parallel plane electrodes a distance d apart between which is a uniform magnetic field H parallel to the electrodes. Let the ion-emitting electrode or cathode be the plane $y=0$. Then the equation of the anode is $y=d$. The Z axis will be oriented parallel to H, and saturation emission of ions from the cathode with negligible initial velocities will be assumed. Evidently the excess V of the potential of any point above that of the cathode and the volume density ρ of space charge will be functions of y only.

With Heaviside-Lorentz units, the fundamental equations involved in the theory are

$$\dot{x}^2+\dot{y}^2=-\frac{2e}{m}V(y), \tag{1}$$

$$\ddot{x}=\frac{e}{mc}\dot{y}H, \tag{2}$$

$$\frac{d^2V}{dy^2}=-\rho(y), \tag{3}$$

$$\dot{y}\rho(y)=j, \tag{4}$$

where j is the constant current density.

[1] L. Page and N. I. Adams, Jr., Phys. Rev. **68**, 126 (1945).

Put $\Phi\equiv-2(e/m)V\geqslant 0$, $\Omega\equiv eH/mc$. Then, from (1) and (2),

$$\dot{y}^2=\Phi-\Omega^2y^2\equiv U,$$

and, after ρ is eliminated from (3) by means of (4), it is found that U satisfies the differential equation

$$\frac{d^2U}{dy^2}+2\Omega^2=\frac{2J}{U^{\frac{1}{2}}}, \tag{5}$$

where $J\equiv(e/m)j$. The boundary conditions at the cathode require that U and dU/dy vanish at $y=0$. The function U must vanish again at cut-off.

The first integral of (5) subject to these boundary conditions is

$$(dU/dy)=(8JU^{\frac{1}{2}}-4\Omega^2U)^{\frac{1}{2}}. \tag{6}$$

To get the second integral put

$$U\equiv\frac{4J^2}{\Omega^4}\sin^4\theta, \quad y\equiv\frac{J}{\Omega^3}\xi.$$

Then (6) becomes

$$d\xi=4\sin^2\theta d\theta,$$

of which the integral is

$$\xi=2\theta-\sin 2\theta, \tag{7}$$

since ξ and θ vanish together. Evidently the zeros of U corresponding to the cathode and to cut-off come at $\theta=0$ and $\theta=\pi$, respectively. At cut-off, then, $\xi=2\pi$.

From the defining equations for $\sin\theta$ and ξ it follows that

$$\frac{\Omega y}{\Phi^{\frac{1}{2}}}=1\bigg/\left(1+4\frac{\sin^4\theta}{\xi^2}\right)^{\frac{1}{2}},$$

$$\frac{Jy^2}{\Phi^{\frac{3}{2}}}=1\bigg/\xi\left(1+4\frac{\sin^4\theta}{\xi^2}\right)^{\frac{3}{2}}.$$

Hence, if we calculate corresponding values of θ and ξ from (7), we can plot the ratio of J to $(2/9)\Phi^{\frac{3}{2}}/y^2$ against the ratio of (eH/mc) to $\Phi^{\frac{1}{2}}/y$, as in Fig. 1. Since the value of J in the absence of a magnetic field is $(2/9)\Phi^{3/2}/y^2$, the ordinate represents the ratio of the current actually existing to that which would exist for the same Φ and y if the magnetic field were not present. Furthermore, since $\Phi=\Omega^2y^2$ is the condition for cut-off, the abscissa represents the ratio of the magnetic field present to that required for cut-off, with the specified Φ and y. In Table I are listed the figures used in plotting the curve. The most interesting feature of the curve is the fact that the current decreases only slightly with increasing magnetic field until cut-off is imminent.

If Φ_D specifies the potential difference between the electrodes when the plane of cut-off $y=D$ coincides with the anode $y=d$, then $\Phi_D=\Omega^2D^2$, and the potential difference between the cathode and a parallel plane at a distance y, is given by

$$\frac{\Phi}{\Phi_D}=\frac{y^2}{D^2}\left(1+4\frac{\sin^4\theta}{\xi^2}\right).$$

Furthermore, the charge density ρ is given by

$$\frac{e}{m}\rho=\frac{\Omega^2}{2\sin^2\theta}=\frac{\Phi_D}{2D^2\sin^2\theta},$$

TABLE I. Dependence of current on magnetic field.

$\frac{eH}{mc}\Big/\frac{\Phi^{1/2}}{y}$	$J\Big/\frac{2}{9}\frac{\Phi^{3/2}}{y^2}$	$\frac{eH}{mc}\Big/\frac{\Phi^{1/2}}{y}$	$J\Big/\frac{2}{9}\frac{\Phi^{3/2}}{y^2}$
0.000	1.000	0.683	0.919
0.070	0.999	0.729	0.905
0.139	0.997	0.771	0.890
0.208	0.993	0.809	0.875
0.275	0.988	0.844	0.860
0.341	0.982	0.901	0.829
0.405	0.974	0.943	0.799
0.466	0.966	0.971	0.771
0.525	0.955	0.993	0.738
0.581	0.944	0.9996	0.720
0.634	0.932	1.0000	0.716

TABLE II. Distribution of potential and charge.

y/D	Φ/Φ_D	$(y/D)^{4/3}$	$P/(\Phi_D/D^2)$	$2/9(y/D)^{-2/3}$	$\frac{(\Phi_d/d^2)}{(\Phi_D/D^2)}$
0.00000	0.00000	0.00000	∞	∞	—
0.00645	0.00097	0.00120	5.237	62.83	23.25
0.0486	0.0145	0.0177	1.447	1.668	6.12
0.1486	0.0655	0.0787	0.764	0.792	2.96
0.3065	0.1768	0.2066	0.553	0.489	1.88
0.5000	0.3513	0.3969	0.500	0.353	1.41
0.6936	0.5639	0.6139	0.553	0.284	1.17
0.8514	0.7682	0.8069	0.764	0.247	1.06
0.9514	0.9172	0.9357	1.447	0.230	1.013
0.9936	0.9880	0.9914	5.237	0.223	1.0008
1.0000	1.0000	1.0000	∞	0.222	1.00000

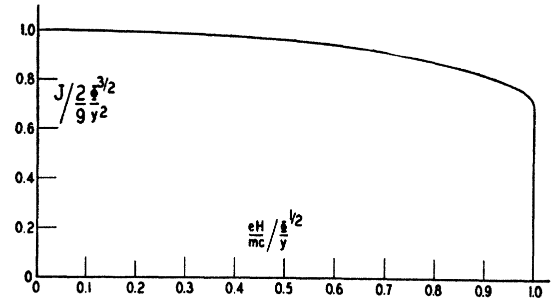

FIG. 1. Variation of relative current density with magnetic field, potential, and distance.

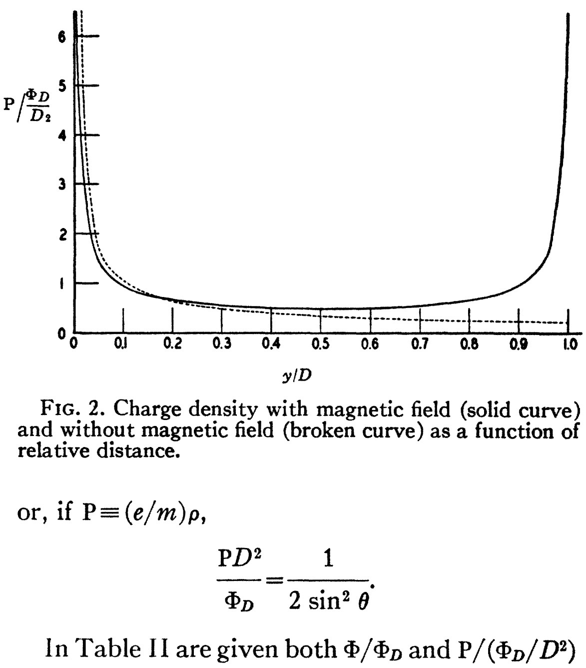

FIG. 2. Charge density with magnetic field (solid curve) and without magnetic field (broken curve) as a function of relative distance.

or, if $P\equiv(e/m)\rho$,

$$\frac{PD^2}{\Phi_D}=\frac{1}{2\sin^2\theta}.$$

In Table II are given both Φ/Φ_D and $P/(\Phi_D/D^2)$ for values of y/D ranging from 0 to 1, that is, from the surface of the cathode to the plane of cut-off. Since the first of these ratios is equal to $(y/d)^{4/3}$ when no magnetic field is present, values of $(y/D)^{4/3}$ are listed for comparison with Φ/Φ_D. The striking feature of the table is that the potential distribution is so little affected by the magnetic field. The charge density becomes infinite at the plane of cut-off as well as at the surface of the cathode, as expected. At both ends, however, nearly the entire rise takes place in a very narrow layer close to the limiting plane. As

$$\frac{Pd^2}{\Phi_d}=\frac{2}{9}\left(\frac{y}{d}\right)^{-2/3},$$

when no magnetic field is present, values of $(2/9)(y/D)^{-2/3}$ are given in the fifth column for comparison. In addition, the ratio of P to Φ_D/D^2 is plotted against y/D in Fig. 2 for both the case where a magnetic field is present (solid curve) and the case where there is no magnetic field (broken curve). The total charge per unit area on the anode is twice that which would be present on a condenser at the same potential as compared with four-thirds as much in case the magnetic field is absent.

In the preceding discussion the plane of cut-off has been supposed to coincide with the anode. Suppose, now, that the potential or charge distribution is desired for a case where the distance d of the anode from the cathode is less than the distance D of the cut-off plane. The potential of the anode being known, Φ_d as well as d is given. Hence, in the specified magnetic field, the quantity

$$M \equiv \Phi_d/\Omega^2 d^2$$

is a *known quantity*. By use of the cut-off relation $\Phi_D = \Omega^2 D^2$,

$$M \equiv \frac{\Phi_d}{\Phi_D}\frac{D^2}{d^2}.$$

Values of this quantity are given in the last column of Table II. From the value of M can be determined the ratio d/D and hence the potential and charge distribution everywhere between the electrodes. For example, if M is found to be 2.96, then $d/D = 0.1486$ from the first column in the table, or the distance between the electrodes is only 14.86 percent of the distance from the cathode to the (virtual) plane of cut-off.

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Space Charge in Cylindrical Magnetron

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The space charge equation for the cylindrical magnetron is solved, the current is obtained as a function of the magnetic field, and the effect of the magnetic field on the distribution of potential and charge is determined.

THE object of this paper is to apply the methods employed in discussing the space charge equation of the diode[1] to the cylindrical magnetron consisting of two coaxial cylindrical electrodes of radii a and b ($a < b$) in a uniform magnetic field H parallel to their common axis. The inner electrode or cathode emits ions (electrons) with negligible initial velocities. These ions are accelerated toward the outer electrode or anode by a radial electric field because of a difference of potential between the electrodes. Saturation emission of ions from the inner electrode is assumed.

While Brillouin[2] has discussed the particular solution of the space charge equation corresponding to circular paths of the ions, very little progress seems to have been made in solving the space charge equation for the more important case of ions originating on the inner electrode.

With polar coordinates r, θ in a plane at right angles to the common axis of the electrodes, the fundamental equations required are, in Heaviside-Lorentz units,

$$\dot{r}^2 + r^2\dot{\theta}^2 = -\frac{2e}{m}V(r), \quad (1)$$

$$\frac{d}{dt}(r^2\dot{\theta}) = -\frac{e}{mc}Hr\dot{r}, \quad (2)$$

[1] L. Page and N. I. Adams, Jr., Phys. Rev. **68**, 126 (1945).
[2] L. Brillouin, Phys. Rev. **60**, 385 (1941); Elec. Comm. **20**, 112 (1941).