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Schematic Treatment of Nuclear Resonances

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A schematic model of competitive nuclear disintegration processes is examined with special attention to the following features: (*a*) The definition of the compound state in a manner independent of the introduction of a "nuclear radius" into the definition; (*b*) the evaluation of the influence of barrier penetration on the parameters entering the dispersion formulas; (*c*) the comparison of the equivalent disintegration probabilities entering as products in the numerators of the dispersion terms with the resonance widths which occur as coefficients of i in the imaginary part of the denominators in the same formulas. It is found that a definition of the compound state can be given in the case considered without the aid of an arbitrarily assigned nuclear radius. The definition of the compound state is arranged to be such as to be nearly independent of the potential barriers affecting the disintegration products. The damping constants turn out to be expressible primarily through the regular radial functions f and are found to depend also on the irregular functions g, inasmuch as the latter determine the linear combinations with which the different damping integrals denoted by $I_n^{(q)}$ combine to give the damping constants Γ. The cross sections can be expressed in the present model in terms of determinants with a finite number of rows and columns. The answer is also transformed into a "dispersion formula" form and it is found that the relation between the resonance width and the disintegration probabilities can be made to be exact in the "isolated level" form but is only an approximation in the representation of the general case that has been used. At the end of the paper a special case is discussed for which the energy dependence of the answer is worked out in terms of elementary functions rather than infinite series.

1. INTRODUCTION

IT is customary and convenient to represent the energy dependence of nuclear reaction cross sections by means of formulas similar to those in the theory of optical dispersion and often referred to as "dispersion formulas." In the vicinity of sharp and isolated resonance energies there is little doubt concerning the correctness of the usual theoretical interpretation[1] of the experimentally observable quantities such as the resonance widths and the values of cross sections for different processes. The analysis of experimental material in energy regions exposed to the combined influence of several resonances is naturally more difficult and the interpretation of the experimentally obtainable parameters in terms of theoretical concepts of nuclear structure is somewhat less definite. Bethe and Placzek[2] as well as Bethe[3] discuss the problem by means of a definition of the level system of the compound nucleus in terms of a preassigned nuclear radius. While this attack has many virtues when the nuclear radius is known, it is doubtless better to free the theory of this somewhat artificial feature as well as to throw light on the nature of the approximations which have to be made in its applications. The discussion of Kalckar, Oppenheimer, and Serber[4] is concerned primarily with situations in which the partial width is large and makes considerable appeal to the analogy with the optical case. Kapur and Peierls[5] define resonance levels somewhat as in earlier work on one body resonances.[6,7] Their levels depend, however, on an assumed nuclear radius and the formula derived for the cross section by Kapur and Peierls contains a length r_0 which must exceed a certain amount. This length enters not only in the damping constants denoted by them as γ_n but also in the term representing the scattering from a hard sphere which is an integral part of their formula. By increasing r_0, say by a factor 2, one can obviously vary the relative magnitudes of contributions among the terms of their formula by considerable amounts. In close proximity of a sharp resonance this arbitrariness of representation makes little difference because in this case the term with the nearly vanishing

[1] G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936). N. Bohr, Nature **137**, 344 (1936).
[2] H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).
[3] H. A. Bethe, Rev. Mod. Phys. **9**, 71, 101–117 (1937).
[4] F. Kalckar, J. R. Oppenheimer, and R. Serber, Phys. Rev. **52**, 273 (1937).
[5] P. L. Kapur and R. Peierls, Proc. Roy. Soc. **A166**, 277 (1938).
[6] G. Breit, Phys. Rev. **40**, 127 (1932).
[7] G. Breit and F. L. Yost, Phys. Rev. **48**, 203 (1935); A. J. F. Siegert, Phys. Rev. **56**, 750 (1939).

resonance denominator is the important one. The representation of the background with which the contribution of the sharp resonance level combines by interference can be varied considerably, however, on account of the presence of r_0 in their formula. This flexibility of representation constitutes progress in that it shows that the representation of experimental results by a dispersion formula is far from unique. It also raises the question as to whether one can represent the cross sections and wave amplitudes in a form that does not involve the employment of arbitrarily assigned lengths and in which the calculation of the parameters entering the final formula does not involve investigations of roots of transcendental equations in the complex domain.

A limited improvement of the situation can be obtained through the employment of the notion of the collision matrix or scattering matrix[8] which has been introduced by J. A. Wheeler for his discussion of resonating group structure and which can be applied to competitive processes in general. Also if one assumes the general form of dependence of cross sections of competitive processes on energy as being of the form of a resonance term superposed on a constant background, the form of the matrices can be established within certain limits. In addition to the relations which follow from the principle of detailed balance which have been found by Wheeler,[9] one finds as a consequence of the "single resonance+background" assumption that an *exact* relation between disintegration probabilities and resonance width can be established. One also finds that the coefficients of the resonance term can be accurately represented in terms of products of disintegration probabilities. The background matrix is found to be expressible in terms of an arbitrary symmetric unitary matrix.

The considerations just mentioned are only a partial help in the general problem. They say nothing about how the potential barriers around the nucleus are to be taken into account and they are based somewhat too heavily on an assumed approximate form of the answer. The way in which barrier penetration affects the formulas representing the cross sections is not sufficiently clear in other work as well. The result of Kapur and Peierls in the interpretation of Konopinski and Bethe[10] is that one should use

$$\Gamma \propto E^{\frac{1}{2}}G^{-2}$$

in the usual notation. On the other hand, the explicit construction of solutions by means of Green's functions gave[11] for sharp resonance and for an interaction taking place between two particles in a thin spherical shell the result

$$f_k^2(kr_0)/k \propto \Gamma$$

in the notation of the latter reference. For some other limiting forms of the assumed interaction, however, the Green's function construction gave[11] a combined occurrence of the functions f, g.

In order to throw light on the questions raised above, it appeared advisable to work out some schematic representations of competitive disintegration processes in which the bothersome features can be more definitely understood. The present paper gives an account of calculations in which the model of the nucleus is sufficiently simple to allow an explicit solution.

The model used below is very schematic and is not intended to be a faithful representation of a nucleus. It takes account, however, in a qualitative way of the following features of actual nuclear processes:

(a) Competition of alternative modes of disintegration. This competition is reproduced only to the extent of having one or another type of disintegration take place at a time.

(b) Effect of barriers on the disintegration probabilities.

(c) Varying conditions of nuclear level spacings and of resonance widths.

The schematic representation is defective and incomplete in the following respects:

(a) The many body interactions are schematically represented by the processes of disappearance of a particle in the nucleus and subsequent reappearance of the same particle or the birth of another

[8] G. Breit, Phys. Rev. **58**, 1068 (1940).
[9] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937).
[10] E. J. Konopinski and H. A. Bethe, Phys. Rev. **54**, 130 (1938).
[11] G. Breit, Phys. Rev. **58**, 506 (1940).

particle which then escapes from the nucleus. These processes are represented in the equations by the occurrence of interaction energies $H_p(r)$ as is seen in Eq. (2.1). The system is treated as though it were a particle with a many valued spin variable having components $\varphi_1, \varphi_2, \cdots, \varphi_N, \varphi$.

(b) The processes of particle disappearance and rebirth are not as complete and general as one could ask for even in the schematic representation which has been employed. Thus, for instance, the matrix elements H_p are taken to be functions only of the distance r from the center of the nucleus. Also direct conversion of an incident particle into an emergent one is not considered so that transitions in the continuum are not taken care of.

(c) The schematic representation employed here does not take into account recoil effects on the residual system when a particle escapes.

(d) In all of the computations the interaction energies H_p and the eigenfunctions u_n are supposed to be real. The matrix (H_{nm}) of the compound state is as a consequence symmetric while in the general case it should be Hermitean.

(e) The residual system described by Eqs. (2.5) is supposed to have a discrete spectrum and no continuum. It would have been better to consider the case of the continuum as well.

(f) The residual system is supposed to be the same independently of which particle leaves the nucleus. This feature would have been very bad if it were not for the fact that the potential energies of the emerging particles are supposed to be different from each other.

The main outline of the calculation is as follows. With the schematic representation of processes of particle absorption by the nuclear system and subsequent creation of the same or other particles introduced by Eq. (2.1) one obtains in terms of the residual system defined by Eq. (2.5) a representation of the asymptotic forms of the radial functions $\times r$ in a form

$$\varphi_q(r) \sim \delta_{pq} f_q(k_q r) + A_q[g_q(k_q r) + i f_q(k_q r)], \tag{1}$$

where f_q, g_q are the regular and irregular functions in the potential fields appropriate to the different disintegration products. One then finds that the amplitudes A_q which are essentially the elements of the collision matrix can be expressed as

$$A_q = B_{pq}/(w_p D) + i\delta_{pq}, \tag{1.1}$$

where the w_p are defined by Eq. (2.3b) and the determinant D is defined by (3.9a) in terms of quantities $\beta_{qq'}$. The definition of the latter involves two steps:

(a) The introduction of the "compound state" by means of the matrix (H_{nm}) and the formation of the matrix of transformation coefficients $a_n{}^r$ from the compound state to the residual. By means of these transformation coefficients one forms projections of the damping integrals $I_n{}^{(q)}$ on to the Hilbert space axes of the compound state and calls these projections $\sigma_q{}^r$. The quantities $\sigma_q{}^r$ and the energy levels of the compound state are used for the definition of $\beta_{qq'}$.

In principle the problem is solved by formula (1.1) in which the determinant D and the cofactor of its (p, q) element which is denoted by B_{pq} are formed by the process just described. At this stage the conversion of the answer to a dispersion formula form could be taken up by an expansion of A_q in terms of the zeros of $1/A_q$ by the method of residues. Such an expansion would have much similarity to the procedure of Kapur and Peierls.[5] In order to carry it through, however, one would have to go into the question of the analytic character of the function A_q in the complex plane.

In order to avoid the consideration of questions of the analytic behavior of functions in the complex plane, the transformation of Eq. (1.1) into a dispersion formula form is carried out by expressing the answer in terms of damping constants and resonance energies which themselves depend on the energy. In this respect there is again a similarity to the procedure of Peierls and Kapur. In the present calculation, however, the determination of the damping constants and energy shifts is reduced to the solution of a system of linear equations.

This system of equations is

$$(\mathbf{W}_r - E_j{}^c)\xi_j{}^r - i\sum_{r'} S^{rr'}\xi_j{}^{r'} = 0. \tag{1.2}$$

Here the W_r are the energies of the compound state defined by

$$\sum_m H_{nm}a_m{}^r = \mathsf{W}_r a_n{}^r, \tag{1.3}$$

with

$$H_{nm} = W_n\delta_{nm} - \sum_q w_q J_{nm}{}^{(q)}, \tag{1.3'}$$

where the $J_{nm}{}^{(q)}$ are double integrals defined by (3.5b). The definition of the compound state by means of Eq. (1.3) has no reference to any assumed nuclear radius. It applies even if all interactions extend to ∞.

The quantities $S^{rr'}$ which occur in Eq. (1.2) are obtainable as

$$S^{rr'} = \sum_q w_q \sigma_q{}^r \sigma_q{}^{r'}, \tag{1.4}$$

where

$$\sigma_q{}^r = \sum_n a_n{}^r I_n{}^{(q)}, \tag{1.4'}$$

with the $a_n{}^r$ defined by (1.3) and the orthogonality relations. The $I_n{}^{(q)}$ are damping integrals obtainable from Eq. (3.5a).

The dispersion formula for the cross section is then

$$\sigma_{p\to q} = (2L+1)\frac{\Lambda^2}{\pi}\left|\delta_{pq}\exp\left[-i\delta_p\right]\sin\delta_p + \sum_j \frac{G_{qj}G_{pj}}{E_j{}^R - E - i\sum_q \Gamma_{qj}}\right|^2, \tag{1.5}$$

where

δ_p = phase shift of wave function f_p describing the pth mode of disintegration,
Λ = wave-length of relative motion,
$L\hbar$ = orbital angular momentum of all the particles,
$G_{qj} = w_q{}^{\frac{1}{2}} \sum_r \xi_j{}^r \sigma_q{}^r$,
$\Gamma_{qj} = |G_{qj}|^2 / \sum_r |\xi_j{}^r|^2$,
$E_j{}^R = \sum_r \mathsf{W}_r |\xi_j{}^r|^2 / \sum_r |\xi_j{}^r|^2$,
$E_j{}^c = E_j{}^R - i\sum_q \Gamma_{qj}$.

The validity of (1.5) is predicated on the possibility of expanding B_{pq}/D in partial fractions in terms of the quantity E^c. The variable E^c is a name for the energy E occurring in the form $\mathsf{W}_r - E$ in the quantities $\beta_{qq'}$ which give the elements of the determinant D. The variable E^c is set equal to the energy E at the end of the calculation. By this strategem one can side step questions of the character of the dependence of the quantities $\sigma_q{}^r$ on E. The energy E still enters the quantities $\sigma_q{}^r$ but is treated in this connection only as a parameter.

The form (1.5) is a legitimate one to use whenever the quotient B_{pq}/D can be considered as the limit of the expressions obtained by employing a finite number of terms in the formula (3.8a) for $B_{qq'}$. This corresponds to the successive inclusion of the levels of the compound nucleus. It is assumed that this procedure converges. It is known to converge in some cases. No general proof of convergence is available and general applicability of this part of the paper is not being claimed.

The quantities G_{qj} have essentially the meaning of disintegration probabilities *via* disintegration mode q for the resonance level j. The sum $\sum_q \Gamma_{qj}$ may be called the resonance width of level j. For sharp and widely spaced levels one has the relation

$$\Gamma_{qj} \backeq |G_{qj}|^2,$$

which gives a ready interpretation of resonance widths essentially as the sum of disintegration probabilities. It is seen, however, that the factor $\sum_r |\xi_j{}^r|^2$ which represents the ratio of the right to the left side of the above approximate equality interferes with the exactness of the relation. It should be stated that the representation of the cross section in terms of a formula of the dispersion type is not unique once one allows the occurrence of coefficients varying with energy.

In Section 7 a transformation of the form (1.1) is carried out in such a way as to represent conditions in the vicinity of an isolated level. The collision cross section is represented by Eq. (7.91) in which the contributions due to the isolated level combine with potential scattering as well as with a background due to other levels. In this representation the relation between resonance width and the

disintegration probabilities is *exact* as shown by Eq. (7.6b) as well as by general arguments applying to such a representation as previously shown.[8]

In Section 8 the way in which the formulas reduce for "weak coupling" is explained. The levels are now sharp in relation to their spacing and here again the relation between resonance width and the disintegration probabilities is exact. The relation of the "isolated level" form of the answer to the more general dispersion form is reexamined and it is shown how the two give the same results. The conditions for weak coupling are very similar to those in the theory of optical dispersion.[12]

In Section 9 the relation of the methods to the one-body problem is discussed. In Section 10 a solution is described in which the energy dependence of the answer is explicitly described by means of elementary functions.

2. EQUATIONS REPRESENTING THE INTERACTION

The simultaneous equations are taken to be

$$\left[\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+E-V(r)\right]\varphi(r)+\sum_{p=1}^{N}H_p(r)\varphi_p(r)=0, \tag{2.1}$$

$$\left[\frac{\hbar^2}{2M_p}\frac{d^2}{dr^2}+E-V_p(r)\right]\varphi_p(r)+H_p(r)\varphi(r)=0; \quad (p=1, 2, \cdots, N). \tag{2.2}$$

Here:

r = distance of any one of the $N+1$ particles from common center.
$\varphi_p(r)=r\times$radial wave function of the pth particle; $4\pi|\varphi_p(r)|^2dr$ = chance that pth particle is in dr.
M_p = mass of the pth particle.
$V_p(r)$ = potential energy of the pth particle in effective central field acting on it. The centrifugal force term $\hbar^2L(L+1)/(2M_pr^2)$ is supposed to be incorporated in $V_p(r)$.
$\varphi(r)=r\times$radial function of particle representing the residual nucleus.
M = mass of particle representing residual nucleus.
E = energy of system.
$V(r)$ = potential energy of particle representing residual nucleus. The centrifugal term is incorporated in $V(r)$.
$H_p(r)$ = interaction energy of pth particle with residual nucleus at distance r.

It is convenient to introduce the following symbols:

$$E_p=E-V_p(\infty), \tag{2.3}$$

$$k_p=(2M_pE_p)^{\frac{1}{2}}/\hbar, \tag{2.3a}$$

$$w_p=2M_p/(\hbar^2k_p)=2/(\hbar v_p), \tag{2.3b}$$

$$v_p=(2E_p/M_p)^{\frac{1}{2}}. \tag{2.3c}$$

The quantity v_p is the velocity of the pth particle at $r=\infty$. It is assumed that

$$V(\infty)>E, \tag{2.4}$$

so that the equations

$$\left[\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+W_n-V(r)\right]u_n(r)=0 \tag{2.5}$$

have a discrete set of proper values W_n and proper functions $u_n(r)$. The latter will be taken to be real and normalized to unity:

$$u_n^*=u_n; \quad \int_0^\infty u_n^2(r)dr=1. \tag{2.6}$$

[12] V. Weisskopf and E. Wigner, Zeits. f. Physik **63**, 54 (1930); V. Weisskopf, Ann. d. Physik **9**, 23 (1931). For the transition to the case of overlapping levels and the effect of the field on the level system through the damping matrix see G. Breit, Rev. Mod. Phys. **5**, 91 (1933); G. Breit and I. S. Lowen, Phys. Rev. **46**, 590 (1934).

The following remarks should be made concerning the schematic representation of interactions described by Eqs. (2.1), (2.2):

(a) The interactions are such that a particle can disappear. In its place there appears then another particle. One of the particles capable of being removed to infinity can interact in the above manner only with the particle described by the wave function $\varphi(r)$ and potential energy $V(r)$. Direct interaction between particles capable of escape is neglected.

(b) The system described by Eq. (2.5) gives a set of energy levels W_n which will be referred to as the levels of the residual nucleus. This terminology is justified by the fact that if the interaction energies H_p were made to vanish, one would have a possibility of simultaneous existence of escaping particles described by Eqs. (2.2) together with a bound particle described by (2.1). It should be remarked, however, that one would be also justified in referring to the W_n in a certain sense as the levels of a compound nucleus because they are the energy levels of a system which results by allowing one of the particles to become converted into a state $\varphi(r)$ and then severing the interaction.

In the interests of definiteness the levels W_n will be referred to as the levels of the residual nucleus. It will be seen later that another system of levels can be introduced so as to correspond more closely with one's intuitive requirements for the level system of the compound nucleus.

(c) It will be noted that the interaction energies $H_p(r)$ correspond to the creation and disappearance of particles at the same value of r. This feature is introduced into the schematic model in the interests of simplicity.

(d) While the schematic model employed here does not represent an actual many body interaction, it has some of the features which are essential for the representation of resonances in systems containing many particles.

3. SOLUTION OF THE EQUATIONS

The solution of the equations can be expressed in terms of functions $f_p(k_p r)$, $g_p(k_p r)$ having the properties of being solutions of

$$\left[\frac{\hbar^2}{2M_p}\frac{d^2}{dr^2}+E-V_p(r)\right]\{f_p(k_p r), g_p(k_p r)\}=0 \tag{3}$$

normalized so that for $r\to\infty$

$$f_p(k_p r)\sim\sin(k_p r-L\pi/2+\delta_p), \tag{3.1a}$$

$$g_p(k_p r)\sim\cos(k_p r-L\pi/2+\delta_p). \tag{3.1b}$$

One has, therefore,

$$f_p' g_p - f_p g_p' = 1. \tag{3.1c}$$

Solutions of the original equations are considered for which the asymptotic forms at $r=\infty$ are

$$\varphi_q\sim\delta_{pq}f_p+A_q(g_q+if_q). \tag{3.2}$$

This choice of the asymptotic form corresponds to there being an incident wave only in the form φ_p, i.e., for particle p. Here δ_{pq} is the Kronecker δ. Straightforward application of the one-dimensional Green's function theory gives

$$\varphi(r)=\sum_n a_n u_n(r), \tag{3.3}$$

$$\varphi_q(r)=\delta_{pq}f_p(k_p r)+w_q\sum_n a_n\Big\{if_p(k_p r)\int_0^\infty f_p(k_p r')H_p(r')u_n(r')dr' + g_p(k_p r)\int_0^r f_p(k_p r')H_p(r')u_n(r')dr'+f_p(k_p r)\int_r^\infty g_p(k_p r')H_p(r')u_n(r')dr'\Big\}, \tag{3.4}$$

where the a_n are determined by the inhomogeneous set of linear equations

$$(E-W_n)a_n+\sum_q w_q[iI_n^{(q)}\sum_m I_m^{(q)}a_m+\sum_m J_{nm}^{(q)}a_m]+I_n^{(p)}=0. \tag{3.5}$$

The quantities I, J occurring above are

$$I_n^{(q)}=\int_0^\infty u_n(r)f_q(k_qr)H_q(r)dr, \tag{3.5a}$$

$$J_{nm}^{(q)}=\int_0^\infty dr\cdot g_q(k_qr)H_q(r)\int_0^r f_q(k_qr')H_q(r')[u_m(r)u_n(r')+u_m(r')u_n(r)]dr'. \tag{3.5b}$$

For large r one obtains the asymptotic forms

$$\varphi_q(r)\sim\delta_{pq}f_q(k_qr)+A_q[g_q(k_qr)+if_q(k_qr)], \tag{3.6}$$

where

$$A_q=w_q\textstyle\sum_n a_nI_n^{(q)}. \tag{3.6a}$$

The above system of equations can be simplified through the introduction of properties of a *compound nucleus*. This is accomplished by means of the matrix having elements

$$H_{nm}=W_n\delta_{nm}-\textstyle\sum_q w_qJ_{nm}^{(q)}, \tag{3.7}$$

and the auxiliary set of equations

$$\textstyle\sum_m H_{nm}a_m^r=\mathsf{W}_ra_n^r. \tag{3.7a}$$

The energy levels W_r are the energy levels of the compound nucleus. The matrix (H_{nm}) is symmetric. If the functions $u_n(r)$ were complex rather than real this matrix would be replaced by a Hermitean one. In terms of the a_n^r one has

$$a_m=\textstyle\sum_r c_ra_m^r, \tag{3.7b}$$

and one has the usual orthogonality relations

$$\textstyle\sum_r a_n^ra_m^r=\delta_{nm}\,;\quad \sum_n a_n^ra_n^s=\delta_{rs}. \tag{3.7c}$$

By means of these relations it is found that

$$iA_q/w_q+\textstyle\sum_{q'}\beta_{qq'}A_{q'}=i\beta_{qp}, \tag{3.8}$$

where

$$\beta_{qq'}=\sum_{r'}\frac{\sigma_q^r\sigma_{q'}^r}{\mathsf{W}_r-E},\quad \sigma_q^r=\textstyle\sum_n a_n^rI_n^{(q)}. \tag{3.8a}$$

The reaction is described by a knowledge of the A_q. The system (3.8) contains, therefore, all the results of immediate interest. It determines the A_q as the solution of N inhomogeneous linear equations on the N unknowns A_q. The coefficients in this system of equations are defined by (3.8a) together with the properties of the compound nucleus determined by (3.7) and (3.7a).

The solution of (3.8) can be verified to be

$$A_q=\frac{B_{pq}}{w_pD}+i\delta_{pq}, \tag{3.9}$$

where

$$D=\det\,|i\delta_{qq'}/w_q+\beta_{qq'}|\,, \tag{3.9a}$$

and B_{pq} is the cofactor of $\beta_{pq}+i\delta_{pq}/w_q$ in D.

In connection with the above formulas it should be observed that

(a) The functions $f_q(k_qr)$, $g_q(k_qr)$ involve the energy E not only in k_qr but also as another variable or parameter. In order not to complicate the notation this dependence of f_q, g_q on E is not indicated. It is nevertheless a strong dependence in many cases.

(b) The functions f_q, g_q have been introduced with the same value of the orbital angular momentum L for all q.

(c) On account of the dependence of $f_q(k_qr)$ on the energy the quantities $I_n^{(q)}$, $J_{nm}^{(q)}$ are functions of the energy. Consequently the matrix elements H_{nm} depend on the energy E and therefore Eq. (3.7a) gives energy levels of the compound nucleus W_r which themselves depend on the energy E. The

transformation coefficients a_n^r representing the change from the proper function system $u_n(r)$ of the residual nucleus to the system of proper functions of the compound nucleus are also functions of the energy of the system.

(d) It is seen from the above that the dependence of the matrix elements of D on the energy is not only that taking place through the occurrence of E in w_p through E_p and the explicit dependence of $\beta_{qq'}$ on E in the combination $\mathsf{W}_r - E$. In addition W_r and the σ_q^r are seen to be functions of the energy.

(e) The quantities $I_n^{(q)}$ depend markedly on barrier penetration on account of the occurrence of f_q under the integral in (3.5a). On the other hand the quantities $J_{nm}^{(q)}$ contain the combination

$$g_q(k_q r) f_q(k_q r'), \quad (r \geqq r').$$

The barrier penetration factors largely compensate in this combination.

It can in fact be verified that if one adds a very high and wide barrier for values of the radius greater than those corresponding to appreciable values of the $u_n(r)$ then the values of the $J_{nm}^{(q)}$ become independent of the height, width, or location of the barrier.

It is seen that the definition of the compound state by means of the matrix satisfies an essential requirement of lack of strong dependence of the compound state on a potential barrier surrounding the whole nucleus.

(f) It will be noted that the definition of the compound state employed here does not depend on an arbitrary introduction of a nuclear radius. Considerations of properties of explicit solutions of one body systems show that essential features in the relation between the mean life and the resonance width are not reproduced if the nuclear radius is arbitrarily defined. In fact if this is done the probability of the escape of a particle from the nucleus is not estimated correctly because the probability of the particle being in the region of negative kinetic energy should be counted as part of the chance of the particle being in the compound state.

(g) The definition of the compound state employed above does not depend on investigations of the somewhat complicated dependence of f_p, g_p on E for complex values of E. The usefulness of such considerations for radioactive states has been shown by Gamow who did not employ arbitrary nuclear radii in his considerations. In Gamow's paper the problem under discussion was one of very sharp states of long lived natural radioactive nuclei. The dependence on complex E could, therefore, be easily followed. The relation of Gamow's picture to a conservative calculation involving real energies is simple[6, 7] and depends only on properties of a Taylor expansion for the coefficients of outgoing and incoming waves. The attempts to formulate along analogous lines a general theory of nuclear resonances which have been made by Peierls and Kapur[5] are naturally more involved because for the general theory it is not sufficient to be concerned with first-order effects of the imaginary part of the energy on the coefficients of outgoing and incoming waves. It may be noted in this connection that Gamow's considerations ended by a complete elimination of the explicit use of complex energies and that these results can be established also without any mention of complex energies.

It is believed, therefore, that the definition of the compound state made in Eqs. (3.7), (3.7a) has advantages of not involving arbitrary nuclear radii or of employing difficult questions of analysis in the investigation of properties of functions in the complex domain. Its main disadvantage is that it is formulated for a special class of systems.

(h) The compound state defined by Eqs. (3.7), (3.7a) could be defined also by asking for a solution of Eqs. (2.1), (2.2) subject to the condition of having the φ_q asymptotic to const. $\times g_q$ at $r = \infty$. For high barriers V_q this condition corresponds to a maximum probability of the particles being inside the nucleus as compared with their density outside. It may be recalled that in the one-body problem[6, 7, 11] the maximum phase shift is obtained when the wave function has a similar asymptotic behavior and that the resonance width in this case is expressible for sharp resonance in terms of the reciprocal of $\int g^2 dr$, the integral being taken up to a suitable r. It is convenient to have the compound state have the property of becoming the one-body compound state and at the same time to have it defined so as to include the many-body case.

4. COMPLEX ENERGIES AND RELATION OF PROBLEM TO VIBRATION THEORY OF CLASSICAL DYNAMICS

In a sense the solution of the problem is completed by Eq. (3.9). This equation does not show, however, the interference of neighboring levels in an obvious form. In the present paragraph transformations of the answer will be made in terms of the complex roots of the determinant D in such a way as to show the interference of levels. These transformations are suggested by the similarity of (3.9) to formulas in the vibration theory of classical dynamics. In the classical theory of small vibrations the vanishing of the determinant of the associated system of linear equations is the feature characteristic of resonance.

The roots of the determinant D will be introduced here by first assigning a real value to the energy, obtaining in terms of it the $J_{nm}^{(q)}$, H_{nm}, W_n, a_n^r and also the $I_n^{(q)}$ as well as the w_q. These quantities will be regarded as parameters in the determinant D the elements of which are

$$b_{qq'}=\beta_{qq'}+i\delta_{qq'}/w_q. \tag{4}$$

The energy E occurs explicitly in the denominators of the terms representing $\beta_{qq'}$ as a sum by means of formula (3.8a). All considerations concerned with complex values of E made below are carried out by allowing only the E in the combinations W_r-E to take on complex values. This is done only as a helpful mathematical device. The value of E in any physical formula is, of course, real. In order to distinguish the E occurring in the denominators W_r-E in Eq. (3.7a) from the real values of the energy it will be referred to as E^c so as to indicate that it will be allowed to take on complex values. Final formulas will involve the special value of E^c *viz.*, $E^c=E$ where E is the real energy for which the other quantities have been evaluated. In this sense the determinant D will be considered as a function of E^c:

$$D=D(E^c).$$

There is a set of roots of this function which will be distinguished by suffixes j so that

$$D(E_j^c)=0. \tag{4.1}$$

The fractions B_{pq}/D in Eq. (3.9) will now be expressed in terms of these roots. It will be noted that the solutions of

$$\sum_q b_{pq}\bar{y}_q=\bar{z}_p \tag{4.2}$$

are

$$\bar{y}_p=\sum_q B_{pq}\bar{z}_q/D. \tag{4.2a}$$

The Eqs. (4.2) can be rewritten as

$$iy_p+\sum_{q,r} s_p^r s_q^r y_q/(\mathsf{W}_r-E)=z_p, \tag{4.2b}$$

where

$$\bar{y}_p=w_p^{\frac{1}{2}}y_p, \quad s_p^r=w_p^{\frac{1}{2}}\sigma_p^r, \quad z_p=w_p^{\frac{1}{2}}\bar{z}_p. \tag{4.2c}$$

The above equations are now meant to hold for all p. The matrix $(S^{rr'})$ is next introduced by letting

$$S^{rr'}=\sum_p s_p^r s_p^{r'} \tag{4.3}$$

and the Eqs. (4.2b) are multiplied by s_p^r and summed over p. One finds

$$(\mathsf{W}_r-E^c)\xi^r-i\sum_{r'} S^{rr'}\xi^{r'}=X^r, \tag{4.4}$$

where ξ^r is defined by

$$(\mathsf{W}_r-E^c)\xi^r=\sum_q s_q^r y_q=\sum_q \sigma_q^r\bar{y}_q, \tag{4.4a}$$

and

$$iX^r=\sum_p s_p^r z_p. \tag{4.4b}$$

The solution of Eqs. (4.4) can be arranged for by introducing a set of transformation coefficients ξ_j^r by means of the equations

$$(\mathsf{W}_r-E_j^c)\xi_j^r-i\sum_{r'} S^{rr'}\xi_j^{r'}=0. \tag{4.5}$$

These equations introduce also a set of complex numbers E_j^c which may be termed the complex eigenvalues of the problem. If $E^c=E_j^c$ then it is possible to satisfy (4.4) with all $X^r=0$. But if all $X^r=0$ then (4.4b) usually requires all z_p to vanish. Hence (4.2b) can be satisfied with all $z_p=0$ and

therefore $D(E_j^c)=0$. Conversely if (4.1) is satisfied by a value of E_j^c then for $E^c=E_j^c$ it is possible to find a set of y_q that satisfy (4.2) for all $z_p=0$ and hence a set of ξ^r for (4.4) with all $X^r=0$. Consequently such an E_j^c will also do for (4.5).

The conclusion that (4.2) is a consequence of (4.5) involves the assumption that if all $X^r=0$ then all $z_p=0$. This is not always the case. Thus, for example, if all the $I_n^{(q)}$ should vanish, then all the s_p^r vanish also and hence the $X^r=0$ for arbitrary finite z_p. In this case D does not contain E^c and it is not possible to make it equal to zero by a proper choice of E^c. In this case, however, all $S^{rr'}$ vanish and hence from (4.5) $E_j^c=\mathsf{W}_r$. The determinant D contains, therefore, indeterminate fractions of the form 0/0 for the solutions of (4.5). In general one cannot expect the quantities s_p^r to be such as to make it possible to satisfy an infinite number of equations $\sum s_p^r z_p=0$ with a finite number of unknowns z_p unless all the z_p vanish.

The system of Eqs. (4.5) has as a consequence orthogonality relations between the coefficients ξ_j^r. These follow in the usual manner by subtracting the result of multiplying the kth equation by ξ_j^r from the result of multiplying the equation by ξ_k^r and summing over all r. One finds in this way that

$$\sum_r \xi_j^r \xi_k^r = \delta_{jk}, \tag{4.5a}$$

provided the ξ_j^r are properly normalized. These are conditions characteristic of orthogonal rather than of unitary transformations. In a well-known manner one derives from (4.5a) the similar set of conditions

$$\sum_j \xi_j^r \xi_j^s = \delta_{rs}. \tag{4.5b}$$

With the help of Eqs. (4.5a), (4.5b) one can determine the coefficients c_j in the expansion

$$\xi^r = \sum_j c_j \xi_j^r, \tag{4.6}$$

so as to satisfy Eq. (4.4). One finds

$$\xi^r = \sum_s \left[\sum_j \xi_j^r \xi_j^s/(E_j^c - E)\right] X^s. \tag{4.6a}$$

Here the quantity E^c has been assigned its physically significant value E. One solves next Eqs. (4.4a) for the $\bar{y}_q$ in terms of the ξ^r and determines therefore through (4.6a) and (4.4b) the expression for $\bar{y}_q$ in terms of the z_p. The last of the three formulas (4.2c) determines z_p in terms of $\bar{z}_p$ and gives therefore a formula for $\bar{y}_p$ in terms of the $\bar{z}_q$. A comparison of the form so obtained with Eq. (4.2a) gives

$$B_{qp}/D = w_p \delta_{pq}/i + \sum_{j,r,s} w_p w_q \sigma_q^r \sigma_p^s \xi_j^r \xi_j^s/(E_j^c - E), \tag{4.7}$$

which when substituted into Eq. (3.9) gives with the aid of (4.4a), (4.2c), and (4.5)

$$A_q = w_q \sum_j \left(\sum_r \sigma_q^r \xi_j^r\right)\left(\sum_s \sigma_p^s \xi_j^s\right)/(E_j^c - E). \tag{4.8}$$

This formula represents A_q as a sum of terms, each term having a pole in the complex plane. The quantity E^c has been replaced by E in Eq. (4.8) because the applications of the formula must be made for real values of the energy.

The quantity A_q/w_q is seen to be symmetric in p and q. The amplitude A_q is introduced by Eq. (3.6) in such a way that the number of systems generated in state q per unit incident system in state p is

$$(v_q/v_p)\,|A_q|^2 = (w_p/w_q)\,|A_q|^2 = w_p w_q |A_q/w_q|^2. \tag{4.8a}$$

The right term of this equation shows that this is also the number of systems which are generated in state p per unit incident system in state q. The scattering matrix thus satisfies the requirement of symmetry which is equivalent to detailed balance at high temperatures.

The numerator of every term in the sum representing A_q/w_q in Eq. (4.8) is a product of two terms, one of which depends only on p and the other only on q. This is to be expected from the general form for A_q/w_q given by Eq. (3.9). When the determinant D vanishes all second minors formed by the cofactors B_{pq} vanish and the cofactors can be represented, therefore, in the form $(B_{pp}B_{qq})^{\frac{1}{2}}$. This

situation is very similar to that occurring in the theory of vibrations of classical dynamics and has already been made use of in connection with the present problem.

5. RESONANCE WIDTH

The relations (4.5) determine a connection between the resonance width and the coefficients occurring in the numerators of the terms in the sum representing A_q/w_q. One defines real numbers E_j^R, E_j^I, x_j^r, y_j^r by

$$E_j^c=E_j^R+iE_j^I, \quad \xi_j^r=x_j^r+iy_j^r. \tag{5}$$

One obtains then from the real and imaginary parts of Eq. (4.5) by multiplying by $(-y_j^r, x_j^r)$ and by (x_j^r, y_j^r) and summing over r the relations

$$E_j^I=-\sum_q \Gamma_{qj}; \quad \Gamma_{qj}=w_q|\sum_r \xi_j^r\sigma_q^r|^2/\sum_r|\xi_j^r|^2, \tag{5.1}$$

$$E_j^R=\sum_r W_r|\xi_j^r|^2/\sum_r|\xi_j^r|^2, \tag{5.2}$$

and one has

$$A_q=\left(\frac{v_p}{v_q}\right)^{\frac{1}{2}}\sum_j \frac{G_{qj}G_{pj}}{E_j^R-E-i\sum_q\Gamma_{qj}}, \tag{5.3}$$

with

$$G_{qj}=w_q^{\frac{1}{2}}\sum_r \xi_j^r\sigma_q^r, \tag{5.4}$$

so that

$$|G_{qj}|^2=\Gamma_{qj}\sum_r|\xi_j^r|^2. \tag{5.5}$$

Substitution of the value of ξ_j^r in terms of x_j^r, y_j^r by means of Eq. (5) into the orthogonality relation (4.5a) for $j=k$ gives

$$\sum_r[|x_j^r|^2-|y_j^r|^2]=1, \tag{5.6}$$

and one has, therefore,

$$\sum_r|\xi_j^r|^2=1+2\sum_r|y_j^r|^2\geqq 1, \tag{5.7}$$

so that in view of Eq. (5.5)

$$|G_{qj}|^2\geqq\Gamma_{qj}. \tag{5.8}$$

It should be remarked that:

(a) Whenever the equality sign applies in Eq. (5.8) one obtains the relation between the resonance width and the numerators in the dispersion formula which holds for the "one-level" theory[1] or for the "one level with background" theory.[8] The latter theory being somewhat more general, comparison is made with it. One has

$$a_{qp}=-(\delta_{pq}+2iA_q)\exp\{-L\pi i+2\delta_q i\}, \tag{5.8a}$$

where a_{qp} is the matrix element of the scattering matrix[8] corresponding to incidence along channel p and emergence along channel q. It was found that close to resonance one can represent

$$a_{qp}\backsimeq\left(\frac{v_p}{v_q}\right)^{\frac{1}{2}}\frac{C_qC_p}{E-E^c}+d_{qp},$$

where d_{qp} represents the "background matrix" and it was also found that

$$|E^c-E^{c*}|=\sum_q|C_q|^2, \tag{5.1, I}$$

where the C_q are in general complex constants. Identifying the coefficient of $(E-E^c)^{-1}$ in these formulas with that in (5.3) one has

$$4|G_p|^2|G_q|^2=|C_q|^2|C_p|^2$$

for all q, p. One has, therefore,

$$2|G_q|^2=|C_q|^2, \tag{5.8b}$$

which together with (5.1, I) gives

$$|E-E^{c*}|=2\sum_q|G_q|^2,$$

which leads to

$$\sum_q\Gamma_{qj}=\sum_q|G_q|^2 \tag{5.9}$$

on account of the first Eq. (5.1) of the present paper. The "one-level" theory is thus seen to correspond to the sign of equality in Eq. (5.8).

(b) It will be seen later that for weak interactions giving small values of the Γ_{qj} one obtains agreement with (5.9) without making any assumption concerning applicability of the "one level with background" approximation. The agreement with (5.9) which one obtains in this case can also be understood as being due to the sharpening of resonances which makes the "one level with background" theory applicable.

6. CROSS SECTIONS

The functions f_q do not have the asymptotic behavior appropriate to regular solutions in the absence of a field. Linear combinations of f_q, g_q are therefore introduced in such a way as to correspond to field-free particles. These combinations will be taken to be F_q, G_q which will be arranged to have asymptotic behaviors.

$$F_q \sim \sin(k_q r - L\pi/2), \quad G_q \sim \cos(k_q r - L\pi/2). \tag{6}$$

In terms of these functions one finds on account of (3.2)

$$\exp[i\delta_p]\varphi_q \sim \delta_{pq}F_p + [\delta_{pq}\exp[i\delta_p]\sin\delta_p + A_q\exp[i(\delta_p+\delta_q)]](G_q+iF_q). \tag{6.1}$$

Remembering that G_q+iF_q is a diverging wave and applying Eq. (6.1) to three-dimensional problems with angular momentum L, the right-hand side of (6.1) can be made to represent the coefficient of $i^L(2L+1)P_L(\cos\theta)/(k_p r)$ in the expression for the qth wave at a large distance. Here θ is the scattering angle and P_L is the Legendre polynomial of order L. The cross section for disintegration *via* mode q due to incidence in mode p is

$$\sigma_{p\to q} = (2L+1)(v_q/v_p)\frac{\Lambda_p^2}{\pi}|\delta_{pq}\exp[-i\delta_p]\sin\delta_p + A_q|^2, \tag{6.2}$$

which becomes in view of Eq. (5.3)

$$\sigma_{p\to q} = (2L+1)\frac{\Lambda_p^2}{\pi}\left|\delta_{pq}\exp[-i\delta_p]\sin\delta_p + \sum\frac{G_{qj}G_{pj}}{E_j^R - E - i\sum_q \Gamma_{qj}}\right|^2. \tag{6.3}$$

This result is very similar to that of Bethe. The main difference is that the quantities $|G_{qj}|^2$ are not exactly the same as the Γ_{qj} but are related to them through Eq. (5.5).

7. AN ISOLATED LEVEL

If one of the levels W_r is located so that the width of other levels does not appreciably overlap it, one can obtain convenient expressions for the A_q directly from Eq. (3.9) without the aid of the secular equations (4.5). The quantities $\beta_{qq'}$ in Eq. (3.8a) depend then critically only on the term involving $1/(\mathsf{W}_r - E)$. The other terms contributing to $\beta_{qq'}$ vary relatively slowly with E and will be lumped into one combination which varies slowly with E. One has thus

$$\beta_{qp} = \sigma_q{}^r\sigma_p{}^r/(\mathsf{W}_r - E) + \epsilon_{qp}, \tag{7}$$

where

$$\epsilon_{qp} = \sum_{s\neq r}\sigma_q{}^s\sigma_p{}^s/(\mathsf{W}_s - E). \tag{7.1}$$

One finds by standard rules for determinants that

$$D = \det|\mu_{qp}| + (\sum M_{qp}\sigma_p{}^r\sigma_q{}^r)/(\mathsf{W}_r - E), \tag{7.2}$$

where

$$\mu_{qp} = i\delta_{pq}/w_p + \epsilon_{qp}, \tag{7.3}$$

and M_{pq} is the cofactor of μ_{pq} in $\det|\mu_{pq}|$. The index p in the present discussion is meant to take on any one of the values 1, 2, $\cdots$, N. Only in the final formulas will it be made to have the value corresponding to the incident state.

One introduces a set of quantities x_q by means of

$$\sum_q \mu_{pq}x_q = \sigma_p{}^r, \tag{7.4}$$

so that

$$x_q=\sum_p M_{qp}\sigma_p{}^r/D',\quad D'=\det|\mu_{pq}|, \tag{7.4a}$$

and one obtains

$$\begin{aligned}\frac{B_{pq}}{D}&=\frac{M_{pq}}{D'}-\frac{x_px_q}{\mathsf{W}_r-E+i\sum_q x_q{}^2/w_q+\sum_{p,q}\epsilon_{pq}x_px_q},\\&=\frac{M_{pq}}{D'}-\frac{x_px_q}{\mathsf{W}_r-E+[\sum_{p,q}M_{pq}\sigma_p{}^r\sigma_q{}^r]/D'}.\end{aligned} \tag{7.5}$$

Here the μ_{pq}, x_q, ϵ_{pq} are all complex. For weak coupling (small H_p and $w_qJ_{nm}{}^{(q)}$) the ix_q are approximately real. The first line of Eq. (7.5) is written so as to show the similarity of the result with (4.7) and (6.3). In the second line the denominator is expressed explicitly in terms of the $\sigma_p{}^r$.

It is an interesting fact that

$$\sum_q\frac{|x_q|^2}{w_q}=-Im\{\sum_{p,q}M_{p,q}\sigma_p{}^r\sigma_q{}^r/D'\}. \tag{7.6}$$

This formula means that the sum of disintegration probabilities occurring in the numerator of (7.5) when properly normalized gives the apparent width of resonance. The quantity ix_p takes place in the numerator of (7.5) of the quantity

$$w_p(\sum_s\sigma_p{}^s\xi_j{}^s)_{j=r}$$

in (4.7) and hence according to (5.4)

$$w_p{}^{\frac{1}{2}}(G_{pj})_{j=r}\to ix_p.$$

It becomes reasonable, therefore, to define

$$w_q{}^{\frac{1}{2}}G_{qr}'=ix_q, \tag{7.6a}$$

and

$$\Gamma_{qr}'=|G_{qr}'|^2 \tag{7.6b}$$

in terms of which (7.5) gives

$$\frac{B_{pq}}{D}=\frac{M_{pq}}{D'}+\frac{w_p{}^{\frac{1}{2}}w_q{}^{\frac{1}{2}}G_{pr}'G_{qr}'}{\mathsf{W}_r'-i\sum_q\Gamma_{qr}'-E}, \tag{7.7}$$

where W_r' is W_r corrected for Dirac's energy shift and is obtainable from Eq. (7.8).

The validity of Eq. (7.7) can be seen and the value of the energy can be derived as follows. Equation (7.4) can be written in view of (7.3)

$$ix_p/w_p+\sum\epsilon_{pq}x_q=\sigma_p{}^r. \tag{7.7a}$$

It will be noted that the ϵ_{pq} and the $\sigma_p{}^r$ are real. The complex conjugate of the above equation is, therefore,

$$-ix_p{}^*/w_p+\sum_q\epsilon_{pq}x_q{}^*=\sigma_p{}^r. \tag{7.7b}$$

One has, therefore,

$$\sum_p|x_p|^2/w_p=\sum_p ix_p(-ix_p{}^*/w_p)=i\sum_p\sigma_p{}^rx_p-i\sum_{p,q}x_p{}^*\epsilon_{pq}x_q, \tag{7.7c}$$

where the value of $-ix_p{}^*/w_p$ has been substituted by means of (7.7b). In view of the fact that ϵ_{pq} is real and also symmetric in p, q the quadratic form occurring on the right side of (7.7c) assumes real values only. Taking the real and imaginary parts of both sides of (7.7c) one obtains, therefore,

$$\sum_p|x_p|^2/w_p=-Im\sum_p\sigma_p{}^rx_p, \tag{7.7d}$$

$$Re\sum_p\sigma_p{}^rx_p=\sum_{p,q}x_p{}^*\epsilon_{pq}x_q. \tag{7.7e}$$

It will be noted in addition that on account of (7.4), (7.4a)

$$(\sum_{p,q}M_{pq}\sigma_p{}^r\sigma_q{}^r)/D'=\sum_p\sigma_p{}^rx_p, \tag{7.7f}$$

Eqs. (7.7d), (7.7f) show that (7.6) is true. Equation (7.7e) gives

$$\mathsf{W}_r'=\mathsf{W}_r+\sum_{p,q}x_p{}^*\epsilon_{pq}x_q. \tag{7.8}$$

These relations show also that

$$\begin{aligned}\sum_p \sigma_p{}^r x_p &= i\sum_p x_p{}^2/w_p+\sum_{p,q} x_p\epsilon_{pq}x_q \\ &= -i\sum_p |x_p|^2/w_p+\sum_{p,q} x_p{}^*\epsilon_{pq}x_q.\end{aligned} \tag{7.9}$$

It will be noted in connection with the above formulas that:

(a) They do *not* involve an approximation of supposing that one level is isolated from other levels. They are put into a form, however, which is convenient for the representation of cross sections in situations and energy regions in which the effect of one level W_r is more pronounced than that of other levels. In such cases the quantities ϵ_{pq} vary relatively slowly and to a first approximation the x_q are constants as a result of Eqs. (7.4). Similarly the "background" M_{pq}/D' occurring in Eq. (7.7) is then approximately constant.

(b) The relation between the numerators of the dispersion term and the imaginary part of its denominator is simpler in the form (7.6b) applying to the isolated level representation (7.7) than in the form (4.7) which corresponds to representation by many levels.

(c) The simple relation (7.6b) between the numerators of the dispersion term and the resonance width is in agreement with the relation (5.1, I) previously derived from rather general considerations for the one-level case. The formulas of the present paragraph may be regarded as a verification of the previous work if they are applied to cases in which W_r' and $\sum \Gamma_{qr}'$ are nearly constant.

(d) The quantity W_r' is a function of the energy not only on account of the dependence of the $\sigma_p{}^r$ and W_r on the energy but also on account of the occurrence of E in ϵ_{pq} as well as the x_q. The latter dependence has its origin in the occurrence of E in combinations with W_s-E in the expression for ϵ_{pq} given by Eq. (7.1).

(e) It appears at first sight as though the isolated level form represents the effect of singling out one of the terms in the sum over complex eigenvalues in (4.7) and lumping all other terms into one contribution considered as a background. This is not the case, however. The last term in (7.7) is not equal to any one term in (4.7).

In fact, the quantity E occurring in combinations W_s-E in the ϵ_{pq} enters both W_r' and the Γ_{qr}'. The denominator of the last term of (7.7) does not have, therefore, the simple form $E_j{}^c-E$. A comparison of the denominator of the second term in the last line of Eq. (7.5) with the formula for D given by Eq. (7.2) shows, in fact, that the resonance denominator in the one-level formula vanishes at all the roots of D as E is varied in the combinations W_r-E, W_s-E. Varying E in the latter manner corresponds to the procedure of introducing the quantity E^c. The resonance denominator in the isolated level formula has, therefore, many roots and consequently is not equal to the denominator in any one of the terms in the "interfering level" formulas (4.7), (5.3), (6.3).

(f) The term M_{pq}/D' which represents the "background" is a complex number which is real only in special cases.

(g) Combining Eqs. (3.9), (6.2), (7.7) one obtains a formula for the collision cross section in the "isolated level" form:

$$\sigma_{p\to q}=(2L+1)\frac{\Lambda_p{}^2}{\pi}\left|\delta_{pq}(i+\exp[-i\delta_p]\sin\delta_p)+\left(\frac{v_p}{v_q}\right)^{\frac{1}{2}}\frac{M_{pq}}{D'}+\frac{G_{pr}'G_{qr}'}{\mathsf{W}_r'-i\sum_q \Gamma_{qr}'-E}\right|^2. \tag{7.91}$$

In this formula the potential scattering term containing the δ_p interferes with the constant amplitude i. In the "interfering levels" formula (6.3) for the cross section there is no such interference. There is also a compensation of the term in the above formula through the term in M_{pq}/D'.

(h) It is clear from the comparisons between the "isolated level" and "interfering levels" formulas that cross-section formulas similar in qualitative physical consequences may correspond to different relations between the apparent disintegration probabilities occurring in the numerators of the dispersion terms and the effective resonance widths determined by the imaginary part of the denominators.

8. WEAK COUPLING

In this section it will be supposed that all

$$\left|\frac{S^{sr}}{\mathsf{W}_s-\mathsf{W}_r}\right|\ll 1 \quad (s\neq r), \tag{8}$$

and this condition will be referred to as "weak coupling" because the compound state is coupled to the continuum through the existence of the quantities σ^{rp} which determine the quantities S^{sr} by means of Eqs. (4.2c), (4.3).

The "interfering levels" formula for A_q leads in this case to

$$A_q=(v_p/v_q)^{\frac{1}{2}}\sum_r(\Gamma_{pr}\Gamma_{qr})^{\frac{1}{2}}/(E_r{}^c-E), \tag{8.1}$$

with

$$\Gamma_{qr}{}^{\frac{1}{2}}\cong w_q{}^{\frac{1}{2}}\sigma_q{}^r, \tag{8.1a}$$

$$E_r{}^c\cong\mathsf{W}_r-i\sum_q\Gamma_{qr}+\sum_{s\neq r}(\sum_p w_p\sigma_p{}^s\sigma_p{}^r)^2/(\mathsf{W}_r-\mathsf{W}_s). \tag{8.1b}$$

In the above formulas the notation is consistent with that of Eqs. (5) to (5.3). The quantity $\Gamma_{qr}{}^{\frac{1}{2}}$ corresponds to G_{qr} of Eq. (5.4). The sign of the $\xi_{qr}{}^{\frac{1}{2}}$ is chosen so as to have $\xi_r{}^r$ approximately equal to unity. In view of the weakness of coupling there is only a slight difference between $E_r{}^c$ and W_r. It is possible, therefore, to label the levels $E_j{}^c$ consistently by the index r of the parent level.

The intermediate steps in the calculation leading from Eq. (5.3) to Eq. (8.1) consist in working out the first power of the small quantity on the left side of Eq. (8) for the Γ_{qr} and for the real part of $E_r{}^c-\mathsf{W}_r$. The formula (8.1) for A_q represents, therefore, a result suitable for estimating the damping constants and the energy shift to a first approximation. It will be noted that in this approximation the inequality in Eq. (5.7) can be replaced by an equality.

For the "isolated level" form one finds for

$$\frac{M_{pq}}{D'}\cong-i\delta_{pq}w_q+w_pw_q\sum_{s\neq r}\sigma_p{}^s\sigma_q{}^s/(\mathsf{W}_s-E) \tag{8.2}$$

$$G_{qr}{}'\cong w_q{}^{\frac{1}{2}}\sigma_q{}^r, \tag{8.3}$$

$$A_q\cong\left(\frac{v_p}{v_q}\right)^{\frac{1}{2}}\left[\sum_{s\neq r}\frac{(\Gamma_{qs}{}'\Gamma_{ps}{}')^{\frac{1}{2}}}{\mathsf{W}_s-E}+\frac{(\Gamma_{pr}{}'\Gamma_{qr}{}')^{\frac{1}{2}}}{\mathsf{W}_r{}'-i\sum_q\Gamma_{qr}{}'-E}\right]. \tag{8.4}$$

The formulas written above are obtained from Eq. (7.7) combined with Eq. (3.9). The approximations are just like those for Eqs. (8.1), (8.1a), (8.1b) with the additional approximation of supposing that E is close to W_r. It was in fact supposed that quantities of the order $(\sigma_p{}^r)^2/|\mathsf{W}_s-E|$ are $\ll 1$. It is seen that Eq. (8.4) agrees with Eq. (8.1) within the approximations made.

One also readily verifies Eq. (7.9) by means of the approximate values for the x_p that correspond to Eq. (8.2).

It is seen that the approximation of weak coupling gives one expressions of the same form as those obtained by Bethe and Placzek for weak coupling.

9. RELATION TO THE ONE-BODY PROBLEM

The representation of resonances which has been obtained in the present paper includes the resonance effects of potential wells. It does not necessarily give an explicit representation or listing of these among the resonance terms of Eq. (6.3). Thus, for example, Eq. (6.3) can give values of the scattering cross section varying rapidly with the energy on account of a rapid variation of δ_p corresponding to a resonance in the potential scattering. Under such conditions some of Γ_{qj} will also vary rapidly and an effect on the real part of $E_r{}^c$ will be present on account of the maximum in the $I_n{}^{(q)}$ and the rapid variation of the $\sigma_r{}^s$. The latter effect can be seen to be present through an inspection of the last term in Eq. (8.1b). It will be noted, however, that the resonance to the potential well does not bring in a term with a resonance denominator in the present representation and that the resonance

occurs partly through a variation of the phase shift, partly through a variation of the real part of the complex energy rather than through the vanishing of the real part of the resonance denominator.

The many-body point of view of nuclear structure emphasized by Bohr makes resonances to potential wells an uninteresting question and it is, therefore, no inconvenience not to have them explicitly indicated by resonance terms.

10. EXPLICIT SOLUTION FOR A SPECIAL CASE

If $N=2$, and if $V(r)=\infty$ when $a<r<\infty$ while

$$H_1, H_2, V, V_1, V_2 = \text{const.} \quad (0<r\leqq a), \tag{10}$$

the quantities $I_n^{(q)}$, $J_{nm}^{(q)}$ are found to be

$$I_n^{(q)} = i_{qn} w_q^{-\frac{1}{2}}; \quad J_{nm}^{(q)} = j_{qnm}/w_q, \tag{10.1}$$

where the values of the symbols are as follows

$$i_{qn} = (-)^n \left(\frac{2}{k_q a}\right)^{\frac{1}{2}} H_q \frac{(2M)^{\frac{1}{2}} \dot{k}_n f_q(k_q a)}{\hbar(\kappa_{q0}^2 - \dot{k}_n^2)}, \tag{10.1a}$$

$$\dot{k}_n = \frac{\pi n}{a}, \quad \kappa_{q0}^2 = \frac{2M}{\hbar^2}[E - V_q(r)], \quad (0<r<a), \tag{10.1b}$$

$$j_{qmn} = \frac{g_q(k_q a)}{f_q(k_q a)} i_{qn} i_{qm} - \frac{2MH_q^2}{\kappa_{q0}^2 - \dot{k}_n^2}\delta_{mn}. \tag{10.1c}$$

The quantities i_{qn}, j_{qmn} have the dimensions of $E^{\frac{1}{2}}$, E, respectively, where E is the energy. The common mass is denoted by M.

For incidence in state 1, i.e., for $p=1$, one finds for the coefficients of the asymptotic forms defined by Eq. (3.2)

$$A_2 = -(k_1/k_2)^{\frac{1}{2}} \alpha_{12}/\Delta, \tag{10.2}$$

where

$$A_1 = -1/b_1 + (1+b_2\alpha_{22})/(b_1\Delta) = [-\alpha_{11} + b_2(\alpha_{12}^2 - \alpha_{11}\alpha_{22})]/\Delta, \tag{10.2a}$$

$$\Delta = (1+b_1\alpha_{11})(1+b_2\alpha_{22}) - b_1 b_2 \alpha_{12}^2, \tag{10.2b}$$

$$\alpha_{qq'} = \sum_n i_{qn} i_{q'n}/(E - \acute{E}_n), \tag{10.2c}$$

$$\acute{E}_n = W_n + H_1^2/[E - W_n - V_1(a) + V(a)] + H_2^2/[E - W_n - V_2(a) + V(a)], \tag{10.2d}$$

$$b_q = [g_q(k_q a) + i f_q(k_q a)]/f_q(k_q a). \tag{10.2e}$$

The quantity W_n is the nth energy level of the residual nucleus:

$$W_n = \hbar^2 \dot{k}_n^2/(2M) = h^2 n^2/(8Ma^2). \tag{10.2f}$$

It should be pointed out in connection with the above results that:

(a) Formulas (10.2), (10.2a) give the quantities A_q for Eq. (6.3) in which the cross sections are expressed in terms of A_q.

(b) Substitution of the symbols involved by means of (10.1a), (10.1b), (10.2c) . . . (10.2f) gives the dependence of the A_q on the energy explicitly through substitution.

(c) The potentials $V_1(r)$, $V_2(r)$ although assumed to be constant in the interval $(0, a)$ are not restricted in any way in the interval (a, ∞). The formulas thus include the influence of potential barriers on the cross sections.

If one assumes that for $r=a$

$$g_2/f_2 \gg g_1/f_1, \tag{10.3}$$

and that

$$b_1 \sum_{n,m} \frac{(i_{2n} i_{1m} - i_{1n} i_{2m})^2}{(\acute{E}_n - \acute{E}_m)(E - \acute{E}_n)} \ll \sum_n \frac{i_{2n}^2}{E - \acute{E}_n}; \tag{10.3a}$$

then it follows from Eq. (10.2) that

$$A_2 \backsimeq -H_1 f_1/(H_2 g_2). \tag{10.4}$$

The assumption of Eq. (10.3) means that the second mode is bottled in by its barrier much more than the first. The assumption of Eq. (10.3a) means that the coupling energy H_1 of the first mode is kept sufficiently small. The form of A_2 given by (10.4) is interesting in that it contains the irregular function for the final state in the denominator and the regular function for the initial state in the numerator.

For

$$V_1(r)=V_2(r)=V_0 \quad (0<r<a), \tag{10.5}$$

the equations for A_1, A_2 reduce to

$$A_1=-\frac{2H_1^2f_1^2(k_1a)}{k_1a\Delta'}, \quad A_2=-\frac{2H_1H_2f_1(k_1a)f_2(k_2a)}{k_2a\Delta'}, \tag{10.6}$$

where Δ' is given by

$$\Delta'-2b_1H_1^2f_1^2/(k_1a)-2b_2H_2^2f_2^2/(k_2a)=\left[\sum_n W_n\left(\frac{\gamma_1}{E-W_{n1}}+\frac{\gamma_2}{E-W_{n2}}+\frac{\gamma_3}{E-W_{n3}}\right)\right]^{-1}, \tag{10.6a}$$

$$=2(\gamma_1x_1\cot x_1+\gamma_2x_2\cot x_2+\gamma_3x_3\cot x_3)^{-1};$$

with

$$W_{n1}=W_n+V_0/2+[(V_0/2)^2+H_1^2+H_2^2]^{\frac{1}{2}}, \tag{10.7}$$

$$W_{n2}=W_n+V_0/2-[(V_0/2)^2+H_1^2+H_2^2]^{\frac{1}{2}}, \tag{10.7a}$$

$$W_{n3}=W_n+V_0, \tag{10.7b}$$

and

$$\gamma_1=(W_n-W_{n2})^{-1}(W_{n1}-W_{n2})^{-1}, \tag{10.8}$$

$$\gamma_2=(W_n-W_{n1})^{-1}(W_{n2}-W_{n1})^{-1}, \tag{10.8a}$$

$$\gamma_3=(W_{n2}-W_n)^{-1}(W_{n1}-W_n)^{-1}=-(H_1^2+H_2^2)^{-1}, \tag{10.8b}$$

$$x_s=[2Ma^2\hbar^{-2}(E+W_n-W_{ns})]^{\frac{1}{2}}; \quad (s=1, 2, 3). \tag{10.9}$$

One sees from Eqs. (10.6), (10.6a) that the numerators of these formulas for the A_q contain the factors $f_q(k_qa)/(k_qa)^{\frac{1}{2}}$ and that the denominators contain imaginary terms in the combinations $2iH_q^2f_q^2/(k_qa)$ so that the damping constants depend essentially on the regular function and the interaction energy H_q. It is of interest to note that Eq. (10.6a) gives an explicit expression in terms of cotangents with the arguments x_s defined by Eq. (10.9).

The example just considered shows that the representation of the scattering amplitudes as a sum of interfering terms can be far from being the simplest.

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A Note on the Relativistic Problem of Uniform Rotation

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A study of uniform rotational motion about an axis is made on the basis of a definition of hydrokinetic character. A solution is found in which the particle speed is linear with distance from the axis of rotation to terms in $(R\omega_0/c)^2$, but approaches the speed of light at great distances. This result is unchanged by the introduction of relativistic accelerated Euclidean axes. Reasons are given for concluding that Ehrenfest's paradox in the problem of the rotating disk, and the question of the "geometry" of the motion, in the sense of general relativity theory, can be answered only on the basis of a theory of the generation of the rotation.

1. INTRODUCTION

THE difficulties presented in the theory of relativity by the "rigid" motion of a body about a fixed axis have been discussed by a number of authors. Among these difficulties, the apparent paradox pointed out by Ehrenfest[1] has received particular attention. For simplicity,

[1] P. Ehrenfest, Physik. Zeits. 10, 918 (1909).