points would require a radical departure of the force between electron and the aluminum nucleus from the Coulomb force. Moreover, it would have to be energy dependent and Z dependent in order to be consistent with the agreement of aluminum points at 2.00 Mev and copper points at 2.27 Mev with the Mott formula. Unfortunately, the data for aluminum at 2.27 Mev were among the last taken before research of this type was interrupted by the war. These measurements with aluminum will be repeated as soon as circumstances permit as they now indicate for the larger angles an interesting divergence from theory.

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Isotope Separation by Thermal Diffusion: The Cylindrical Case

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All of the theoretical treatments of the thermal separation column published up to the present time have been restricted by the assumption that the column consists of two parallel plane walls, one hot and the other cold. Actually, nearly all of the separation columns used in practice consist of two concentric cylinders, the inner cylinder often being simply a hot wire. The theoretical treatment of the plane case which was given by Furry, Jones, and Onsager is here extended to include the cylindrical case. The extension is carried through in general, that is, for a gas whose physical properties are arbitrary functions of the temperature. It is found that the extended treatment is formally very similar to that already given for the plane case. The difficulty of the calculations is enormously increased, however, by the explicit appearance in the characteristic differential equation of the radius as a function of the temperature. The solution is here carried through in detail only for a perfect gas whose viscosity, thermal conductivity, and diffusivity have the same temperature dependences as those of a Maxwellian gas. Exact numerical solutions for a few cases have been obtained, but the computations were so tedious that it was found desirable to develop approximate methods of solution. Two different kinds of approximate solutions are given: a series solution useful when the ratio of radii is not larger than about four or five, and an asymptotic solution valid when the ratio of radii is large, as in the case of the hot-wire types of separation column.

THE theory of the functioning of the new apparatus[1] for isotope separation by thermal diffusion has been investigated by various writers.[2–7] The quantitative agreement between

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† This paper was received for publication on the date indicated but was voluntarily withheld from publication until the end of the war.

[1] K. Clusius and G. Dickel, Naturwiss. **26**, 546 (L) (1938); Zeits. f. physik. Chemie **B44**, 397 (1939).

[2] L. Waldmann, Zeits. f. Physik **114**, 53 (1939).

[3] W. H. Furry, R. Clark Jones, and L. Onsager, Phys. Rev. **55**, 1083 (1939). This will be referred to as FJO.

[4] W. van der Grinten, Naturwiss. **27**, 317 (L) (1939).

[5] P. Debye, Ann. d. Physik **36**, 284 (1939).

[6] J. Bardeen, Phys. Rev. **57**, 35 (1940) (further considerations based on reference 3).

[7] For a general account of the subject, given mostly from

theory and experiment has been found to be reasonably good.[8] It therefore seems worth while to try to improve the theoretical treatment by dispensing with an idealization which has been involved in all of the calculations so far published: namely, the assumption that the processes take place between plane walls. In actual practice one is usually concerned with concentric cylinders. In some cases[9] the radii of the cylinders are so nearly equal that the results for plane walls are fairly accurate; in many cases, however, the inner cylinder is simply a fine wire, and the "plane case" formulas are completely inapplicable.

We shall first show how the formulation in terms of a master function, which was used in FJO, can be extended to the cylindrical case. We then specialize the formulas to the Maxwellian case, in which heat conductivity, viscosity, and diffusivity are proportional to the first, first, and second powers of the absolute temperature, and carry out various specific calculations on this basis.

The present paper is devoted entirely to the task of determining for the cylindrical case the constants H, K_c, and K_d which enter into the transport equation of the thermal separation column.

I. GENERAL FORMULATION

We shall carry through the formulation in strict analogy to the argument in FJO, Eqs. (6)–(26). For convenience of comparison, we shall number the analogous equations here as (106)–(126). The notation is the same, with two exceptions: (a) Instead of the Cartesian coordinate x, we now have the variable r of cylindrical coordinates, the limits being r_1 and r_2; we assume $r_1 > r_2$, corresponding to $T_1 < T_2$, because we take the inner cylinder to be the hot one. (b) The symbol Q has a different meaning, $2\pi Q$ being the heat flow by conduction per unit length of the cylinders, in cal./cm-sec.

the theoretical point of view, see R. Clark Jones and W. H. Furry, Rev. Mod. Phys., submitted for publication. This article includes a summary of the results of the present paper.

[8] A. O. Nier, Phys. Rev. **57**, 30 (1940). The calculations made in the present paper show that the agreement is even better than is indicated by Nier.

[9] For example, in the case described in reference 8.

The equations of the temperature field are now

$$2\pi Q = 2\pi r\lambda(-dT/dr),$$

$$Q \log (r_1/r_2) = \int_{T_1}^{T_2} \lambda dT, \qquad (106)$$

$$(\partial/\partial r) = -(Q/r\lambda)(\partial/\partial T).$$

For $\mathbf{v}$ parallel to z, $\partial v/\partial z = \partial v/\partial\varphi = 0$, the hydrodynamical equation in cylindrical coordinates is

$$r^{-1}(\partial/\partial r)r\eta(\partial v/\partial r) = (dp/dz) + \rho g, \qquad (107)$$

and substitution from (106) gives

$$(Q^2/\lambda r^2)(d/dT)(\eta/\lambda)(dv/dT) = (dp/dz) + \rho g. \qquad (108)$$

The boundary conditions are, as before,

$$v(T_1) = v(T_2) = 0. \qquad (109)$$

The equation for the flux of species 1 is, of course, just Eq. (10) of FJO:

$$\mathbf{J}_1 = \rho[\mathbf{v}c_1 + D(-\text{grad } c_1 + \alpha c_1 c_2 \text{ grad log } T)]. \qquad (110)$$

(In order that this equation be strictly correct, the density ρ should be considered as the density the gas would have if all of the molecules were of species 1.)

The formulation given in FJO is based on certain simplifying assumptions which are not really essential and the use of which can be avoided by a procedure given by Bardeen.[10] Although the argument of Bardeen can readily be extended to the present case, we shall, for simplicity and brevity, adhere to the procedure of FJO.

We accordingly apply the stationary condition,

$$\text{div } \mathbf{J}_1 = 0, \qquad (111)$$

which, when we use (110) and (106), becomes

$$(\partial/\partial T)(\rho D/\lambda)[(\partial c_1/\partial T) - (\alpha c_1 c_2/T)] = (\lambda\rho r^2/Q^2)[v(\partial c_1/\partial z) - D(\partial^2 c_1/\partial z^2)]. \qquad (112)$$

As in FJO, we omit the term in $(\partial^2 c_1/\partial z^2)$; the effect of longitudinal diffusion is of course to be included later in a simpler way. We use, then, the equation

$$(\partial/\partial T)(\rho D/\lambda)[(\partial c_1/\partial T) - (\alpha c_1 c_2/T)] = (\lambda\rho r^2/Q^2)v(\partial c_1/\partial z). \qquad (112')$$

[10] J. Bardeen, Phys. Rev. **58**, 94 (L) (1940). The extension of this argument to the cylindrical case is given essentially by changing the variable x to r and replacing ρ by $r\rho$ throughout.

In terms of the function G defined by

$$(\partial c_1/\partial z)G(z, T)=(r\lambda Q^3/\rho D)J_{1r} \quad (113a)$$

$$=Q^4[(\partial c_1/\partial T)-(\alpha c_1c_2/T)], \quad (113b)$$

(112′) can be written

$$(\partial/\partial T)(\rho D/\lambda)(\partial c_1/\partial z)G(z, T)=\lambda\rho Q^2r^2v(\partial c_1/\partial z). \quad (114)$$

The final assumption made by FJO is that $(\partial^2c_1/\partial z\partial T)=0$. By just the same arguments as were used in FJO (Eqs. (15)–(17)), this leads to two consequences: The total transport of gas along the tube is zero, and G is a function of T only. Equation (114) can then be solved for v:

$$v(T)=(1/\lambda\rho Q^2r^2)(\partial/\partial T)(\rho DG(T)/\lambda). \quad (118)$$

By substitution of (118) in (108) and differentiation with respect to T, we obtain the differential equation for $G(T)$:

$$\frac{d}{dT}\frac{1}{\lambda r^2}\frac{d}{dT}\frac{\eta}{\lambda}\frac{d}{dT}\frac{1}{\lambda\rho r^2}\frac{d}{dT}\left(\frac{\rho D}{\lambda}G(T)\right)=g\frac{d\rho}{dT}. \quad (120)$$

The boundary conditions can readily be obtained from (113a), (109), and (118):

$$G(T_1)=G(T_2)=G'(T_1)=G'(T_2)=0. \quad (121)$$

We can now obtain the expression for τ_1, the upward transport of species 1. By (106) we have

$$\tau_1=2\pi\int_{r_2}^{r_1}\rho c_1vrdr=(2\pi/Q)\int_{T_1}^{T_2}\rho\lambda c_1vr^2dT. \quad (115)$$

Substitutions of (118) in (115) gives

$$\tau_1=(2\pi/Q^3)\int_{T_1}^{T_2}c_1(d/dT)(\rho DG(T)/\lambda)dT. \quad (122)$$

By partial integration and use of (121) and (113b) we now obtain the transport equation in standard form,

$$\tau_1=Hc_1c_2-K_c\partial c_1/\partial z, \quad (123)$$

with coefficients given by

$$H=-(2\pi/Q^3)\int_{T_1}^{T_2}(\rho D\alpha/\lambda T)G(T)dT, \quad (124)$$

and

$$K_c=(2\pi/Q^7)\int_{T_1}^{T_2}(\rho D/\lambda)\{G(T)\}^2dT. \quad (125)$$

In order to obtain the correct transport equation, we must add to (123) a term which gives the effect of diffusion along the tube, since this effect was omitted when (112) was replaced by (112′). The term required is $-K_d\partial c_1/\partial z$, where

$$K_d=2\pi\int_{r_2}^{r_1}\rho Drdr.$$

On using (106), we obtain

$$K_d=(2\pi/Q)\int_{T_1}^{T_2}\lambda\rho Dr^2dT. \quad (126)$$

The problem for the cylindrical case has now been given a formulation which is, formally, precisely analogous to that of the plane case. The practical difficulties of solution, however, are enormously increased. This is due to the appearance in the differential equation (120) of the factor r, which must be obtained as a function of T by the integration of Eq. (106). This leads to complicated calculations even when the temperature dependences of the properties of the gas are assumed to be very simple.

II. THE MAXWELLIAN CASE

Formulation in Terms of Dimensionless Quantities

We assume that the quantities

$$(\lambda/T),\ (\eta/\lambda),\ (\rho D/\lambda),\ (\rho T)$$

are all independent of T. From (106) we then obtain by integration

$$2\log r=-(\lambda/QT)T^2+\text{const.} \quad (1)$$

We now introduce the dimensionless variable t and the dimensionless parameters t_1, t_2:

$$t=(\lambda/QT)^{\frac{1}{2}}T,\quad t_i=(\lambda/QT)^{\frac{1}{2}}T_i. \quad (2)$$

The quantities t_1 and t_2 can be determined from

$$t_2/t_1=T_2/T_1, \quad (3)$$

$$t_2^2-t_1^2=2\log(r_1/r_2). \quad (4)$$

To save writing we introduce a length r_0, defined by writing (1) in the form

$$r^2=r_0^2\exp(-t^2). \quad (5)$$

We also introduce a new master function γ which differs from G by a constant factor:

$$G(T)=-(QT/\lambda)^{\frac{1}{2}}(\lambda^4\rho g/\eta D)r_0^4\gamma(t). \quad (6)$$

The differential equation (120) now takes the form

$$\frac{d}{dt}t^{-1}\exp(t^2)\frac{d^2}{dt^2}\exp(t^2)\frac{d\gamma}{dt}=t^{-2}, \qquad (7)$$

with the boundary conditions

$$\gamma(t_1)=\gamma(t_2)=\gamma'(t_1)=\gamma'(t_2)=0. \qquad (8)$$

The formulas for H, K_c, and K_d reduce to

$$H=2\pi\cdot\{\rho^2gt^3/\eta\}\cdot r_0^4\int_{t_1}^{t_2}(\alpha\gamma/t)dt, \qquad (9)$$

$$K_c=2\pi\cdot\{\rho^3g^2t^7/\eta^2D\}\cdot r_0^8\int_{t_1}^{t_2}\gamma^2dt, \qquad (10)$$

$$K_d=2\pi\{\rho D/t\}\cdot r_0^2\int_{t_1}^{t_2}t^2\exp(-t^2)dt. \qquad (11)$$

In these expressions the quantities in curly brackets are independent of the temperature.

The results of the computations may be stated conveniently in terms of the following definitions of dimensionless "shape factors" h, k_c, k_d:

$$H=(2\pi/6!)\{\alpha\rho^2g/\eta\}_1\cdot r_1^4\cdot h, \qquad (12)$$

$$K_c=(2\pi/9!)\{\rho^3g^2/\eta^2D\}_1\cdot r_1^8\cdot k_c, \qquad (13)$$

$$K_d=2\pi\cdot\{\rho D\}_1\cdot r_1^2k_d, \qquad (14)$$

$$h=6!t_1^3\exp(2t_1^2)\int_{t_1}^{t_2}(\alpha/\alpha_1)(\gamma/t)dt, \qquad (15)$$

$$k_c=9!t_1^7\exp(4t_1^2)\int_{t_1}^{t_2}\gamma^2dt, \qquad (16)$$

$$k_d=t_1^{-1}\exp(t_1^2)\int_{t_1}^{t_2}t^2\exp(-t^2)dt. \qquad (17)$$

Here the quantities in curly brackets are not independent of the temperature, and have been given the subscript 1 to indicate that they are to be evaluated at the lower temperature T_1. In the extreme cylindrical case this is the most reasonable simple choice of a temperature characteristic of the gas as a whole, since only a very small part of the gas is at temperatures near T_2. Equation (15) has been written in a form which can be applied when the quantity α varies with the temperature; α_1 is the value of α at the temperature T_1.

The numerical factors in Eqs. (12)–(14) have been chosen to correspond to those in the formulas for the plane case as given by FJO, with the correlations $B\rightarrow 2\pi r_1$, $2w\rightarrow r_1$.

For the "nearly plane case," r_1/r_2 not very large, it is convenient to define the shape factors h', k_c', k_d':

$$H=(2\pi/6!)\{\alpha\rho^2g/\eta\}\cdot\tfrac{1}{2}(r_1+r_2)(r_1-r_2)^3(2u)^2h', \qquad (18)$$

$$K_c=(2\pi/9!)\{\rho^3g^2/\eta^2D\}\cdot\tfrac{1}{2}(r_1+r_2)(r_1-r_2)^7(2u)^2k_c', \qquad (19)$$

$$K_d=2\pi\cdot\{\rho D\}\cdot\tfrac{1}{2}(r_1+r_2)(r_1-r_2)\cdot k_d', \qquad (20)$$

$$h'=6!\cdot\bar{t}^3\cdot 2[\exp(-\tfrac{1}{2}t_1^2)+\exp(-\tfrac{1}{2}t_2^2)]^{-1}\cdot[\exp(-\tfrac{1}{2}t_1^2)-\exp(-\tfrac{1}{2}t_2^2)]^{-3}\cdot(2u)^{-2}\int_{t_1}^{t_2}(\alpha/\bar{\alpha}_1)(\gamma/t)dt. \qquad (21)$$

$$k_c'=9!\cdot\bar{t}^7\cdot 2[\exp(-\tfrac{1}{2}t_1^2)+\exp(-\tfrac{1}{2}t_2^2)]^{-1}\cdot[\exp(-\tfrac{1}{2}t_1^2)-\exp(-\tfrac{1}{2}t_2^2)]^{-7}\cdot(2u)^{-2}\cdot\int_{t_1}^{t_2}\gamma^2dt, \qquad (22)$$

$$k_d'=\bar{t}^{-1}\cdot 2[\exp(-t_1^2)-\exp(-t_2^2)]^{-1}\cdot\int_{t_1}^{t_2}t^2\exp(-t^2)dt. \qquad (23)$$

Here the quantities in curly brackets are to be evaluated at the arithmetical mean temperature $\bar{T}=(T_2+T_1)/2$. We have used the abbreviations

$$\bar{t}=(t_2+t_1)/2, \qquad (24)$$

$$u=(\Delta T/2\bar{T})=(T_2-T_1)/(T_2+T_1)=(t_2-t_1)/(t_2+t_1). \qquad (25)$$

In the limit $(r_1-r_2)/r_1\rightarrow 0$, the factors h', k_c', k_d' approach limits

$$\begin{aligned} h'&\rightarrow f(2u)=1+u^2/7+u^4/21+\cdots,\\ k_c'&\rightarrow 1,\\ k_d'&\rightarrow 1+u^2/3, \end{aligned} \qquad (26)$$

and Eqs. (18)–(20) become identical with the results of FJO for the plane case, with the identifications $B\rightarrow\pi(r_1+r_2)$, $2w\rightarrow r_1-r_2$. With these identifications, the coefficients of h', k_c', and k_d' in Eqs. (18)–(20) are identical with the $H^{(0)}$, $K_c^{(0)}$, and $K_d^{(0)}$ of reference 7.

Since by (7) and (8), the function γ depends only on t, t_1, and t_2, it follows from (15)–(17) and (21)–(23) that the shape factors h, k_c, and k_d, and h', k_c', k_d' depend only on t_1 and t_2. By writing

Eqs. (3) and (4) in the form

$$t_1^2=\frac{2\log(r_1/r_2)}{(T_2/T_1)^2-1}, \tag{3'}$$

$$t_2^2=\frac{2\log(r_1/r_2)}{1-(T_1/T_2)^2}, \tag{4'}$$

it is evident that t_1 and t_2 are in turn functions only of the two ratios r_1/r_2 and T_2/T_1. *The shape factors are therefore functions only of the two numbers* r_1/r_2 *and* T_2/T_1. Our problem is to determine the values of the shape factors for any given values of these two ratios. This is done by computing the shape factors as functions of the convenient dimensionless parameters t_1 and t_2, whose values are related to those of r_1/r_2 and T_2/T_1 by (3′) and (4′).

The Formal Expression for the Master Function

In order to obtain the solution of (7) which satisfies (8), we must obtain a particular integral of (7) and four linearly independent integrals of the homogeneous equation

$$\frac{d}{dt}t^{-1}\exp(t^2)\frac{d^2}{dt^2}\exp(t^2)\frac{dz}{dt}=0. \tag{7'}$$

We denote the particular integral by y, and, since $z=constant$ obviously satisfies (7′), we may denote the integrals of (7′) by u, v, w, 1. The solution of (7) which satisfies (8) is then

$$\gamma=\epsilon/\delta, \tag{27}$$

where

$$\epsilon=\begin{vmatrix} y & u & v & w & 1\\ y_1 & u_1 & v_1 & w_1 & 1\\ y_2 & u_2 & v_2 & w_2 & 1\\ y_1' & u_1' & v_1' & w_1' & 0\\ y_2' & u_2' & v_2' & w_2' & 0\end{vmatrix}, \tag{28}$$

$$\delta=\text{cofactor of } y \text{ in } \epsilon. \tag{29}$$

Here the prime means differentiation with respect to t, and subscripts 1, 2 indicate that the function is evaluated at t_1, t_2, respectively. (The use of u as one of the integrals of (7′) should not be confused with the use of the same symbol as the parameter defined by (25).)

In carrying out the computation of γ, the determinant (28) may be manipulated in any of the usual ways, and its order may be reduced; but in so doing it is convenient to impose the following conditions: (a) The sign affixed to the resulting determinant must be positive. (b) The function $y(t)$ must occur in only one element, and with coefficient +1. Under these conditions one may still use Eq. (27), with δ defined as the cofactor of the element which contains $y(t)$.

The functions y, u, v, w can all be expressed in terms of exponentials and error functions. For shortness and flexibility we shall introduce the following special notations:

$$e=\exp(-t^2),\quad e_i=\exp(-t_i^2),\quad (i=1,2), \tag{30}$$

$$f=\int^t \exp(-x^2)dx,\qquad f_i=\int^{t_i}\exp(-x^2)dx,\quad (i=1,2), \tag{31}$$

$$g=\int^t \exp(-2x^2)dx,\qquad g_i=\int^{t_i}\exp(-2x^2)dx,\quad (i=1,2). \tag{32}$$

The lower limits are left unspecified, but are to be the same throughout any expression in which these symbols appear. We also define

$$\begin{aligned} f_{0t}&=\int_0^t \exp(-x^2)dx,\\ f_{t\infty}&=\int_t^\infty \exp(-x^2)dx,\\ f_{12}&=\int_{t_1}^{t_2}\exp(-x^2)dx,\\ f_{it}&=\int_{t_i}^{t}\exp(-x^2)dx=-f_{ti},\\ f_{0i}&=\int_0^{t_i}\exp(-x^2)dx,\\ f_{i\infty}&=\int_{t_i}^\infty \exp(-x^2)dx,\quad (i=1,2); \end{aligned} \tag{33}$$

and g_{0t}, $g_{t\infty}$, etc., are to have corresponding meanings.

A set of solutions suitable for use in (28) is

$$y=\tfrac{1}{2}ef-g;\quad u=f^2;\quad v=f;\quad w=e. \tag{34}$$

Equation (27) holds independently of our choice of the lower limit of the integrals. We may also replace (34) by a set in which an arbitrary linear combination of u, v, w, 1 is added to y, and in which u, v, w are replaced by any three such linear combinations which, together with 1, make a linearly independent set.

III. THE NEARLY PLANE CASE

We now consider the case in which r_2 is not small compared to r_1, so that the plane case formulas retain the significance of first approximations. The factors h', k_c', k_d' defined in Eqs. (18)–(23) can here be expressed as power series in $\log (r_1/r_2)$, the leading terms being given by (26).

From (3) and (4) we see that if $\log (r_1/r_2)$ is small, and T_2/T_1 is not too close to unity, then t_1, t_2, and, *a fortiori*, t are small numbers. We proceed by expanding γ in ascending powers of these quantities. As a first step it is desirable to choose the set of solutions y, u, v, w, 1 in such a way that the first terms in their series expansions are linearly independent, in order to avoid needless complication and heavy cancellation in the calculations. This can be done by considering the equation

$$\frac{d}{dt}t^{-1}\frac{d^3\gamma}{dt^3}=t^{-2}, \tag{35}$$

to which (7) reduces if we set $\exp (t^2)\rightarrow 1$. This is, apart, from a constant dimensional factor, the equation for the Maxwellian plane case as used by FJO. For this equation the simplest choice of a set of functions y, u, v, w, 1 is obviously

$$\begin{aligned} &y=-t^3/6,\\ &u=t,\quad v=t^2,\quad w=t^4. \end{aligned} \tag{36}$$

We therefore wish to take our set of functions in the forms:

$$\begin{aligned} y&=-(t^3/6)\{1+p_2t^2+p_4t^4+\cdots\},\\ u&=t\{1+a_2t^2+a_4t^4+\cdots\},\\ v&=t^2\{1+b_2t^2+b_4t^4+\cdots\},\\ w&=t^4\{1+d_2t^2+d_4t^4+\cdots\}. \end{aligned} \tag{37}$$

This can be accomplished by setting

$$\begin{aligned} y&=\tfrac{1}{2}f(e+1)-g,\\ u&=f,\\ v&=1-e,\\ w&=6(1-e-f^2). \end{aligned} \tag{38}$$

We then have

$$\begin{aligned} &p_2=-7/10, && p_4=3/10, && \cdots,\\ &a_2=-1/3, && a_4=1/10, && \cdots,\\ &b_2=-1/2, && b_4=1/6, && \cdots,\\ &d_2=13/15, && d_4=61/140, && \cdots. \end{aligned} \tag{39}$$

Using (37), we can carry out the expansion of the determinants ϵ and δ, defined by (28) and (29), in terms of the quantities:

$$\epsilon(klmn)=\begin{vmatrix} t^k & t^l & t^m & t^n & 1\\ t_1^k & t_1^l & t_1^m & t_1^n & 1\\ t_2^k & t_2^l & t_2^m & t_2^n & 1\\ kt_1^{k-1} & lt_1^{l-1} & mt_1^{m-1} & nt_1^{n-1} & 0\\ kt_2^{k-1} & lt_2^{l-1} & mt_2^{m-1} & nt_2^{n-1} & 0 \end{vmatrix}; \tag{40}$$

$$\delta(lmn)=\begin{vmatrix} t_1^l & t_1^m & t_1^n & 1\\ t_2^l & t_2^m & t_2^n & 1\\ lt_1^{l-1} & mt_1^{m-1} & nt_1^{n-1} & 0\\ lt_2^{l-1} & mt_2^{m-1} & nt_2^{n-1} & 0 \end{vmatrix}. \tag{41}$$

The result is:

$$\begin{aligned} -6\epsilon=\epsilon(3124)&+[d_2\epsilon(3126)+p_2\epsilon(5124)]\\ &+[(a_4-a_2p_2)\epsilon(3524)\\ &+(b_4-b_2d_2)\epsilon(3164)+d_4\epsilon(3128)\\ &\quad+p_4\epsilon(7124)+d_2p_2\epsilon(5126)]+\cdots, \end{aligned} \tag{42}$$

$$\begin{aligned} \delta=\delta(124)&+[a_2\delta(324)+d_2\delta(126)]\\ &+[a_4\delta(524)+a_2d_2\delta(326)\\ &+(b_4-b_2d_2)\delta(164)+d_4\delta(128)]+\cdots. \end{aligned} \tag{43}$$

Here each quantity in square brackets is a homogeneous function of t_1, t_2, t, whose degree exceeds by two that of the preceding quantity.

One can without much effort evaluate those of the determinants (41) which are involved in the three terms written in (43). It is convenient to write the results in the notation introduced in (24), (25):

$$\begin{aligned} \delta(124)&=64\bar{t}^5u^4,\\ \delta(234)&=\bar{t}^2(1-u^2)\delta(124),\\ \delta(126)&=\bar{t}^2(5+3u^2)\delta(124),\\ \delta(146)&=\bar{t}^4(15-2u^2+3u^4)\delta(124),\\ \delta(128)&=2\bar{t}^4(7+14u^2+3u^4)\delta(124),\\ \delta(245)&=\bar{t}^4(5-6u^2+u^4)\delta(124),\\ \delta(236)&=3\bar{t}^4(3-2u^2-u^4)\delta(124). \end{aligned} \tag{44}$$

The calculation of the quantities $\epsilon(klmn)$ is facilitated by the fact that these functions and their first derivatives vanish at $t=t_1$ and $t=t_2$, so that $(t-t_1)^2(t-t_2)^2$ is a factor of $\epsilon(klmn)$. One can see by inspection what powers of t occur and what coefficient t^k has, and this usually suffices to

TABLE I. Series coefficients for the nearly plane case with α = constant.

u	T_2/T_1	f	h_1	h_2	k_{c1}	k_{c2}	k_{d1}	k_{d2}
0.0	1	1	0	−0.174	0	−0.213	0	0
0.1	11/9	1.001	0.101	−0.169	0.172	−0.198	−0.033	−0.0002
0.2	3/2	1.006	0.203	−0.154	0.344	−0.153	−0.066	−0.001
0.3	13/7	1.013	0.307	−0.128	0.516	−0.078	−0.098	−0.002
0.4	7/3	1.024	0.415	−0.092	0.688	+0.028	−0.129	−0.003
0.5	3/1	1.039	0.527	−0.044	0.861	+0.163	−0.158	−0.005
0.6	4/1	1.061	0.648	+0.018	1.033	+0.329	−0.186	−0.007

determine the form of $\epsilon(klmn)$. For example, we see that

$$\begin{aligned}\epsilon(3126) &= \delta(126)t^3 + l_0 + l_1 t + l_2 t^2 + l_6 t^6 \\ &= (At^2 + Bt + C)(t-t_1)^2(t-t_2)^2.\end{aligned}$$

The requirements that the coefficient of t^3 be $\delta(126)$ and that the coefficients of t^4 and t^5 vanish provide three equations from which A, B, C can be determined.

We shall write the results in terms of the quantities t and u, and a variable s:

$$\begin{gathered} s = (2t - t_2 - t_1)/(t_2 + t_1); \\ t = \bar{t}(1+us), \\ (t-t_1)^2(t-t_2)^2 = \bar{t}^4 u^4 (1-s^2)^2, \\ dt = \bar{t}u\,ds, \\ t_1 \leqslant t \leqslant t_2, \quad -1 \leqslant s \leqslant 1. \end{gathered} \tag{45}$$

We find:

$$\begin{aligned}
\epsilon(3124) &= -\delta(124)(\bar{t}^3 u^4/4)(1-s^2)^2, \\
\epsilon(3126) &= \bar{t}^2(15 + 2u^2 + 6us + u^2 s^2)\epsilon(3124), \\
\epsilon(5124) &= \bar{t}^2(-10 + 2u^2 - 4us)\epsilon(3124), \\
\epsilon(3524) &= \bar{t}^4\{-5 + 6u^2 - u^4 \\ &\qquad -4(1-u^2)us\}\epsilon(3124), \\
\epsilon(3164) &= \bar{t}^4\{45 - 24u^2 - u^4 + 12(3-u^2)us \\ &\qquad +2(3-u^2)u^2 s^2\}\epsilon(3124), \\
\epsilon(3128) &= \bar{t}^4\{70 + 56u^2 + 3u^4 \\ &\qquad +8(7+2u^2)us + 2(14+u^2)u^2 s^2 \\ &\qquad +8u^3 s^3 + u^4 s^4\}\epsilon(3124), \\
\epsilon(5126) &= \bar{t}^4\{50 - 10u^2 + 4u^4 \\ &\qquad +40us + 2(5+u^2)u^2 s^2\}\epsilon(3124), \\
\epsilon(7124) &= \bar{t}^4\{-105 - 14u^2 \\ &\qquad +3u^4 - 4(21+2u^2)us \\ &\qquad -28u^2 s^2 - 4u^3 s^3\}\epsilon(3124).
\end{aligned} \tag{46}$$

From (27), (42), (43), (39), (44), and (46) we obtain:

$$\begin{aligned}
\gamma = \{\bar{t}^3 u^4 (1-s^2)^2/24\} \cdot \{1 + \bar{t}^2[-2 - (u^2/5) \\
-(12us/5) - (13u^2 s^2/15)] \\
+\bar{t}^4[2 + (4u^2/15) + (139u^4/2100) \\
+(24us/5) + (104u^3 s/525) + (24u^2 s^2/5) \\
+(239u^4 s^2/3150) + (16u^3 s^3/7) \\
+(61u^4 s^4/140)] + \cdots\}.
\end{aligned} \tag{47}$$

The integrals required for h' and k_c' can be evaluated by use of the formulas

$$\int_{-1}^{1}(1-s^2)^2 s^n ds = \begin{cases} 0 \text{ for odd } n, \\ \dfrac{16}{(n+1)(n+3)(n+5)} \text{ for even } n \end{cases} \tag{48}$$

$$\int_{-1}^{1}(1-s^2)^4 s^n ds = \begin{cases} 0 \text{ for odd } n, \\ \dfrac{768}{(n+1)(n+3)(n+5)(n+7)(n+9)} \text{ for even } n, \end{cases} \tag{49}$$

and the integration for k_d is of course performed by expanding $\exp(-t^2)$ in power series. The other factors occurring in (21)–(23) must also be expressed as power series in $\bar{t}$, and multiplied by the series obtained by integration. The results can then be expressed as power series in $\log(r_1/r_2)$ by using the relation

$$2\bar{t}^2 u = \log(r_1/r_2), \tag{50}$$

which follows from (4), (24), and (25).

We finally obtain the series:

$$\begin{aligned} h' = h_0(u) + h_1(u)\cdot\ln(r_1/r_2) \\ + h_2(u)\cdot\{\ln(r_1/r_2)\}^2 + \cdots, \end{aligned} \tag{51}$$

$$\begin{aligned} k_c' = 1 + k_{c1}(u)\cdot\ln(r_1/r_2) \\ + k_{c2}(u)\cdot\{\ln(r_1/r_2)\}^2 + \cdots, \end{aligned} \tag{52}$$

$$\begin{aligned} k_d' = 1 + (u^2/3) + k_{d1}(u)\cdot\ln(r_1/r_2) \\ + k_{d2}(u)\cdot\{\ln(r_1/r_2)\}^2 + \cdots. \end{aligned} \tag{53}$$

For α independent of T we get:

$$\begin{aligned}
h_0(g) &= f(2u) = 1 + 0.143u^2 + 0.048u^4 + \cdots, \\
h_1(g) &= (9u/10)f(2u) + (23/30u)\{f(2u) - 1\} \\
&= 1.010u + 0.165u^3 + 0.059u^5 + \cdots, \\
h_2(g) &= \{-(17/60) + (3499/8400)u^2\}f(2u) \\
&\quad + \{(23/30u^2) + (53/72)\}\cdot\{f(2u) - 1\} \\
&\quad - (37/80u^2)\{f(2u) - 1 - (u^2/7)\} \\
&= -0.174 + 0.496u^2 + 0.088u^4 + \cdots.
\end{aligned} \tag{54}$$

These coefficients can of course be computed readily enough when α is any polynomial in T, or any power series which converges rapidly in the range in question. For example, for $\alpha \propto T$ we obtain

$$\begin{aligned}
h_0(u) &= 1, \\
h_1(u) &= 176u/210 = 0.838u, \\
h_2(u) &= -(73/420) + (571/1575)u^2 \\
&= -0.174 + 0.363u^2.
\end{aligned} \tag{55}$$

The coefficients which occur in k_c and k_d are

$$k_{c1}(u)=284u/165=1.721u.$$
$$k_{c2}(u)=-(703/3300)+(339044/225225)u^2 \quad (56)$$
$$=-0.213+1.505u^2,$$

$$\begin{aligned} k_{d1}(u)&=-(u/3)\{1-(u^2/5)\}\\ &=-0.333u+0.067u^3,\\ k_{d2}(u)&=-(u^2/15)\{(1/3)-(u^2/7)\}\\ &=-0.022u^2+0.010u^4. \end{aligned} \quad (57)$$

The amount of labor required to obtain successive terms in this series formulation increases very rapidly, so that it does not seem worth while to go beyond the terms given here. The three terms we have obtained are presumably adequate at least up to $r_1/r_2=3$; some comparisons with the results of numerical computations from the exact formulas are given later. The radii of convergence of the series of course remain unknown.

Some values of the various coefficients are given in Table I. The quantities h_1 and h_2 are for α independent of temperature. The table is not extended beyond $u=0.6$ because large values of T_2/T_1 are not practicable in the nearly plane case.

IV. SOLUTION BY NUMERICAL INTEGRATION

The integral in Eq. (17) can be expressed easily in terms of known functions. The result is

$$k_d=\tfrac{1}{2}-(t_2/2t_1)\exp(t_1^2-t_2^2)+\tfrac{1}{2}f_{12}. \quad (58)$$

The values of the integrals in (15) and (16) can likewise be worked out, for any chosen simple temperature dependence of α, in terms of exponentials, error functions, and a small number of new functions defined by integrals. Such a formulation would have the aesthetic advantage of involving only the values at t_1 and t_2 of various functions of a single variable. The formulas would, however, be so extremely complicated that the only way to give them any practical meaning would be to carry out a quite tedious process of numerical evaluation for each given pair of values t_1, t_2. Actually the most feasible procedure for determining the behavior of the functions h and k_c seems to be to choose various specific pairs of values of t_1, t_2, plot the function γ for each such pair, and perform numerical integrations *ad hoc* according to (15) and (16). This is essentially what we shall do. In order to carry out such a program with some economy of effort, it is desirable to have available two different algebraic translations of the formula (27) for the function γ.

Formulation with t_1 as Lower Limit of All Integrals

If t_1 is taken as the lower limit of all the integrals used in (31) and (32) to define the symbols used in writing (34), then several elements in the second and fourth rows of the determinant (28) vanish automatically. The evaluation of the determinants ϵ and δ is then not difficult, and one obtains from (27):

$$\gamma=\tfrac{1}{2}(e+e_1)f_{1t}-g_{1t}-Af_{1t}^2+C_1(e_1-e-2t_1f_{1t}), \quad (59)$$

where

$$A=\frac{2f_{12}(t_2e_1-t_1e_2-t_1t_2f_{12})-\tfrac{1}{2}(e_1-e_2)^2-2(t_2-t_1)g_{12}}{2f_{12}\{(t_1+t_2)f_{12}-(e_1-e_2)\}}; \quad (60)$$

$$C_1=\frac{f_{12}(3e_2+e_1+2t_2f_{12})-4g_{12}}{4\{(t_1+t_2)f_{12}-(e_1-e_2)\}}. \quad (61)$$

This formulation is convenient for dealing with a set of cases in which t_1 remains the same.

Formulation with t_2 as Lower Limit

The evaluation of the determinants is of course simplified just as much if t_2 is taken as the lower limit in (31) and (32). For convenience we reverse the limits of all integrals, with corresponding sign changes, in writing the result:

$$\gamma=g_{t2}-\tfrac{1}{2}(e+e_2)f_{t2}-Af_{t2}^2+C_2(2tf_{t2}+e_2-e) \quad (62)$$

where

$$C_2=\frac{f_{12}(3e_1+e_2-2t_1f_{12})-4g_{12}}{4\{(t_1+t_2)f_{12}-(e_1-e_2)\}}. \quad (63)$$

This formulation would be more convenient for dealing with a set of cases in which t_2 remains the same. It can also be used as a starting point in obtaining an approximate solution for cases in which $r_1\gg r_2$.

The Extreme Cylindrical Case

The parameter t_2 occurs in (62) in two quite different ways: (a) As a factor in certain terms in numerators and denominators; (b) in the exponential e_2 and as a limit of integrals. The limit $(r_2/r_1)\to 0$ corresponds to $t_2\to\infty$; the result of setting $t_2\to\infty$ in (62) is not, however, a very good approximation to any practical case, since $(r_2/r_1)=10^{-3}$ corresponds only to $t_2\sim 4$. A good approximation is rather to be obtained by setting $t_2\to\infty$ only where it occurs as described in (b); we thus set

$$f_{t2}\to f_{t\infty},\quad g_{t2}\to g_{t\infty},\quad e_2\to 0. \tag{64}$$

This procedure introduces errors which are only of order $(r_2/r_1)^2$. The fractional errors in the results are then of order $(r_2/r_1)^2$, unless the terms retained cancel heavily. Strong cancellation will indeed occur for low values of t_2 (less than about 2.5), and in practice this is what limits the application of the formulas obtained by these approximations.

The physical basis for the distinction between the two ways in which t_2 appears is quite simple. The finite radius of the wire affects the values of H, K_c, and K_d: (a) because it exerts a restraining effect on the convection, and (b) because the wire occupies a certain amount of cross-sectional area. The second effect is obviously of the order $(r_2/r_1)^2$, and it is the second effect which we are ignoring in the approximations (64). The terms which we are retaining are inversely proportional to t_2, and therefore depend only logarithmically on r_1/r_2.

We may show by a simple analogy that the restraining effect of the wire on the convection may be expected to depend logarithmically on r_1/r_2. Consider the case of ordinary pressure-driven lamellar flow between two concentric cylinders of radii r_1 and r_2. In the approximation in which we ignore terms of the order $(r_2/r_1)^2$ compared with unity, the total rate of flow F is given by[11]

$$F=F_0\left(1-\frac{1}{\log(r_1/r_2)}\right), \tag{65}$$

where F_0 is the rate of flow (given by Poiseuille's formula) for $r_2/r_1=0$. We see from this relation that for a ratio of radii as high as 100, the presence of the wire reduces the flow by 21.7 percent.

When (64) is substituted into (62), (60), and (63), we obtain after some algebraic manipulation the expression

$$\gamma\cong\frac{\gamma_\infty-(\gamma''/t_2)}{1-(\delta''/t_2)}, \tag{66}$$

where

$$\gamma_\infty=g_{t\infty}-\tfrac{1}{2}ef_{t\infty}+(g_{1\infty}+t_1f_{1\infty}{}^2-e_1f_{1\infty})(f_{t\infty}/f_{1\infty})^2 +(\tfrac{3}{2}e_1f_{1\infty}-2g_{1\infty}-t_1f_{1\infty}{}^2)(f_{t\infty}/f_{1\infty}), \tag{67}$$

$$\gamma''=\delta''(g_{t\infty}-\tfrac{1}{2}ef_{t\infty})+(t_1g_{1\infty}-\tfrac{1}{4}e_1{}^2)(f_{t\infty}/f_{1\infty})^2 +(\tfrac{3}{2}e_1f_{1\infty}-2g_{1\infty}-t_1f_{1\infty}{}^2)(e/2f_{1\infty}), \tag{68}$$

$$\delta''=(e_1/f_{1\infty})-t_1. \tag{69}$$

We now make a further approximation similar to those indicated in (64) by replacing t_2 by ∞ as the upper limit of the integrals in (15) and (16). We thus obtain

$$h\cong 6!t_1{}^3\exp(2t_1{}^2)\cdot\frac{h_\infty-(h''/t_2)}{1-(\delta''/t_2)}, \tag{70}$$

$$k_c\cong 9!t_1{}^7\exp(4t_1{}^2)\cdot\frac{k_\infty-(2k_1''/t_2)+(k_2''/t_2{}^2)}{\{1-(\delta''/t_2)\}^2}, \tag{71}$$

with

$$h_\infty=\int_{t_1}^\infty(\alpha/\alpha_1)(\gamma_\infty/t)dt,\qquad h''=\int_{t_1}^\infty(\alpha/\alpha_1)(\gamma''/t)dt, \tag{72}$$

$$k_\infty=\int_{t_1}^\infty\gamma_\infty{}^2dt,\qquad k_1''=\int_{t_1}^\infty\gamma_\infty\gamma''dt,\qquad k_2''=\int_{t_1}^\infty\gamma''^2dt. \tag{73}$$

Equations (67)–(73) express h and k asymptotically in terms of the quantities h_∞, h'', k_∞, k_1'', k_2'', which are functions of t_1 only. The sole dependence on t_2 is that indicated by the explicit appearance of t_2 in (70) and (71).

It is important to remember that (70) and (71) do *not* simply indicate the first terms in power series or asymptotic series in $(1/t_2)$: the error

[11] See, for example, L. Page, *Introduction to Theoretical Physics*, second edition (D. Van Nostrand Company, Inc., New York, 1935), p. 260.

involved in using (70) and (71) is not given by a power of $(1/t_2)$, but is of the order of $\exp(-t_2^2) \sim (r_2/r_1)^2$, as we have already discussed in some detail.

The formulas (70) and (71) are adapted, and much more strongly so than (59), to the treatment of sets of cases in which t_1 is held fixed and t_2 is given various values. It is interesting to note what such cases have in common physically. From Eq. (2) we see that if Q and T_1 are given, t_1 is determined. Thus if the outer radius r_1, the lower temperature T_1, and the heat dissipated by conduction per unit length per second, $2\pi Q$, are given, the factors H, K_c, K_d are completely determined by (12)–(14), except for the dependences of the factors h, k_c, k_d on t_2. Any two cases of such a set have the same temperature field throughout the region which is occupied by gas in both cases. The specification of r_2 then fixes the inner boundary of this region and, by (3) and (4), the temperature at this boundary.

In experimental language, one may say that such a set of cases is realized when one has a number of identical outer tubes, maintained at the same temperature, but containing coaxial wires of various sizes, provided that the temperatures of the wires are adjusted so that the heat flow by conduction is the same per unit length in all the tubes.

Numerical Data for Use in the Asymptotic Formulas

The functions γ_∞ and γ'' can be tabulated readily, since excellent tables are available for all the functions involved.[12–14] In order to avoid interpolation, we obtained $f_{t\infty}$ from Burgess's tables[12] and $g_{t\infty}$ from the British Association Tables.[13] The elimination of interpolation in this manner was a considerable saving of time because heavy cancellation made it necessary to obtain numbers from the tables accurate to nine places for the smaller values of t. In the worst case, the use of nine significant figures led to results accurate to three.

In Table II we give the values of γ_∞ and γ'', so that they may be available for numerical use in applying (70) with any specified temperature dependence of α.

In Table III are listed the coefficients (72) and (73), the values h_∞ and h'' being for α independent of temperature.

Comparison of Exact and Approximate Results

The function γ was tabulated for $t_1 = \frac{1}{2}$ and several values of t_2, the calculations being based on the exact formula (59). The coefficients h and k_c were then calculated by numerical integration from (15) and (16), for the case $\alpha \equiv \alpha_1$.

The comparison of these exact results with the results obtained from the approximate formulas (70), (71) is given in Table IV. It is seen that the agreement is excellent for $(r_1/r_2) = 20$, and is tolerably good even for $(r_1/r_2) = 6.5$.

In Table V we compare the exact results with the results obtained from the series expressions (51) and (52) for the nearly plane case; again we take $\alpha \equiv \alpha_1$. The table also shows the comparison between the results obtained from (58) and those from (53), for the shape factor k_d'. The comparison is made in terms of the quantities h', k_c', k_d', which differ from h, k_c, k_d only by easily computed factors (cf. Eqs. (21)–(23) and (15)–(17)). It is seen that the agreement is excellent for $(r_1/r_2) = 1.45$, is fairly good for $(r_1/r_2) = 2.72$, and is passable even for $(r_1/r_2) = 6.5$.

These comparisons indicate that the two types of approximation taken together cover the whole range fairly well.

Tables for the Extreme Cylindrical Case

By using the results given in Table III, we can compute from (70) and (71) the values of h and k_c for a chosen value of $(r_1/r_2) \gg 1$ and for certain values of $(T_2/T_1) = (t_2/t_1)$ which are determined by the available values of t_1. By graphical interpolation we can then obtain h and k_c for other values of T_2/T_1. Tables VI and VII were constructed in this way. Table VIII gives the

[12] J. Burgess, Trans. Roy. Soc. Edinburgh **39**, 257 (1899). This article contains tables of $H_1(t) = (2/(\pi)^{\frac{1}{2}})f_{0t}$:

$t = 0.000$ to 1.250, interval 0.001, 9 places,
$t = 1.000$ to 1.500, 0.001, 15 places,
$t = 1.500$ to 3.000, 0.002, 15 places,
$t = 3.0$ to 5.0, 0.1, 15 places.

[13] *British Association Mathematical Tables* (Cambridge University Press, London, 1931), Vol. 1. On pp. 60–71 there is a table of $Hh_0(2t) = 2g_{t\infty}$:

$2t = -7.0$ to 6.6, 0.1, 10 places.

[14] F. W. Newman, Trans. Camb. Phil. Soc. **13**, 146 (1883). This article contains tables of the descending exponential e^{-x}:

$x = 0.000$ to 15.349, 0.001, 12 places.

TABLE II. Values of γ_∞ and γ''.

t	$t_1=0.5$ $\gamma_\infty\times10^6$	$t_1=0.5$ $\gamma''\times10^6$	$t_1=0.8$ $\gamma_\infty\times10^7$	$t_1=0.8$ $\gamma''\times10^7$	$t_1=1.0$ $\gamma_\infty\times10^7$	$t_1=1.0$ $\gamma''\times10^7$	$t_1=1.2$ $\gamma_\infty\times10^8$	$t_1=1.2$ $\gamma''\times10^8$
0.5	0	0						
0.6	187	283						
0.7	542	835						
0.8	867	1360	0	0				
0.9	1073	1714	465	792				
1.0	1141	1863	1242	2148	0	0		
1.1	1096	1831	1826	3216	155	286		
1.2	975	1672	2078	3731	391	730	0	0
1.3	816	1439	2037	3738	543	1032	450	891
1.4	650	1181	1805	3393	585	1133	1069	2150
1.5	497	931	1486	2866	543	1075	1403	2871
1.6			1153	2288	456	925	1428	2980
1.7	261	523	854	1746	357	742	1256	2678
1.8					264	564	1002	2187
1.9	122	263	418	911	186	410	745	1668
2.0					126	288	524	1207
2.1	52	120	181	423	83	196	353	837
2.3	20	50	71	178	33	83	144	364
2.5	7	19	25	68	12	31	53	142
2.7	2	7	8	24	4	9	18	53
2.9		2	2	8	1	2	6	19
3.1				2			2	6

values of k_d calculated from (58) for the same values of r_1/r_2 and T_2/T_1.

V. DISCUSSION

Behavior of the Transport Coefficients in the Extreme Cylindrical Case

A rather surprising feature of the results given in Tables VI and VII is the great difference between the dependences of H and K_c on T_2/T_1 in the highly cylindrical case and in the plane case. Whereas in the plane case both coefficients increase (as $(\Delta T/\bar{T})^2$) with increasing T_2/T_1, in the highly cylindrical case H is rather insensitive to this ratio and K_c *decreases* strongly as T_2/T_1 increases. This result would hardly be expected without calculation, but it can be made plausible by physical considerations. For high values of T_2, the gas near the hot wire becomes highly conductive of heat and very viscous. This has the effect of reducing the convective flow. Now H depends both on the convective flow, which is decreased, and on the temperature gradient, which is increased. K_c, on the other hand, depends only on the convective flow, and indeed quadratically.

This behavior of H and K_c should be of considerable utility in practice. It seems likely that it depends rather strongly on the temperature dependences of the gas coefficients, and for this reason calculations for other dependences are desirable.

It is probable that the value of H would be rather strongly influenced by any marked temperature dependence which the factor α might have. Since the assumption that α depends on a power of the temperature is quite artificial and probably decidedly lacking in verisimilitude, numerical integrations were not carried through for such cases. Whenever the temperature dependence of α for a given gas may become reasonably well known empirically, it can be used together with the results tabulated in Table II to carry out an *ad hoc* numerical integration for the value of H.

It is easily shown[15] that the effectiveness of a unit length of column is measured by $H^2/(K_c+K_d)$. By adjusting the value of r_1, however, it is always possible to make K_d any specified multiple of K_c. The power efficiency of a column may thus be measured by H^2/K_cQ. For any given gas this ratio is independent of r_1, provided that the temperature T_1 is held constant, as will usually be the case in practice, and is proportional to $h^2t_1^2/k_c$. This factor, which may be considered as a measure of the relative power efficiency, is tabulated in Table IX.

Perhaps the most interesting feature of the

[15] See reference 7, Eq. (333).

TABLE III. Values of h_∞, h'', etc.

t_1	0.5	0.8	1.0	1.2
h_∞	0.838×10^{-3}	1.113×10^{-4}	0.2565×10^{-4}	0.5314×10^{-5}
h''	1.409×10^{-3}	2.076×10^{-4}	0.5113×10^{-4}	1.132×10^{-5}
k_∞	0.729×10^{-6}	2.178×10^{-8}	0.1575×10^{-8}	0.897×10^{-10}
k_1''	1.221×10^{-6}	4.029×10^{-8}	0.3112×10^{-8}	1.892×10^{-10}
k_2''	2.056×10^{-6}	7.47×10^{-8}	0.6161×10^{-8}	3.995×10^{-10}
δ''	1.3327	1.5070	1.6390	1.7809

TABLE IV. Comparison of exact and approximate results for the extreme cylindrical case with $t_1=\frac{1}{2}$.

t_2	r_1/r_2	h(exact)	h(from (70))	k_c(exact)	k_c(from (71))
3	79.8	0.984×10^{-1}	0.984×10^{-1}	0.356×10^{-2}	0.356×10^{-2}
2.5	20.1	0.875×10^{-1}	0.872×10^{-1}	0.286×10^{-2}	0.287×10^{-2}
2	6.52	0.666×10^{-1}	0.594×10^{-1}	0.170×10^{-2}	0.145×10^{-2}

TABLE V. Comparison of exact and approximate results for the nearly plane case with $t_1=\frac{1}{2}$.

t_2	r_1/r_2	h' (exact)	h' (series)	k_c' (exact)	k_c' (series)	k_d' (exact)	k_d' (series)
1	1.45	1.127	1.127	1.207	1.209	0.996	0.991
1.5	2.72	1.475	1.522	1.964	2.024	0.930	0.919
2	6.52	2.066	2.35	4.02	4.09	0.814	0.745

results tabulated in Table IX is the fact that the entries vary as little as they do. The largest entry is only 1.73 times the smallest. The table further indicates that with a small temperature ratio, the concentric tube type of construction is the more efficient, whereas for gases which permit a large temperature ratio, the hot wire type of construction is more efficient.

Comparison with Experiment

A rather extensive comparison of experiment with the theory presented in this paper, as well as with other aspects of the theory of the separation column, is given in Part IV of reference 7. In particular, we present there a detailed comparison with the results of Clusius and Dickel, of Nier, and of Taylor and Glockler. The agreement is found to be good.

In order to avoid unnecessary duplication of material, we shall present here only the comparison with the experimental work of Nier.[8]

The column used by Nier had the specifications

$$r_1=2.458\text{ cm};\quad r_2=1.746\text{ cm};\quad T_1=300^\circ\text{K};\quad T_2=573^\circ\text{K};\quad \log(r_1/r_2)=0.342;\quad u=0.313. \tag{74}$$

At the mean temperature 436.5°, we find from the recent work of Trautz and Sorg[16] that the coefficient of viscosity of methane is

$$\eta=1.51\times10^{-4}\text{ poise,} \tag{75}$$

and from the same reference we find that at this temperature the coefficient of viscosity varies as $T^{0.78}$. That is to say, the power by which η depends on the temperature is

$$n=0.78. \tag{76}$$

We have further

$$\rho=4.47\times10^{-4}\text{ g/cm}^3 \tag{77}$$

if we consider that methane is chiefly $C^{12}H_4$.

The value of D may now be estimated on the basis of the inverse power model from a formula given by one of the writers.[17] From Eqs. (29) and (31), and Table I of reference 17, we find from (76) that

$$D=1.406\eta/\rho=0.475\text{ cm}^2/\text{sec.} \tag{78}$$

The value of α has been determined experi-

TABLE VI. Values of h with α=constant.

T_2/T_1 \ r_1/r_2	15	25	40	60	100
2	0.059	0.059	—	—	—
3	0.091	0.098	0.100	0.101	0.100
4	0.092	0.103	0.109	0.113	0.116
5	0.075	0.092	0.103	0.108	0.114
6	—	—	0.085	0.093	0.102

TABLE VII. Values of k_c.

T_2/T_1 \ r_1/r_2	15	25	40	60	100
2	0.0144	0.0184	—	—	—
3	0.0095	0.0130	0.0162	0.0183	0.0207
4	0.0045	0.0068	0.0088	0.0105	0.0128
5	0.0022	0.0034	0.0046	0.0056	0.0072
6	—	—	0.0025	0.0031	0.0040

TABLE VIII. Values of k_d.

T_2/T_1 \ r_1/r_2	15	25	40	60	100
2	0.61	0.60	—	—	—
3	0.75	0.73	0.70	0.68	0.67
4	0.91	0.87	0.83	0.81	0.78
5	1.08	1.01	0.97	0.94	0.90
6	—	—	1.11	1.07	1.03

[16] M. Trautz and K. Sorg, Ann. d. Physik **10**, 81 (1931).
[17] R. Clark Jones, Phys. Rev. **58**, 111 (1940).

mentally by Nier,[18] who finds

$$\alpha = 0.0077. \tag{79}$$

These data are all for a pressure of one atmosphere.

Substituting the data (74)–(79) in Eqs. (18)–(20), we find

$$\begin{aligned} H/h' &= 2.585\times 10^{-5}\text{ g/sec.},\\ K_c/k_c' &= 1.041\times 10^{-2}\text{ g-cm/sec.},\\ K_d/k_d' &= 1.997\times 10^{-3}\text{ g-cm/sec.}\end{aligned} \tag{80}$$

Furthermore, we find upon substituting (74) in Eqs. (51)–(57)

$$h' = 1.109;\quad k_c' = 1.176;\quad k_d' = 0.998. \tag{81}$$

Combining (80) and (81), we have finally

$$\begin{aligned} H &= 2.866\times 10^{-5}\text{ g/sec.},\\ K_c &= 1.224\times 10^{-2}\text{ g-cm/sec.},\\ K_d &= 1.992\times 10^{-3}\text{ g-cm/sec.}\end{aligned} \tag{82}$$

On the basis of the transport equation

$$\tau_1 = Hc_1c_2 - (K_c+K_d)(\partial c_1/\partial z), \tag{83}$$

it has been shown by Furry, Jones, and Onsager[3] that the logarithm of the equilibrium separation factor is

$$\log q_e = \frac{HL}{K_c+K_d}, \tag{84}$$

where L is the length of the column. Since it follows directly from (18)–(20) that the coefficients H, K_c, and K_d are proportional, respectively, to the second, fourth, and zeroth power of the pressure, we have from (82) and (84)

$$\log q_e = \frac{1.710/P^2}{1+0.1628/P^4}, \tag{85}$$

where P is measured in atmospheres, and where we use $L = 7.3$ meters.

Nier found that his three experimental points could be fitted exactly by the formula

$$\log q_e = \frac{1.34/P^2}{1+0.126/P^4}. \tag{86}$$

If we multiply numerator and denominator of this expression by 1.283, we have

$$\log q_e = \frac{1.72_0/P^2}{1.283+0.161_7/P^4}. \tag{87}$$

Comparison of (87) with (85) indicates that the assumption of a parasitic remixing (cf. Eq. (70) of reference 3) given by

$$K_p/K_c = 0.283, \tag{88}$$

leads to a discrepancy between theory and experiment of 0.6 percent. The excellence of this check is of course fortuitous, since the physical constants used are not known to this accuracy; in particular, the probable error in the measurement of α is several percent.

It will be remembered that Nier found a discrepancy of about 10 percent, and a value of K_p/K_c equal to[19] about 0.78. The improved check and the much smaller value of K_p/K_c found in the calculation given here are due primarily to the inclusion of the corrections (81) for the cylindricity of the apparatus, and secondarily, to the use of the better viscosity data of Trautz and Sorg.

TABLE IX. Values of $h^2t_1^2/k_c$ with α = constant.

T_2/T_1 \ r_1/r_2	1	15	25	40	60	100
2	0.46	0.44	0.41	—	—	—
3	0.54	0.59	0.59	0.57	0.57	0.56
4	0.54	0.68	0.67	0.66	0.66	0.64
5	0.51	0.58	0.67	0.71	0.71	0.69
6	0.48	—	—	0.61	0.65	0.68

[18] A. O. Nier, Phys. Rev. **56**, 1009 (1939).

[19] Nier actually found $K_p/K_c = 0.63$, but this larger value follows if we match the coefficients of P^{-2} and P^{-4} with equal fractional errors, instead of placing all of the burden on one of them.