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On the Theory of the Slowing Down of Neutrons in Heavy Substances[1]

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The rigorous solution of the integral equation for the stationary energy distribution in the case of slowing down without capture is given. Its relation to the average energy loss is discussed (Sections 1, 2). In Section 3, the problem of mixtures is treated. It is shown, in particular, that for mixtures of a light and a heavy substance concentrations exist for which the energy distribution is at the beginning entirely determined by the energy loss in the heavy substance and at the end by the energy loss in the light substance. The effects of capture are discussed in Sections 4–6. It is shown that the solution assumes a simpler form if the ratio of the mean free paths for scattering and capture varies slowly over energy regions of the order of the average energy loss. The case of $1/v$—capture is treated in detail (Section 5); rapidly varying capture is discussed in Section 6. In Section 7, a simplified treatment, based on the concept of neutron age, is given, and its limitations are discussed. Section 8 contains the discussion of the effects of the chemical binding on the energy distribution. An expression for the mean square distance of diffusion and its mass dependence is derived (Section 9).

The process of slowing down neutrons by elastic collisions has been studied chiefly in hydrogeneous substances[2–4] where the protons, because of their large scattering cross section, are the chief agents for slowing down. However, for a number of problems, such as the production of small energy changes for the investigation of nuclear energy levels,[5–7] as well as the study of the diffusion in the atmosphere of neutrons, produced by the cosmic radiation,[8] the knowledge of the energy distribution of neutrons slowed down by collisions with nuclei of higher mass is essential. An expression for the distribution in energy of neutrons which have suffered a given number of collisions has been given by Condon and Breit.[9] In the present paper the derivation of the general energy distribution with and without capture will be attempted.

1. The Integral Equation of the Slowing Down Process

We consider a homogeneous medium of infinite extension in which per second Q neutrons of energy E_0 are produced. The energy of these neutrons is changed by collisions. We ask for the stationary energy distribution. So long as we are not interested in the spatial distribution of the neutrons, the distribution of the sources in space is immaterial. We will first suppose that no capture of neutrons occurs. If we measure the energy by $x=E_0/E$, and use for convenience a logarithmic energy scale, we may write for the number of neutrons between $\log x$ and $\log x+d(\log x)$:

$$\rho(x)dx/x=Q[l(v)/v]K(x)dx/x. \tag{1}$$

Introduction of the quantity $K(x)$ defined by Eq. (1) considerably simplifies the following analysis. In (1), v represents velocity; $l(v)$, mean free path for scattering. $l(v)/v$ is the lifetime of a neutron of velocity v against a scattering collision. The quantity $K(x)(dx/x)$ represents the average number of collisions a neutron experiences in the interval between $\log x$ and $\log x+d\log x$.

We further denote by $k(x, x')dx/x$ the chance that a neutron shall be in the interval dx after one collision, if its energy before the collision was E_0/x'. The function $k(x, x')dx/x$ will depend on the interaction. For its integral we have

$$\int_0^\infty k(x, x')dx/x=1.$$

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* This paper was received for publication on the date indicated but was voluntarily withheld from publication until the end of the war.

[1] Part of the results has been presented at the 1939 Washington meeting of the American Physical Society, cf. Phys. Rev. **55**, 1130 (1939).

[2] E. Amaldi and E. Fermi, Phys. Rev. **50**, 899 (1936).

[3] E. Fermi, *Zeeman Verhandelingen* (s-Gravenhage, M. Nijhoff, 1935), p. 128.

[4] E. Fermi, Ricerca Scient. [2] **7**, 13 (1936).

[5] B. Pontecorvo and G. C. Wick, Ricerca Scient. [1] **7**, 134, 220 (1936).

[6] E. Fermi and F. Rasetti, Ricerca Scient. **9**, 472 (1938).

[7] J. Hornborstel and F. A. Valente, Phys. Rev. **55**, 108 (1939).

[8] H. A. Bethe, S. A. Korff, and G. Placzek, Phys. Rev. **57**, 573 (1940).

[9] E. U. Condon and G. Breit, Phys. Rev. **49**, 229 (1936).

Under stationary conditions the number of collisions in an interval dx/x must be equal to the number of collisions which bring a neutron into this interval. The number of collisions which bring a neutron from the interval dx'/x' into the interval dx/x is

$$K(x')k(x, x')(dx/x)(dx'/x').$$

Therefore we get as condition for a stationary distribution:

$$K(x) = k(x, 1) + \int_0^\infty K(x')k(x, x')dx'/x'. \quad (2)$$

The first term on the right-hand side takes care of the neutrons which reach the interval dx/x directly in one collision from the source.

If the collision is always connected with a loss in energy, we may narrow the limits of the integral in (2), writing

$$K(x) = k(x, 1) + \int_1^x K(x')k(x, x')dx'/x'. \quad (2a)$$

If capture occurs, it has to be accounted for by multiplying the collision function $k(x, x')$ with the relative probability of scattering, *viz.*, $f(x') = l(x')/l_s(x')$, where l is the total mean free path (for capture and scattering) and l_s the mean free path for scattering alone:

$$K(x) = f(1)k(x, 1) + \int_1^x f(x')K(x')k(x, x')dx'/x'. \quad (3)$$

$K(x)dx/x$ remains the average number of collisions in the interval dx/x, which may now be of either the scattering or capturing type. (1) remains unchanged if we understand by l the total mean free path.

I. SLOWING DOWN WITHOUT CAPTURE

2. Substances Containing a Single Type of Nuclei Only

(*A*) *Protons*

If the neutron energy is large compared to the vibration quanta of bound protons ($\sim\frac{1}{3}$ ev) the protons can be considered free and at rest. In this case the energy distribution after one collision is constant in the energy scale, which gives:

$$k(x, x') = \begin{cases} x'/x & \text{for } x > x' \\ 0 & \text{for } x < x', \end{cases} \quad (4)$$

so that (2a) becomes:

$$K(x) = \frac{1}{x}\left\{1 + \int_1^x K(x')dx'\right\}. \quad (5)$$

(5) is solved by $K(x) = 1$, which, introduced into (1), gives Fermi's[7, 8] well-known result.

(*B*) *Nuclei of Atomic Weight M*

For free nuclei initially at rest and an angular distribution of the scattered neutrons spherical in the center of gravity system, the energy distribution after one collision is again constant in the energy scale, but extends now only from E_0 to pE_0, where

$$p = (M - 1/M + 1)^2. \quad (6)$$

This gives

$$k(x, x') = \begin{cases} \dfrac{1}{1-p}\dfrac{x'}{x} & \text{for } px < x' < x \\ 0 & \text{for } x'/p < x \text{ and } x' > x, \end{cases} \quad (7)$$

and

$$K(x) = \frac{1}{(1-p)x}\left\{1 + \int_1^x K(x')dx'\right\} \quad \text{for } 1 < x < p^{-1}, \quad (8a)$$

$$K(x) = \frac{1}{(1-p)x}\int_{px}^x K(x')dx' \quad \text{for } x > p^{-1}. \quad (8b)$$

(8a) can be solved directly; its solution is

$$K_0(x) = \frac{1}{1-p}x^{p/(1-p)}. \quad (9)$$

The solution of (8b) is a function of x which is equal to its own mean value in the interval between px and x. If (8b) were valid for all values of x, its only solution would obviously be $K = \text{const}$. We shall see in the following that this is the asymptotic solution of (8b) for large x. Furthermore, we see from (8a, b) that K is discontinuous for $x = p^{-1}$; indeed we have

$$K_1(p^{-1}) - K_0(p^{-1}) = -p/(1-p)$$

and hence from (9):

$$K_1(p^{-1}) = \frac{p}{1-p}(p^{-1/(1-p)} - 1). \quad (10)$$

The physical meaning of this discontinuity is, of course, that for values of x larger than p^{-1} the neutrons can no longer enter the energy interval directly from the source.

In order to solve (8b), we shall make use of a substitution which will map the region of the variable x between 1 and ∞ on the interval between 1 and $1/p$. We call $K_n(x)$ the solution of (8b) for values of $p^{-n} \leqslant x \leqslant p^{-(n+1)}$, introduce the variable

$$z = \frac{p^{1/(1-p)}}{1-p} \log p^n x \tag{11}$$

which varies between $z=0$ for $x=p^{-n}$ and

$$z = p^{1/(1-p)} \log p^{-1/(1-p)} = \zeta \tag{11a}$$

for $x = p^{-(n+1)}$, and put

$$K_n(x) = K_0(x) J_n(z). \tag{12}$$

Substituting (11) and (12) into (8b) we get

$$\begin{gathered} J_n(z) = J_n(0) - \int_0^z J_{n-1}(z')dz' \quad \text{for } n>0, \\ J_0(z) = 1 \end{gathered} \tag{13}$$

and by recurrent application of (13)

$$J_n(z) = \sum_{m=0}^{n} J_{n-m}(0) \frac{(-z)^m}{m!}. \tag{14}$$

(14) expresses the value of the function we look for in terms of its values at the edges of the intervals. We have now to determine the latter values. We see from (8a, b) that K is continuous except for $x=p^{-1}$. Hence we have

$$J_n(0) = J_{n-1}(\zeta) \quad \text{for } n>1, \tag{15}$$

and because of (10)

$$J_1(0) = J_0(\zeta) - p^{1/(1-p)} = 1 - p^{1/(1-p)}. \tag{16}$$

We write now (14) for $z=\zeta$ and introduce (15) into it. This gives

$$J_{n-1}(\zeta) = J_n(0) = \sum_{m=0}^{n-1} J_{n-m-1}(0) \frac{(-\zeta)^m}{m!} \quad \text{for } n>1. \tag{17}$$

From (16) and (17) we can now determine $J_n(0)$ by iteration. The result is

$$J_n(0) = A_n(\zeta) - p^{1/(1-p)} A_{n-1}(\zeta), \tag{18}$$

$$\begin{gathered} A_n(\zeta) = \sum_{m=0}^{n-1} (m-n)^m \frac{\zeta^m}{m!}, \\ A_0(\zeta) = 1, \quad A_{-1}(\zeta) = 0. \end{gathered} \tag{18a}$$

Equation (18) together with (14), (13), and (9) contains the complete solution of the integral Eq. (8a, b). We shall first investigate its asymptotic behavior. As shown in Appendix I, the polynomial $A_n(\zeta)$ has for large n the asymptotic value

$$A_{n\,\mathrm{as}}(\zeta) = \frac{v_1^n}{1+\log v_1} + \frac{v_2^n}{1+\log v_2}, \tag{19}$$

where v_1 and v_2 are the two roots of the equation

$$-v \log v = \zeta, \tag{20}$$

which for $\zeta \rightarrow 0$ go to 0 and 1, respectively. From the definition (11a) of ζ, we see that

$$v_1 = p^{1/(1-p)}. \tag{21}$$

Now we have to find v_2. In general, it is not easy to find v_2 from v_1 (see Section 3). In the present case, however, it is possible. The result is:

$$v_2 = p^{p/(1-p)}. \tag{22}$$

From (21) and (22) one also gets the relation

$$v_2 = v_1/p = v_1^p. \tag{23}$$

We can now find the asymptotic value for the polynomial $J_n(0)$. From (18), (19), and (21) we have, taking also (23) into account:

$$J_{n\,\mathrm{as}}(0) = \frac{1-p}{1+\log v_2} v_2^n. \tag{24}$$

The term with v_1 has dropped out. In order to find $J_{n\,\mathrm{as}}(z)$ we may introduce (24) into (14) and replace the upper limit n of the sum in (14) by infinity. This gives:

$$J_{n\,\mathrm{as}}(z) = \frac{1-p}{1+\log v_2} v_2^n \exp(-z/v_2), \tag{25}$$

and, by making use of the definition (11) of z and of the relation (23)

$$J_{n\,\mathrm{as}} = \frac{1-p}{1+\log v_2} x^{-p/(1-p)}. \tag{26}$$

Now we can find the asymptotic expression for K by simply putting (26) and (9) into (12). We get at once:

$$K_{\mathrm{as}} = \frac{1}{1+\log v_2} = \frac{1}{1+[p/(1-p)] \log p}. \tag{27}$$

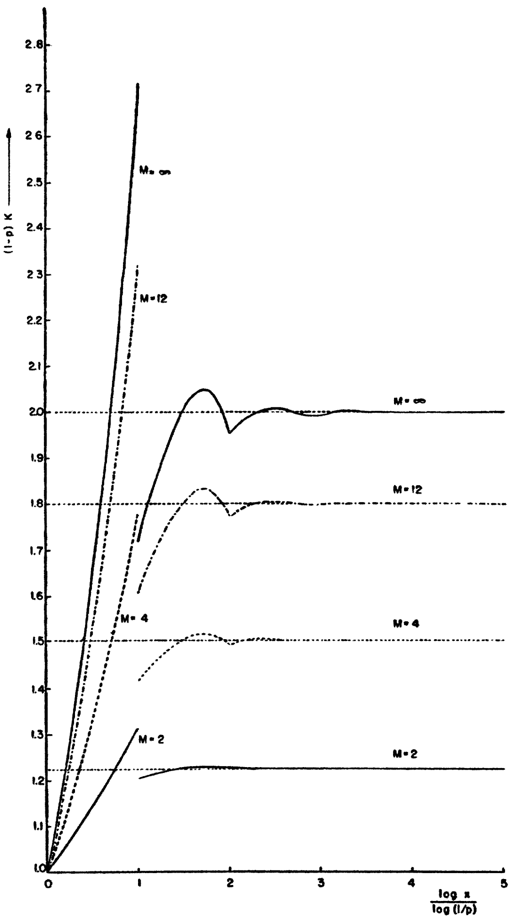

FIG. 1.

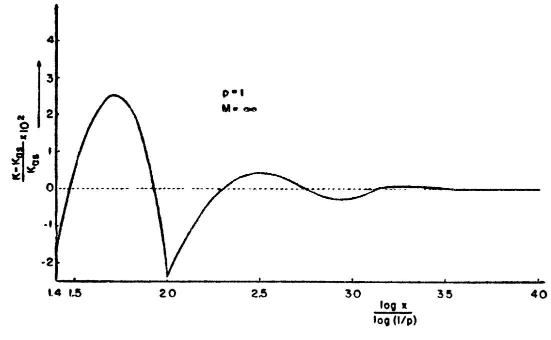

FIG. 2.

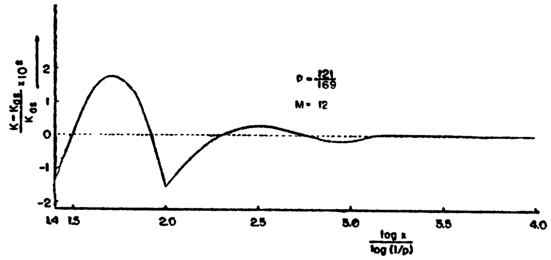

FIG. 3.

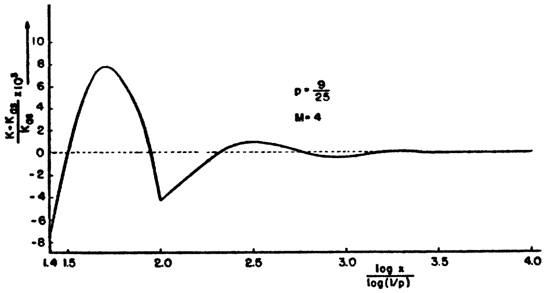

FIG. 4.

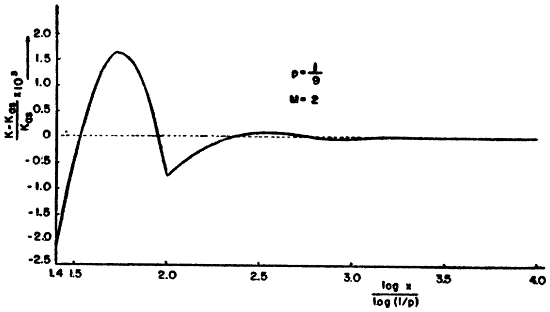

FIG. 5.

So K_{as} is a constant. Its physical significance is very simple: it is the reciprocal mean logarithmic energy loss in one collision.[10]

This means simply that the average number of collisions $K(x)dx/x$ in a given energy interval is asymptotically equal to the logarithmic width dx/x of that interval by the logarithmic width of the interval which is in the average traversed by one collision.

In order to see the dependence on M in a convenient direct way, the following expansion is useful:

$$K_{as}=\frac{1}{1+[p/(1-p)]\log p}=\frac{M}{2}+\frac{1}{3}+\frac{1}{18M}\left(1+\frac{16}{15M}+\frac{8}{45M^2}+\cdots\right). \tag{27a}$$

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[10] The mean logarithmic energy loss in one collision is given by

$$(\log x)_{Av}=\int_1^p (\log x')k(x',1)\frac{dx'}{x'}=\frac{1}{1-p}\int_1^{1/p}\log x'dx'=1+[p/(1-p)]\log p. \tag{28}$$

TABLE I. The values of K for the extrema between $y=0$ and $y=4$.

M	K	$\delta(y)\cdot 10^3$								
∞	$M/2$	-500	359	-141	25.4	-23.8	4.36	-2.74	0.83	-0.41
12	6.3384	-444	290	-108	17.8	-15.6	2.64	-1.55	0.44	-0.19
4	2.3513	-335	180	-58.6	7.95	-6.03	0.82	-0.43	0.10	-0.04
2	1.3787	-184	74	-16.7	1.63	-0.91	0.088	-0.034	0.0053	-0.001
$y(M=2)$		0		1	1.75	2.00	2.54	2.95	3.36	3.81
$y(M=\infty)$		0		1	1.72	2.00	2.50	2.94	3.31	3.78

From this expansion it is seen that $K_{\mathrm{as}} \approx \frac{1}{2}M+\frac{1}{3}$ is a good approximation already for moderate values of M. Indeed, even for $M=2$ the error is only 3.3 percent.

If we want to assume that K is asymptotically equal to a constant, then the value of the constant can be found by simpler ways. As shown in Appendix II, the integral equation (8a, b) can be transformed into the equivalent form

$$K(x)=\frac{1}{1-p}\left\{1+p\int_1^x K(x')\frac{dx'}{x'}\right\} \quad \text{for } px<1, \tag{29a}$$

$$K(x)=1+\frac{p}{1-p}\int_{px}^x K(x')\frac{dx'}{x'} \quad \text{for } px>1. \tag{29b}$$

Putting K in (29b) equal to the constant K_{as}, we find at once (27).[11] The behavior of the complete solution and the way in which the constant asymptotic value of K is reached is shown in Figs. 1–5, which give K as a function of $y=\log x/\log(1/p)$. This measure of the energy gives, for different masses, equal extensions to the maximal energy regions which may be traversed in one collision. As we have seen already, K is discontinuous for $x=1/p$, i.e., $y=1$. Therefore, as can be shown from (13), its nth derivative is discontinuous for $x=p^{-n}$ or $y=n$.

Figure 1 gives $(1-p)K$ as function of y for $y=0$ to $y=4$ for the masses $M=\infty$, 12, 4, and 2. K first rises according to (9), drops discontinuously at $y=1$, goes on rising to a maximum, reaches a minimum with discontinuous tangent at $y=2$, and approaches with oscillations of rapidly decreasing amplitudes the asymptotic constant value K_{as}. The oscillations can be better seen from Figs. 2–5, where the relative deviation from the asymptotic value, i.e., $(K-K_{\mathrm{as}})/K_{\mathrm{as}}$ is plotted on an enlarged scale for the same masses for $y=1$ to $y=4$. Table I gives numerical values for the extreme deviations of K from its asymptotic value. The first two columns give the masses and K_{as}, and the following 9 columns show the relative deviation from the asymptotic value:

$$\delta(y)\cdot 10^3=(K-K_{\mathrm{as}})/K_{\mathrm{as}}\cdot 10^3$$

for $y=0$, 1, and all the following extrema up to $y=4$. The values of y at which K is extreme are only slightly different for different masses; they are given in the last two rows for $M=2$ and $M=\infty$ only. (Table I.)

It is seen that the deviations are smaller for light nuclei. It must, however, be remembered that the maximum energy loss in one collision is smaller for heavy nuclei, so that the deviations, if plotted in the energy scale, extend farther for light nuclei. For a heavy nucleus K varies in the first interval by a factor e, but the interval comprises only the region between E_0 and $E_0[1-(4/M)]$, while e.g., for deuterium, the variation in the first interval is only 37 percent, but the interval extends from E to $E/9$. Since one has experimentally always to do with neutrons of a certain width in initial energy, the deviations of K from the constant value will, in general, be directly observable for light nuclei only. However, the importance of these variations lies not so much in the direct effects near E_0 but rather in the fact that they will start anew from every point where capture becomes important.

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[11] Formally equivalent to this derivation is the following one: The number of neutrons going per second from the region $x<x_0$ to the region $x>x_0$ must be equal to the number produced per second, i.e., to Q. This gives:

$$\frac{1}{1-p}\int_{x_0}^{x_0/p}\frac{dx}{x^2}\int_{px}^{x_0}K(x')dx'=1, \tag{30}$$

from which again (27) results by putting $K=K_{\mathrm{as}}$. Equation (30) can, of course, also be obtained from the integral equation by using together both forms (8b) and (29b).

3. Mixtures

If the medium in which the neutrons are slowed down contains nuclei of different atomic weights M_s, the energy distribution after one collision [cf. (7)] will be determined by the function:

$$k(x,x')=\mu\left(\frac{x'}{x}\right)\frac{x'}{x}\sum_s\frac{c_s(x')}{1-p_s}\mu\left(\frac{p_sx}{x'}\right). \tag{31}$$

The function $\mu(u)$ is 1 for $u<1$, 0 for $u>1$ and discontinuous for $u=1$. $c_s(x')=l(x')/l_s(x')$, where $l(x')$ is the total mean free path and $l_s(x')$ the mean free path for scattering at a nucleus of type s. c_s is a measure of the concentration, $\sum_s c_s=1$. Introducing the kernel (31) into the general Eq. (2), we get:

$$xK(x)=\sum_s\frac{1}{1-p_s}\left\{c_s(1)\mu(p_sx)+\int_{\lambda(p_sx)}^{x}c_s(x')K(x')dx'\right\}, \tag{32}$$

where

$$\lambda(u)=\mu(u)+u\{1-\mu(u)\}=\begin{cases}1 & \text{for } u<1\\ u & \text{for } u>1.\end{cases} \tag{33}$$

As in the discussion of Eq. (3), we see that $K(x)$ is discontinuous at all the points $x=p_s^{-1}$, the discontinuity being

$$K_+(p_s^{-1})-K_-(p_s^{-1})=-\frac{c_s(1)p_s}{1-p_s}. \tag{34}$$

If we arrange the quantities p_s according to their magnitude, calling the largest of them p_1, then in the interval $1<x<p_1^{-1}$ Eq. (32) becomes

$$xK(x)=\sum_s\frac{1}{1-p_s}\left\{c_s(1)+\int_1^x c_s(x')K(x')dx'\right\}. \tag{35}$$

This equation is solved by

$$K(x)=\sum_s\frac{c_s(1)}{1-p_s}\exp\left\{\sum_s\frac{p_s}{1-p_s}\int_1^x c_s(x')\frac{dx'}{x'}\right\}. \tag{36}$$

From this solution we can proceed to construct the complete solution in a manner similar to, though more complicated than, the treatment of the case of a single substance. We shall do this later for a significant particular case, and shall now derive the asymptotic solution.

We use the fact, that, as shown in Appendix II, Eq. (32) may be transformed into

$$K(x)=1+\sum_s\frac{p_s}{1-p_s}\left\{c_s(1)\mu(p_sx)+\int_{\lambda(p_sx)}^{x}c_s(x')K(x')\frac{dx'}{x'}\right\}. \tag{37}$$

For $x>p_k^{-1}$ (p_k denoting the smallest p), (37) reads:[12]

$$K(x)=1+\sum_s\frac{p_s}{1-p_s}\int_{p_sx}^{x}c_s(x')K(x')\frac{dx'}{x'}. \tag{38}$$

If all the mean free paths are constant or vary all in the same way with energy, the c_s are constants. In this case, (38) is again asymptotically solved by a constant K_∞, for which we find from (38) (with constant c_s):

$$K_\infty=\left[1+\sum_s\frac{c_sp_s}{1-p_s}\log p_s\right]^{-1}. \tag{39}$$

As can be easily checked this constant is again equal to the reciprocal average logarithmic energy loss in one collision. For variable mean free path it is easily seen that (39), with the c_s depending on x, is a good approximation provided the quantities c_s vary slowly enough with energy, so that the variation of (39) in a region of the extension of the average logarithmic energy loss is small.

We now derive the complete solution for the special case of a mixture of hydrogen with another substance, the concentration of which, in the above mentioned units, we denote by c. If the mean free paths are constant, the equation for this case becomes [cf. (32)]:

$$xK(x)=\left(1+\frac{cp}{1-p}\right)\left\{1+\int_1^x K(x')dx'\right\}\quad\text{for } px<1. \tag{40a},$$

$$xK(x)=\frac{c}{1-p}\int_{px}^{x}K(x')dx+(1-c)\left\{1+\int_1^x K(x')dx'\right\}\quad\text{for } px>1. \tag{40b}$$

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[12] If hydrogen ($p=0$) is present in the mixture, this region does not exist, but since the term referring to hydrogen in the sum in Eq. (37) drops out because of the p in the numerator, (38) is equally valid in this case for $x>p_{k-1}^{-1}$.

The equivalent form (37) of these equations is:

$$K(x)=\begin{cases}\left(1+\dfrac{cp}{1-p}\right)\left\{1+\displaystyle\int_1^x K(x')\frac{dx'}{x'}\right\} & \\ & px<1 \quad \text{(41a)}\\ 1+\dfrac{cp}{1-p}\displaystyle\int_{px}^x K(x')\frac{dx'}{x'} & px>1. \quad \text{(41b)}\end{cases}$$

The solution of (40a) or (41a) is according to (36):

$$K_0(x)=\left(1+\frac{cp}{1-p}\right)x^{cp/(1-p)}. \tag{42}$$

If we denote again by $K_n(x)$ the solution for $p^{-n}<x<p^{-n+1}$, the substitution

$$K_n(x)=K_0(x)G_n(u), \tag{43}$$

$$u=cv_2^{c-1}z \tag{43a}$$

[for z see Eq. (11)], where u runs from 0 to

$$\xi=cv_2^{c-1}\zeta, \tag{44}$$

leads to

$$G_n(u)=\sum_{m=0}^{n} G_{n-m}(0)\frac{(-u)^m}{m!}, \tag{45}$$

$$G_n(0)=A_n(\xi)-\frac{cpv_2^c}{1-p-cp}A_{n-1}(\xi), \tag{46}$$

where the polynomial A is defined by (18a). While (45) is identical with the corresponding Eq. (14) for the polynomials $J_n(z)$, (46) differs from (18) by the different variable (ξ instead of ζ) and the different factor of A_{n-1}. The latter arises from the fact that the discontinuity at $x=p^{-1}$ has now a different magnitude. The solution $K_n(x)$ is again an oscillating function. We introduce into (46) the asymptotic expression (19) for the A_n:

$$A_{n\,\mathrm{as}}(\xi)=\frac{s_1^n}{1+\log s_1}+\frac{s_2^n}{1+\log s_2}, \tag{47}$$

where s_1 and s_2 are now the solutions of

$$-s\log s=\xi \tag{48}$$

which for $\xi\to 0$ go to 0 and 1, respectively.

Introducing (47) into (46), we have

$$G_{n\,\mathrm{as}}(0)=\frac{s_2^n}{1+\log s_2}\frac{1-p}{1-p+cp}+\frac{s_1^n}{1+\log s_1}\left(1-\frac{cp}{1-p+cp}\frac{s_2}{s_1}\right) \tag{49}$$

and, using (43), (43a), (42), and (6),

$$K_{\mathrm{as}}(p^{-n})=\frac{1}{1+\log s_2}+\frac{(s_1/s_2)^n}{1+\log s_1}\times\left\{1+\frac{cp}{1-p}\left(1-\frac{s_2}{s_1}\right)\right\}. \tag{50}$$

Now we have to find the solutions $s_{1,2}$ of Eq. (48). From (20), (22), and (44) we see that

$$s_2=v_2^c=p^{cp/(1-p)}, \tag{51}$$

but, in contrast to the previous case, it now does not seem possible to express the other solution s_1 as a function of p and c in closed form.

However, part of the discussion can be done without explicit knowledge of s_1. We put

$$\frac{s_1}{s_2}=\frac{\log s_2}{\log s_1}=p^{\epsilon}. \tag{52}$$

Hence

$$\log s_1=p^{-\epsilon}\log s_2=\log s_2+\epsilon\log p. \tag{53}$$

This gives, if we insert the value of s_2 from (51) and put $p^{-1}=q$:

$$\frac{q^{\epsilon}-1}{q-1}=\frac{\epsilon}{c}. \tag{54}$$

Equation (54) has only one solution $\epsilon\geqslant 1$. For $c=0$, $\epsilon=\infty$; for $c=1$, $\epsilon=1$.

Introduction of (53) and (54) into (49) and (50) gives

$$G_{n\,\mathrm{as}}(0)=\frac{1-p}{1-p+cp}\left\{\frac{s_2^n}{1+\log s_2}+\frac{(1-\epsilon)s_1^n}{1+\log s_1}\right\}, \tag{49a}$$

$$K_{n\,\mathrm{as}}(q^n)=\frac{1}{1+\log s_2}+\frac{(\epsilon-1)q^{-n\epsilon}}{\epsilon\log q-\log s_2-1} \tag{50a}$$

and, using (49a), (43), (43a), (45), (51), (53), (54):

$$K_{\mathrm{as}}(x)=\frac{1}{1-\dfrac{c}{q-1}\log q}+\frac{(\epsilon-1)x^{-\epsilon}}{\left(\epsilon-\dfrac{c}{q-1}\right)(\log q)-1}. \tag{55}$$

From (54) it can be proved that the factor of $x^{-\epsilon}$ in (55) is always positive. Hence, K_{as} decreases with increasing x and goes finally over into the constant K_∞, given by the first term of (55) in agreement with the general expression (39). Various expansions for ϵ can be derived from (54) for various cases. In particular, for $1-c=c_2\ll 1$, $1-p=\alpha\ll 1$, one finds:

$$\epsilon-1=\frac{2c_2}{\alpha}\{1+O(\alpha)+O(c_2)\}. \tag{56}$$

Equation (55) then becomes

$$K_{\mathrm{as}}(x)=\frac{1}{\alpha/2+c_2+O(\alpha_1{}^2c_2{}^2,\ \alpha c_2)}+\frac{2c_2x^{-[1+(2c_2/\alpha)]}(1+O(\alpha,\ c_2)}{\alpha\{\alpha/2+c_2+O(\alpha^2,\ c_2{}^2,\ \alpha c_2)\}}. \tag{57}$$

Since $\alpha\ll 1$, we can apply this asymptotic expression for K already for values of x not noticeably different from one. Therefore, we can interpret (57) in the following way: K first reaches very rapidly the value $K_{\mathrm{as}}(1)=2/\alpha$, which is the final value if no hydrogen were present. With increasing x, the influence of hydrogen comes into play and K decreases until it finally reaches the constant value given by the first term of (57), i.e., the reciprocal mean logarithmic energy loss of a neutron in the mixture. This variation of K is especially important in the case $c_2\gg\alpha/2$, for which there is a considerable difference between the energy loss in the pure heavy substance and in the mixture, since the latter is entirely due to hydrogen. (57) here becomes

$$K_{\mathrm{as}}=\frac{1}{c_2}+\left(\frac{2}{\alpha}-\frac{1}{c_2}\right)x^{-[1+(2c_2/\alpha)]}. \tag{57a}$$

At the beginning the neutron density is here entirely determined by the heavy substance, at the end, entirely by hydrogen.[13] The transition takes place for $x_0\approx(2c_2/\alpha)\exp[\alpha/2c_2]$, or in numbers n of intervals: $n_0\approx(1/2c_2)\log(2c_1/\alpha)$, which is always a large number when the above approximation is valid. Moreover, as one can see from (32) to (36), the function K will in the first intervals, where it is still oscillating, be exactly identical with the function for the pure heavy substance, as shown in Figs. 1 and 2, and the transition to the entirely different final value will only start long after the oscillations have died down.

In the more general case that c_2 is no longer small compared to one the transition will take place earlier, but, in order that K should be determined by hydrogen from the very beginning, it is necessary, as we can see from (42), that $1-c_2=c\ll\alpha$, while for K to be determined by hydrogen only in the end it is, of course, sufficient that the mean energy loss in the mixture is the same as in hydrogen, *viz.*, $c_2\gg\alpha/2$. In the opposite case $c_2\ll\alpha/2$, (57) becomes

$$K_{\mathrm{as}}=\frac{2}{\alpha}\left\{1+\frac{2c_2}{\alpha}(x^{-[1+(2c_2/\alpha)]}-1)\right\} \tag{57b}$$

and the deviation from the case of the pure heavy substance is small throughout.[14]

The effects discussed here may be considered as characteristic in general for the slowing down process in a mixture of a light and a heavy substance.

II. SLOWING DOWN WITH CAPTURE

4. The Equations

Putting (7) into (3), we have

$$K(x)=\frac{1}{(1-p)x}\left\{f(1)+\int_1^x f(x')K(x')dx'\right\},\qquad px<1 \tag{59a}$$

$$K(x)=\frac{1}{(1-p)x}\int_{px}^x f(x')K(x')dx',\qquad px>1. \tag{59b}$$

The equivalent transformed equations are (cf. Appendix II):

$$K(x)=\frac{1}{1-p}\left\{1-g(1)+\int_1^x(p-g(x'))K(x')\frac{dx'}{x'}\right\},\qquad px<1 \tag{60a}$$

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[13] It is to be remembered that the neutron density is $Q(l/v)K(x)dx/x$, where $l=c_2l_H$ is the average mean free path, so that the factor c_2 drops out.

[14] This case requires some attention in connection with the treatment of capture. We consider the integral $\int_1^xK(x')dx'$ which enters into Eq. (40b). Writing

$$K(x)=K_\infty+M(x)$$

we find from (57b)

$$\int_1^\infty M(x)dx=2/\alpha; \tag{58}$$

c_2 has dropped out. In spite of this, (58) is *not* valid for $c_2=0$. This can be easily seen by noticing that the second part of (57b) contains c_2 as factor and therefore vanishes throughout for $c_2=0$. Actually, the integral (58) in this case is of the order $-\frac{1}{3}$ because of the contributions of the first intervals where the asymptotic expression (57b) is not yet applicable.

$$K(x)=1-g(1)-\int_1^x g(x')K(x')\frac{dx'}{x'}+\frac{p}{1-p}\int_{px}^x f(x')K(x')\frac{dx'}{x'},\quad px>1 \tag{60b}$$

where $g(x)=1-f(x)=1-l(x)/l_s(x)$. The solution of (59a) or (60a) is:

$$K(x)=\frac{f(1)}{1-p}x^{p/(1-p)}\times\exp\left[-\frac{1}{1-p}\int_1^x g(x')\frac{dx'}{x'}\right]. \tag{61}$$

For hydrogen ($p=0$) (59a) is valid everywhere, and therefore the complete solution in this case reduces to

$$K(x)=f(1)\exp\left[-\int_1^x g(x')\frac{dx'}{x'}\right], \tag{61a}$$

which is in accordance with Fermi's[3] result.

The solution in the region $px>1$ can in principle be found by repeated integration from (61), in the fashion of Section 2. This procedure has, however, only limited interest since in most of the practical cases the capture sets in only for energies considerably lower than the initial energy. Therefore, it is sufficient to investigate how the *asymptotic* solution of the equation without capture is modified by the capture. This is done by considering (59b) and (60b) valid for all values of x. The methods of solution will be different according as the variation of g in one interval is slow or rapid.

Before entering into these methods, it will be useful to discuss Eq. (60b) somewhat more closely. Multiplying (60b) with

$$a=1+p\log p/(1-p)$$

the average logarithmic energy loss in one collision, we have

$$S(x)=aP(x)+\frac{p}{1-p}\int_{px}^x f(x')S(x')\frac{dx'}{x'}, \tag{62}$$

where

$$S(x)=aK(x)$$

is the ratio between K and the asymptotic solution of the equation without capture (see (27)) and gives thus the relative reduction of K by the capture, and

$$P(x)=1-g(1)-\int_1^x g(x')K(x')\frac{dx'}{x'}$$

is the probability that the neutron is not captured below x.

Adding $p\log p/1-p$ to both sides of (62) and dividing by a, we obtain

$$S(x)=P(x)+\frac{p}{(1-p)a}\int_{px}^x\{f(x')S(x')-S(x)\}\frac{dx'}{x'}. \tag{63}$$

In the presence of the second term in (63) consists the essential difference between the mechanism of capture in heavy substances and in hydrogen, for which it is zero. It is of special importance if the capture is large, or small but rapidly varying, in which cases it entirely modifies the solution. Its physical meaning is that the effect of capture occurring at a point x_1 on the value of S at a point x_2 depends on the distance between these points. This may be understood by first considering the case in which the total change of S in the region considered is small. In this case all the capturing points may be considered as independent sources of neutrons taken away of amount $g(x_1)(dx_1/x_1)$, the change in S due to every point x, will then be given by the initial solution of the equation without capture and thus exhibit considerable variations, as shown in Fig. 1, which will extend a few intervals beyond x_1. If the total change is no longer small, these variations will in turn influence each other.

If capture occurs only above a certain energy, S will approach a constant value a few intervals below this energy, whereupon the second term in (63) will vanish, so that the value of the constant is given by

$$S=P=1-\frac{1}{a}\int g(x')S(x')\frac{dx'}{x'}. \tag{64}$$

If the change in S is small, we may replace $S(x')$ in the integral by one and have

$$S=1-\frac{1}{a}\int g(x')\frac{dx'}{x'}. \tag{64a}$$

If the capture occurs in a number of discrete regions, separated by regions without capture as may be the case with absorption lines, we may apply (64a) for every region separately, writing

$$S_k - S_{k-1} = \frac{S_k}{a}\int_k g(x')\frac{dx'}{x'}, \tag{64b}$$

where S_k is the value of S before entering the capturing region k and the integral has to be taken over this region. From (64b) we may find the final S even if the total change in S is large, provided the changes due to every single region are small.

5. Slowly Varying Capture

We first consider the case of constant capture. Here, the assumption that the capture vanishes in the first intervals is not valid. However, it can be easily seen that the solution obtained by considering (59b) and (60b) valid everywhere will differ from the asymptotic solution of (59a, b) only by a constant factor. If we use the homogeneous form (59b) of the equation, the solution will contain anyhow an undetermined constant factor, which, besides, is immaterial for the following. (59b) with constant f

$$S(x) = \frac{f}{(1-p)x}\int_{px}^{x} S(x')dx' \tag{65}$$

is solved by

$$S(x) = Cx^{-\nu_0}, \tag{66}$$

where C and ν_0 are constants.

Putting (66) into (65), we obtain the following equation for ν_0:

$$\frac{1}{f} = \frac{p^{1-\nu_0}-1}{(1-p)(\nu_0-1)}. \tag{67}$$

Equation (67), which is similar to (54), has only one solution $\nu_0 \geqslant 0$. For $f=1$, $\nu_0=0$, for $f=0$, $\nu_0=\infty$. Of the various possible expansions for ν_0, the following is of particular physical interest.

The right side of (67) can be written as an integral, as follows:

$$\frac{1}{f} = \frac{1}{1-p}\int_p^1 y^{-\nu_0}dy. \tag{67a}$$

Putting now

$$y^{-\nu_0} = \sum_{s=0}^{\infty}\frac{(-\nu_0 \log y)^s}{s!},$$

we obtain the following expansion of $1/f$ in powers of ν_0:

$$\frac{1}{f} = \sum_{s=0}^{\infty}\frac{a_s \nu_0{}^s}{s!},$$

$$a_s = \frac{(-1)^s}{1-p}\int_p^1 (\log y)^s dy$$

$$= s!\left\{1 - \frac{1}{q-1}\sum_{m=1}^{s}\frac{(\log q)^m}{m!}\right\}. \tag{68}$$

It is easily seen (cf. (28)) that the coefficients a_s are the average $s-t$ powers of the logarithmic energy loss in one collision.

Now we put

$$b_s = \frac{a_s}{s!} = 1 - \frac{1}{q-1}\sum_{m=1}^{s}\frac{(\log q)^m}{m!},$$

$$w = \frac{1}{f} - 1 = \frac{g}{f} = \frac{l_s}{l_c}$$

and have

$$w = \sum_{s=1}^{\infty} b_s \nu_0{}^s.$$

Reverting this series, we obtain:

$$\nu_0 = \frac{w}{a}\left\{1 - \frac{b_2}{a^2}w + \frac{2b_2 - b_3 a}{a^4}w^2 - \frac{b_4 a^2 - 5b_3 b_2 a + 5b_2{}^3}{a^6}w^3 + \cdots\right\}. \tag{69}$$

The coefficients of the powers of w in the brackets of (69) are of the order of one. They are exactly one in the case of hydrogen, since for this case (cf. (61a)) $\nu_0 = g = w/(1+w)$. For large masses, $1-p=\alpha \ll 1$, we have, as is best directly seen from the integral in (68), $b_s = \alpha^s/(s+1)!$, which makes the coefficients in (69) 1, 2/3, 5/9, 34/135, etc. Thus the expansion (69) will always rapidly converge if $w \ll 1$. Breaking off at the term S will give an approximately correct result for S if $(w^{s+2}/a)\log x \ll 1$.

It is now of interest to calculate the second term in (63) for constant capture. Introducing (66) into the integral and using (67), one obtains

$$P(x) = gS(x)/\nu_0 a. \tag{70}$$

Thus $S(x) - P(x)$ is small compared to $P(x)$ if g or w is small. For large w, however, $\nu_0 a/g$ is large and the second term $S-P$ will be even large compared to P.

We now consider the case of slowly variable capture. Neglecting the variation of f in one interval, we have

$$d \log S/d \log x = \nu_0(x),$$

$\nu_0(x)$ being determined by (67). According to our assumption there is no capture in the first intervals, hence $S(1)=1$ and consequently

$$S(x)=\exp\left[-\int_1^x \nu_0(x')\frac{dx'}{x'}\right]. \tag{71}$$

If, in particular, w is small throughout, we may use the expansion (69) for ν_0. Its first term will be sufficient if

$$\int_1^x w^2(x')dx'/x' \ll a.$$

This condition is for small a, i.e., large masses, perfectly compatible with the first term being large compared to one, *viz.*,

$$\int_1^x w(x')dx'/x' \gg a.$$

Thus, for large masses and small w, the expression

$$S(x)=\exp\left[-(1/a)\int_1^x w(x')dx'/x'\right] \tag{71a}$$

will have a wide range of validity. We now may improve (71) by correcting for the neglected variation of f within one interval. Differentiating (60b), and putting

$$d \log S/d \log x = \nu(x)$$

we obtain

$$\nu(x)=1-\frac{f(x)}{1-p}\left\{1-p\frac{K(px)f(px)}{K(x)f(x)}\right\}. \tag{72}$$

Putting

$$\nu=\nu_0+\nu_1,$$

where ν_0 is given by (6) and ν_1 is the correction term to be determined, we develop f and ν in the interval:

$$\frac{f(px)}{f(x)}\approx 1-\log q\frac{d \log f(x)}{d \log x}, \tag{73}$$

$$\frac{K(px)}{K(x)}=\exp\int_{px}^x \nu(x')\frac{dx'}{x'}\approx\exp\left\{\nu(x)\log q-\frac{d\nu(x)}{d \log x}\frac{(\log q)^2}{2}\right\}$$
$$\approx\exp\left\{(\nu_0+\nu_1)\log q-\frac{d\nu_0}{d \log x}\frac{(\log q)^2}{2}\right\}\approx q^{\nu_0}\left\{1+\nu_1\log q-\frac{d\nu_0}{d \log x}\frac{(\log q)^2}{2}\right\}, \tag{74}$$

where we have replaced $d\nu/d \log x$ by $d\nu_0/d \log x$ and assumed that

$$\nu_1\frac{\log q}{2}\frac{d\nu_0}{d \log x}\ll 1.$$

Introducing (73) and (74) into (71) and using (67), we obtain finally

$$\nu_1=-\frac{\mu}{1-\mu}\frac{d}{d \log x}\left(\log f+\nu_0\frac{\log q}{2}\right), \tag{75}$$

where

$$\mu=\frac{fq^{\nu_0-1}}{1-p}\log q$$

which by (67) goes over into

$$\mu=\left(\nu_0+\frac{p-g}{1-p}\right)\log q. \tag{76}$$

If $\alpha\ll 1$, $g\ll 1$, (75) reduces to

$$\nu_1=-\tfrac{1}{3}dg/d \log x. \tag{75a}$$

An example for slowly varying capture is the case of the mean free path for scattering being constant and the mean free path for capture proportional to the velocity ($1/v$ law for the capture cross section). This makes

$$w=(x/x_c)^{\frac{1}{2}}, \tag{77}$$

where x_c is a constant. Thus we may use w as a measure of the energy, putting $S(x)=\varphi(w)$, and have for the solution:

$$\varphi(w)=\exp -2\int_0^w \{\nu_0(w')+\nu_1(w')\}\frac{dw'}{w'}, \tag{78}$$

where ν_0 and ν_1 are given by (67), (69), and (75). We may do still a little better, since it is here possible to take the variations of the capture within one interval not only approximately but exactly into account. For this purpose we introduce (77) into (59b), which gives the equation

$$w^2\varphi(w)=\frac{2}{1-r^2}\int_{rw}^w \frac{w'\varphi(w')}{1+w'}dw', \tag{79}$$

where

$$r=p^{\frac{1}{2}}=(M-1)/(M+1).$$

We now expand $\varphi(w)$ in powers of w

$$w^2\varphi(w)=\sum_{s=0}^{\infty}\beta_s w^s,\quad \beta_0=1 \tag{80}$$

and introduce this expansion into both sides of (79). Comparison of the coefficients of equal powers of w on both sides gives after some calculation

$$\beta_s=(-1)^s\lambda_s\prod_{k=1}^{s}(1-\lambda_k)^{-1}, \tag{81}$$

where

$$\lambda_s=\frac{2}{s+2}\,\frac{1-r^{s+2}}{1-r^2}.$$

The expansion (80), (81) is of some use for small masses. For larger masses, however, the coefficients β_s increase rapidly with increasing s, and in the limit of $\alpha\rightarrow 0$ one finds

$$\beta_s=\frac{(-1)^s}{s!}\left(\frac{\alpha}{2}\right)^s,$$

so that

$$\lim_{\alpha\rightarrow 0}\varphi(w)=e^{-4w/\alpha}.$$

For large masses, therefore, an expansion of log φ rather than φ will be useful, as is already suggested by (78) with (69). Its coefficients can be determined by introducing it into (79) or also by rearranging (80). One finds

$$\varphi(w)=\exp -\frac{2}{1-r}\left\{\frac{1+r+r^2}{1+2r}w-\frac{1+3r+2r^3}{2(1+2r)^2}w^2-\cdots\right\}. \tag{82}$$

The coefficients of (82) are very near to the coefficients obtained by the approximate treatment, which result from putting (69) and (75) into (78). This may be seen by expanding both in powers of α or $1/M$.

It will now be of interest to compare the effects of 1/v-capture in light and heavy slowing down agents. As in the discussion of (71), we see that the higher terms in (82) will in the case of large masses only come into play at energies for which already most of the neutrons have been captured. Expressing r by M, we have

$$\varphi(w)=\exp\left[-(M+\tfrac{1}{3})w+v\tfrac{1}{3}Mw^2\right], \tag{83}$$

where terms of the order w^2 and Mw^3 have been neglected.

For hydrogen, on the other hand, we have from (61a) and (77)

$$\varphi(w)=1/(1+w)^2. \tag{84}$$

Going over to the neutron density as given by (1), and denoting by $\psi(v)$ the neutron density in the velocity scale we have

$$\psi(v)dv=-\rho(x)\frac{dx}{x}=\frac{2Ql}{a}\varphi(w)\frac{dv}{v^2}$$

$$=\frac{2Ql_s}{a(1+w)}\varphi(w)\frac{dv}{v^2}. \tag{85}$$

Putting $w=v_c/v$, where v_c is the velocity at which the mean free paths for scattering and capture become equal, we obtain for large M, using (27a):

$$\psi(v)dv=Ql_s(M+\tfrac{2}{3})\times\exp\left[-\left(M+\frac{4}{3}\right)\frac{v_c}{v}+\frac{1}{3}M\left(\frac{v_c}{v}\right)^2\right] \tag{86}$$

and for hydrogen

$$\psi(v)dv=\frac{2Ql_s v dv}{(v+v_c)^3}; \tag{87}$$

$\psi(v)$ has a maximum which lies for hydrogen at $v=(v_c/2)$ and for large M at about $v=(2/M)v_c$. It also may be noted from (86) and (87) that the neutron density in the energy scale has no maximum in the case of hydrogen, while for larger M a maximum occurs at about $(3/M)^2E_c$. In connection with the fact, that the capture in heavy substances already comes into play for energies at which σ_c is of the order of σ_s/M rather than of the order σ_s as in hydrogen, it has been proposed[1] to use slowing down experiments in heavy materials for the detection and measurement of small capture cross sections. It may further be noted that the low energy cut-off is for large M considerably sharper than for hydrogen, which opens experimental possibilities of a different kind. By slowing neutrons down in a heavy substance containing a small and variable admixed amount of boron or another element with $1/v$ capture, it may be possible to produce a continuous neutron spectrum with variable lower limit, which would be very useful for the analysis of nuclear energy levels. While this method is not in principle different from the usual boron absorption method of neutron spectroscopy, it is not subject to the well-known difficulties which make the analysis of boron absorption data impracticable as soon as thicker layers have to be employed.

6. Rapidly Varying Capture

If f does no longer vary slowly within one interval, the treatment of the preceding section will not hold. In this case it is necessary to solve the equations rigorously. This is not difficult, if the occurrence of capture is limited to one or even a few intervals. Assuming that the capture is negligible for $x < x_0$, we have for the region $x_0 < x < x_0/p$:

$$S(x) = \frac{1}{(1-p)x} \int_{x_0}^{x} f(x')S(x')dx', \tag{88}$$

with

$$S(x_0) = 1.$$

(88) can be solved by differentiation with the result:

$$\begin{aligned} S(x) &= e^{\chi(x)} \left\{ 1 - \frac{p}{1-p} \int_{x_0}^{x} e^{-\chi(x')} \frac{dx'}{x'} \right\}, \\ \chi(x) &= \frac{1}{1-p} \left\{ p \log \frac{x}{x_0} - \int_{x_0}^{x} g(x') \frac{dx'}{x'} \right\}. \end{aligned} \tag{89}$$

It can be shown from (89) that the solution for rapidly varying g differs from the case of slowly varying g even if $g \ll 1$. The solution in the following intervals $((x_0/p) < x < (x_0/p^2)$, etc.) can be found from (89) by successive integrations. After the capturing region has been passed, S will be subject to fluctuations and finally tend to a constant value, which, with the help of (64), may be derived directly from the solution in the capturing region without explicit knowledge of S in the fluctuating region in between.

A case of rapidly varying g is the capture by absorption lines. Its discussion will be given in a separate paper.

7. A Simplified Treatment and Its Limitations

We shall now illustrate some of the results obtained above by comparing the actual slowing down mechanism with a simpler idealized one. If the loss in energy, which takes place in jumps of different magnitude occurring at different times, were instead continuous, the velocity of a neutron would be a function of its age. (Actually, to every age corresponds a velocity distribution and *vice versa*; these distributions can, at least for constant mean free path, be explicitly derived, the velocity spread for any given age being of the order v_{Av} for hydrogen and of the order $v_{Av}/\sqrt{M}$ for large masses.[15] We find the relation between velocity and age by putting the relative change in velocity dv/v equal to one-half of the average logarithmic energy loss a in one collision times the probability that a collision occurs, which is

$$\begin{aligned} [v/l_s(v)]dt: \quad dv/v &= -\tfrac{1}{2}avdt/l_s(v), \\ dt &= -(2/a)l_s(v)dv/v^2. \end{aligned} \tag{90}$$

We now ask for the number $N(t)dt$ of neutrons of age between t and $t+dt$. If no capture occurs, all ages are equally probable, and $N(t)$ is therefore equal to Q, the number of neutrons produced per second. Including capture, we have obviously:

$$N(t)dt = Q \exp\left[-\int_0^t \frac{dt'}{\tau_c(t')} \right] dt, \tag{91}$$

where $\tau_c = [l_c(v)/v]$ is the lifetime of the neutron against capture, which is in general a function of velocity. We now obtain at once the velocity distribution by introducing (90) into (91):

$$\psi(v)dv = -N(t)dt = \frac{2Ql_s(v)}{a} \times \exp\left[-\frac{2}{a} \int_v^{v_0} w(v') \frac{dv'}{v'} \right] \frac{dv}{v^2}, \tag{92}$$

where l_s/l_c has again been denoted by w. (92) is identical with Eq. (71a) except for a factor $l_s/l = 1 + w$. For large M this factor is of no importance, so that the simplified treatment will here be justified under the conditions of validity of (71a), *viz.*, w slowly varying and

$$\int_v^{v_0} w^2(v') \frac{dv'}{v'} \ll \frac{a}{2},$$

which is also rather obvious from the assumptions. For hydrogen, on the other hand, the changes of the neutron distribution caused by capture may be given wrongly by (92) even if they are small, as is the case for $1/v$ capture,

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[15] G. Placzek, Phys. Rev. (to appear later).

where the change of ψ by the capture is in first approximation $-3w\psi$, while (92) gives $-2w\psi$. This is connected with the fact that the velocity spread for any given age in hydrogen is larger than in heavy substances.

8. Modifications at High and Low Energies

Our results will break down at high and low energies because of changes in the mechanism of the collision. As soon as the neutron energy exceeds the energy of the first excited level of the nucleus in question, inelastic collisions will occur, which make the slowing down much more rapid. Another, less important, effect at high energies is the deviation of the angular distribution from spherical symmetry in the center of gravity system. This effect may occur as soon as the neutron wave-length becomes of the order of nuclear dimensions and may change somewhat the collision function $k(x, x')$.

At low energies the collision function will be modified by the chemical binding as soon as the energy transfer to a free nucleus in one collision becomes of the order of the vibration quanta, thus, for large M: $E\sim\frac{1}{2}M\hbar\omega$. These energies are of the order of several volts for heavy substances.

While the binding completely changes the collision function for $E<\frac{1}{2}M\hbar\omega$, it will not affect the average energy loss unless the energy itself becomes of the order of the vibration frequencies.[16] This may be seen from a schematic example. We consider the collision of a neutron with a particle of atomic weight $M\ll 1$, elastically bound in an isotropic field of force of vibration frequency ω, and being in the ground state $n=0$ before the collision. We assume the neutron energy E to be large compared to $\hbar\omega$, but small compared to $\frac{1}{2}M\hbar\omega$.

For this case the cross sections σ_{0n} for the transition $0\rightarrow n$, which can be found from the theory of the scattering of neutrons by bound protons,[17] reduce to

$$\sigma_{0n}=\sigma_f\cdot\frac{1}{(n+1)!}\left(\frac{4E}{\hbar\omega}\right)^n, \tag{93}$$

where σ_f is the cross section for a fixed nucleus, which for large M is not appreciably different from the cross section of the free nucleus. Only the transitions $0\rightarrow 0$ and $0\rightarrow 1$ are of importance, and hence the energy loss is given by

$$\hbar\omega\sigma_{01}/\sigma_f=2E/M. \tag{94}$$

Thus, instead of frequent small losses of maximum amount $4E/M$ we have a rare large loss of amount $\hbar\omega$, the average remaining $2E/M$.

The formulae for the energy distribution will therefore remain valid down to energies of the order of the vibration quanta if the capture is small and slowly varying within regions of the order of the vibration quanta so that only the average energy loss matters. In the opposite case, however, the deviations will come into play already for energies of the order $M/2$ times the vibration quanta. This effect may play a role in the study of absorption lines.

9. Spatial Distribution

The spatial distribution may be characterized by the average square distance of diffusion $(r^2)_{\text{Av}}$ as a function of initial and final energy. Generalizing the standard treatment as given by Fermi[4] and Bethe,[18] one finds

$$(r^2)_{\text{Av}}(x)=2\int_1^x\frac{l_s^2(x')K(x')}{1-(\cos\vartheta)_{\text{Av}}}\frac{dx}{x'}, \tag{95}$$

where ϑ is the scattering angle in the system of the observer. Equation (95) is valid if $\log x$ is large compared to the average logarithmic energy loss and l_s varies slowly within regions of the order of the latter. The denominator contains the effect of the angular assymetry of the scattering in the system of the observer. In our case the scattering is isotropic in the center of gravity system. Therefore we have

$$(\cos\vartheta)_{\text{Av}}=\tfrac{1}{2}\int_{-1}^{1}\cos\vartheta d\cos\theta, \tag{96}$$

where θ is the scattering angle in the center of gravity system. We now express $\cos\vartheta$ by $\cos\theta$:

$$\cos\vartheta=\frac{1+M\cos\theta}{(1+M^2+M\cos\theta)}. \tag{97}$$

Integrating (96) with (97), one finds

$$(\cos\vartheta)_{\text{Av}}=2/3M. \tag{98}$$

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[16] I am indebted to Professor H. A. Bethe for pointing this out to me.

[17] N. Arley, Kgl. Danske. Vid. Sels. Medd. **16**, 1 (1938).

[18] H. A. Bethe, Rev. Mod. Phys. **9**, 69 (1937).

Putting for K the inverse average logarithmic energy loss and using (98), we write (95) in the form:

$$(r^2)_{\mathrm{Av}}(x)=C\int_1^x l_s^2(x')\frac{dx'}{x'},\qquad C=\frac{2}{\left(1+\dfrac{p}{1-p}\log p\right)\left(1-\dfrac{2}{3M}\right)}. \tag{99}$$

For $M=1$,

$$C=6 \tag{99a}$$

which gives the well-known formula for hydrogen; for $M=2$

$$C=4.1. \tag{99b}$$

The decrease is caused by the diminuition of the forward asymmetry of scattering, which in this case overcompensates the change in energy loss. For $M>2$, C increases again and becomes for $M\gg 1$

$$C=M+4/3. \tag{99c}$$

For mixtures, (99) becomes (notations of Section 3):

$$(r^2)_{\mathrm{Av}}(x)=2\int_1^x \frac{l_s^2(x')}{\left(1+\sum_k\dfrac{c_k(x')p_k}{1-p_k}\log p_k\right)\left(1-\dfrac{2}{3}\sum_k\dfrac{c_k(x')}{M_k}\right)}\frac{dx'}{x'}. \tag{100}$$

This gives for air[8] $C=15.6$ and 15.8 for low and high energies, respectively. From (99) it may be seen that the effectiveness of hydrogen in slowing down the neutrons in a small space is caused by its exceptionally large scattering cross section rather than to its small mass. If its mass were 2 or 4, the distance required for slowing down would be even smaller.

It remains now to discuss the capture in mixtures, which presents many interesting features and is also of practical importance. The treatment will be given in another connection. It has already been shown in Section 3 that in mixtures of a heavy and a light substance there are for certain concentrations enormous deviations from the asymptotic value of K, which extend over many intervals. Without entering into details, it may therefore be concluded already that in such cases the mutual correlation of the effects of capture occurring at different energies will be of primary importance and act over distances much wider than for single substances.

It is a pleasure to thank Professor H. A. Bethe for many interesting discussions.

APPENDIX I

Asymptotic Evaluation of $A_n(\zeta)$

We write the polynomial

$$A_n(\zeta)=\sum_{m=0}^{n-1}(m-n)^m\frac{\zeta^m}{m!}$$

as the difference of two infinite series:

$$A_n(\zeta)=T_n(\zeta)-B_n(\zeta), \tag{I1}$$

$$T_n(\zeta)=\sum_{m=0}^{\infty}(m-n)^m\frac{\zeta^m}{m!}, \tag{I1a}$$

$$B_n(\zeta)=\sum_{m=0}^{\infty}\frac{(m\zeta)^{m+n}}{m!}. \tag{I1b}$$

Both series converge if $\zeta<e^{-1}$. $T_n(\zeta)$ can be evaluated rigorously. We put

$$\zeta=-v\log v. \tag{I2}$$

Equation (I2) has two roots v_1 and v_2, which for $\zeta\rightarrow 0$ tend to 0 and 1, respectively. Expanding ζ in powers of $\log v_2$ and introducing this expansion into (I1a) one obtains[19]

$$T_n(\zeta)=\frac{v_2^{\,n}}{1+\log v_2}. \tag{I3}$$

The series $B_n(\zeta)$ consists of positive terms, which, if n is large, have a sharp maximum. Hence we may get the asymptotic value of $B_n(\zeta)$ for large n by using Stirling's formula and replacing the sum by an integral. Putting

$$1+\frac{m}{n}=y^2,\quad \log\zeta=-(1+\gamma)$$

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[19] Polya-Szegö, *Aufgaben und Lehrsatze aus der Analysis* (Verlagsbuchhandlung Julius Springer, Berlin, 1926).

we get

$$B_{n\,\mathrm{as}}=\left(\frac{2n}{\pi}\right)^{\frac{1}{2}}\int_{1}^{\infty}\{(1-y^{-2})e^{-\gamma}\}^{ny^2}dy,$$
$$=\left(\frac{2n}{\pi}\right)^{\frac{1}{2}}\int_{1}^{\infty}e^{F(y)}dy. \qquad (\mathrm{I}4)$$

The integrand has a sharp maximum, the location of which we determine as follows:

We have

$$F(y)=ny^2\{-\gamma+\log(1-y^{-2})\}. \qquad (\mathrm{I}5)$$

From $F'(y_M)=0$, we get

$$\frac{1}{y_M{}^2-1}=-\gamma+\log(1-y_M{}^{-2}). \qquad (\mathrm{I}6)$$

Putting

$$\zeta=te^{-t}, \qquad (\mathrm{I}7)$$

we have

$$-\gamma=1-t+\log t$$

and hence may write (I6):

$$1-t+\log t=1-\frac{y_M{}^2}{1-y_M{}^2}+\log\frac{y_M{}^2}{1-y_M{}^2}, \qquad (\mathrm{I}8)$$

from which we get:

$$y_M{}^2=t/(1-t). \qquad (\mathrm{I}9)$$

Equation (I7) goes over into (I2) by putting $t=-\log v$ and hence has two solutions $t_1\geqslant 1$ and $t_2\leqslant 1$ which tend to ∞ and 0, respectively, for $\zeta\rightarrow 0$. Since $y^2=1+(m/n)$ is positive, we see from (I9) that the solution applying to our case is $t_1=-\log v_1$.

We now evaluate B_n by integration around the maximum:

$$B_{n\,\mathrm{as}}(\zeta)=\left(\frac{4n}{-F''(y_M)}\right)^{\frac{1}{2}}\exp[F(y_M)]. \qquad (\mathrm{I}10)$$

From (I5, 6, 9) we have

$$F(y_M)=\frac{-ny_M{}^2}{y_M{}^2-1}=-nt_1,$$
$$F''(y_M)=-4n(t_1-1)^2.$$

Hence we get

$$B_{n\,\mathrm{as}}(\zeta)=\frac{\exp(-nt_1)}{t_1-1}=-\frac{v_1{}^n}{1+\log v_1}. \qquad (\mathrm{I}11)$$

And finally, from (I1, 3, I1):

$$A_{n\,\mathrm{as}}(\zeta)=\frac{v_1{}^n}{1+\log v_1}+\frac{v_2{}^n}{1+\log v_2}. \qquad (\mathrm{I}17)$$

APPENDIX II

Transformation of the Integral Equation

Differentiation of (32) gives

$$xK'(x)=\sum_s\frac{p_s}{1-p_s}\{c_s(x)K(x)$$
$$-(1-\mu(p_sx))c_s(p_sx)K(p_sx)\}. \qquad (\mathrm{II}1)$$

Equation (II1) is not valid at the points $x=p_s{}^{-1}$, for which K is discontinuous. K' itself is discontinuous for the points $x=p_s{}^{-2}$. We multiply with dx/x and integrate from 1 to x, taking the discontinuities into account. The left-hand side gives:

$$\int_1^x K'(x')dx'=K(x)-K(1)$$
$$-\sum_{s,}(1-\mu(p_sx))\{K_+(p_sx)-K_-(p_sx)\}.$$

From (36) we have

$$K(1)=\sum_s\frac{c_s(1)}{1-p_s}.$$

Introducing (34) we get

$$\int_1^x K'(x')dx'=K(x)-1-\sum_s\frac{c_s(1)\mu(p_sx)p_s}{1-p_s}.$$

Integration of the right-hand side of (II1) gives

$$\int_1^x K'(x')dx'=\sum_s\frac{p_s}{1-p_s}\int_{\lambda(p_sx)}^x c_s(x')K(x')\frac{dx'}{x'}.$$

Hence,

$$K(x)=1+\sum_s\frac{p_s}{1-p_s}\left\{c_s(1)\mu(p_sx)\right.$$
$$\left.+\int_{\lambda(p_sx)}^x c_s(x')K(x')\frac{dx'}{x'}\right\}. \qquad (\mathrm{II}2)$$

If only one type of nucleus is present, this reduces to Eq. (29a, b).

The same transformation, applied to the equation with capture, gives the result:

$$K(x)+\int_1^x g(x')K(x')\frac{dx'}{x'}$$
$$=f(1)+\sum_s\frac{p_s}{1-p_s}\left\{f(1)c_s(1)\mu(p_sx)\right.$$
$$\left.+\int_{\lambda(p_sx)}^x f(x')c_s(x')K(x')\frac{dx'}{x'}\right\}. \qquad (\mathrm{II}3)$$