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Classical Theory of the Point Electron*

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The difficulties of the classical theory of the electron are examined and methods to eliminate them are given. It is shown that the whole theory can be derived from a division of the total field created by a point charge in two parts, one which reacts on the generating particle and accounts for the emission of radiation, another which does not react on the praticle but acts on other particles. There are several types of motions of the particles depending on the kind of field they generate, fields which are always solutions of Maxwell's equations. Only three types of motions are, apparently, physically interesting: (*a*) motions with positive or negative kinetic energy in which the particles radiate, and (*b*) radiationless motions analogous to the stationary motions of quantum theory. It is shown that the field picture of Faraday and Maxwell must be revised because not all the electric actions between particles can be considered as arising from their interaction with a field. The whole theory of the particles and the field can be derived from an action principle and boundary conditions for the equations of motion of the particles and the field.

INTRODUCTION

1. THE classical theory of the point electron presents serious difficulties arising from the divergences of the field on the particle's world line: (*a*) The energy of the field diverges and (*b*) the force acting on the particle is infinite, if it is taken as the Lorentz force corresponding to the total field at the particle's position.

The divergence of the force is perhaps the worst difficulty. Dirac[1] has found a method to overcome this difficulty and to justify the equations of motion which were obtained approximately from the Lorentz theory by neglecting divergent terms and the terms which depend on the particle's inner structure. Following Dirac's work, Pryce[2] has shown how it is possible to get a finite energy of the electromagnetic field. However, these theories have some unsatisfactory features. First, they introduce infinite energies of unknown nature in order to cancel out the divergencies of the electromagnetic ones. Second, they do not lead to unique equations of motion and stress-tensors. The second inconvenience is connected with the first one, because the lack of unicity is a consequence of the indeterminateness of the non-electromagnetic energies and momenta.

Later, Dirac[3] worked out the theory in a more elegant way by introducing the so-called λ-process which has been already used by Wentzel.[4] The λ-process enabled Dirac to build a Hamiltonian formalism. This method has the inconvenience of being of an entirely formal nature, and it seems impossible to base on it a satisfactory quantum theory.

However, there is a very simple method, suggested by electrostatics, that has not been worked out until now and which leads to a very simple classical theory and can also be used in quantum theory.[5] The divergencies of the total field on the electron's world line are due only to the divergence of the electron's own field. A similar divergence occurs already in electrostatics but it does not lead to any difficulties. Let us consider the static field created by n point charges Q_i; the force acting on Q_i is

$$\mathbf{F}_i = -\sum_{j\neq i} \frac{e_i e_j}{r_{ij}^3}\mathbf{r}_{ij}, \tag{1}$$

the summation being carried over all the other particles; $\mathbf{r}_{ij}$ is the position vector of Q_j in relation to Q_i, and r_{ij} is the distance between Q_i and Q_j.

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[1] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938).

[2] M. H. L. Pryce, Proc. Roy. Soc. **A168**, 389 (1938).

[3] P. A. M. Dirac, Ann. de l'Inst. H. Poincaré **9**, 13 (1939); Proc. Roy. Soc. **A180**, 1 (1942).

[4] G. Wentzel, Zeits. f. Physik **86**, 479, 635 (1933); **87**, 726 (1934).

[5] M. Schönberg, Phys. Rev. **67**, 122 (1945) and **67**, 193 (1945).

The energy W of the field is

$$W=\frac{1}{2}\sum_i\sum_{j\neq i}\frac{e_ie_j}{r_{ij}}. \quad (2)$$

Frenkel[6] made an attempt to extend this method to the general dynamical case by assuming that there is no self-force. But it is easily seen that it is not possible to get the Larmor loss without introducing a self-force. This was already shown by Synge's work.[7] We must look for some invariant way of splitting the particle's field in two parts: one which reacts on it and accounts for the Larmor loss; another which does not act on the generating particle, though influencing the motion of the other particles. Such a division of the field was already proposed by Leite Lopes and Schönberg.[8] It will be the basic assumption of this paper.

2. Besides the divergence difficulties there are others of a different nature, related to the stability of motions of electric particles. It has been assumed that there is no classical possibility of stable accelerated motions of charged particles because of the radiation losses; the existence of stationary motions would be characteristic of quantum mechanics. However, Fokker[9] has shown a long time ago that it is possible to develop a classical theory of the motion of a system of charged particles in which there are no radiation losses, provided it is assumed that the force between two particles is half-retarded and half-advanced. Fokker's theory did not include the possibility of radiative processes and therefore was not duly considered, until recently, when Wheeler tried to use it as the starting point of a general theory of the dynamics of charged particles. It turns out that Fokker's interaction between two charged particles is precisely that part of their interaction which is due to the fields called attached fields by Leite Lopes and Schönberg,[8] the part of a particle's field which does not react on it.

Fokker's theory leads to satisfactory results when applied to the motion of a particle in an external field and presumably also in the case of the motion of a system of particles which corresponds to the stationary motions of quantum mechanics. On the other hand Eliezer[10] has shown that the Lorentz-Dirac equations of motion do not give always satisfactory results when applied to the motion of systems of particles as well as to the motion of a particle in an external field. We shall see that it is possible to formulate a consistent set of principles from which result both the Lorentz-Dirac equations and the Fokker equations, the type of equations of motion depending on the type of motion, without giving up Maxwell's equations for the field which are always satisfied. Though the field created by a particle is always a solution of the Maxwell equations, their behavior at the boundary is not always the same. The nature of the boundary conditions imposed on the solution of the field equations determines the type of the equations of motion of the particles.

The same general principles which conciliate the Lorentz-Dirac equations with Fokker's equations lead also to a satisfactory theory of motions with negative kinetic energy. It is generally assumed that the investigation of such motions belongs to quantum theory, but this point of view is not quite satisfactory because the existence of negative kinetic energies is not a quantum effect. If we try to develop a classical theory of such states of motion, there are considerable difficulties arising from the impossibility of "hiding" such states by means of the Pauli principle, as it was done by Dirac in his theory of the positron. However, it turns out that the equations of motion of a particle with negative kinetic energy are the same as those of a particle with opposite charge and positive kinetic energy, even taking in account the reaction of radiation, provided it is assumed that particles with negative kinetic energies generate advanced fields in non-stationary states of motion. The fact of generating advanced fields gives to these particles properties similar to those of holes, because the advanced fields result from the superposition of converging spherical waves, so that the particles which generate such fields behave as sinks of energy.

[6] J. Frenkel, Zeits. f. Physik **32**, 518 (1925).
[7] J. L. Synge, Proc. Roy. Soc. **A177**, 118 (1940).
[8] J. L. Lopes and M. Schönberg, Phys. Rev. **67**, 122 (1945).
[9] A. D. Fokker, Zeits. f. Physik **58**, 386 (1929).
[10] J. Eliezer, Proc. Camb. Phil. Soc. **39**, 173 (1943).

PART I. NON-STATIONARY MOTIONS

General Definitions

3. We shall consider only charged point particles without electric or magnetic moments. Since the magnetic moment of the electron is a quantum effect our theory will be applicable to electrons. The motions of a system of charged particles in which there is radiation of energy will be called non-stationary motions, those in which there is no radiation of energy will be called stationary motions.

We shall assume that any particle has only two kinds of momentum: the kinetic one $G^{\mu}{}_{\text{kin}}$ and the electromagnetic one $G^{\mu}{}_{\text{el}}$

$$G^{\mu}{}_{\text{kin}}=mc\frac{dx^{\mu}}{ds}; \tag{3}$$

however, in general we will not have

$$G^{\mu}{}_{\text{el}}=\frac{e}{c}A^{\mu}{}_{\text{act}}; \tag{4}$$

m is the rest mass of the particle, e its electric charge, and $A^{\mu}{}_{\text{act}}$ the potentials of the field which acts on the particle. If we allow ds to take both positive and negative values, we shall have both positive and negative kinetic energies, the sign of the kinetic energy being the same of ds

$$ds=\epsilon(dx^{\mu}dx_{\mu})^{\frac{1}{2}},\quad \epsilon=\pm 1. \tag{5}$$

We shall call attached field of a point particle half the sum of its retarded and advanced fields.

In our tensor notation the metric tensor $g^{\mu\nu}$ has the components

$$g^{00}=1,\quad g^{11}=g^{22}=g^{33}=-1,$$
$$g^{01}=g^{02}=g^{03}=g^{12}=g^{13}=g^{23}=0.$$

The field tensor $F^{\mu\nu}$ is taken in such a way that

$$F_{01}=E_x,\quad F_{02}=E_y,\quad F_{03}=E_z,$$
$$F_{32}=H_x,\quad F_{13}=H_y,\quad F_{21}=H_z,$$

E and **H** being the electric and magnetic vectors. The field $F_{\mu\nu}$ is given in terms of the potential by the relation:

$$F_{\mu\nu}=\partial A_{\nu}/\partial x^{\mu}-\partial A_{\mu}/\partial x^{\nu},$$
$$F^{\mu\nu}{}_{\text{at}}=\tfrac{1}{2}(F^{\mu\nu}{}_{\text{ret}}+F^{\mu\nu}{}_{\text{adv}}). \tag{6}$$

$F^{\mu\nu}$ is a field tensor and A^{μ} the corresponding potential. Taking in account the expressions of the advanced and retarded potentials due to Dirac[1]

$$A^{\mu}{}_{\text{ret}}(z)=2e\int_{-\infty}^{s'}\frac{dx^{\mu}}{ds}\delta(\{x^{\mu}-z^{\mu}\}\{x_{\mu}-z_{\mu}\})\epsilon ds, \tag{7}$$

$$A^{\mu}{}_{\text{adv}}(z)=2e\int_{s'}^{+\infty}\frac{dx^{\mu}}{ds}\delta(\{x^{\mu}-z^{\mu}\}\{x_{\mu}-z_{\mu}\})\epsilon ds. \tag{8}$$

we get

$$A^{\mu}{}_{\text{at}}(z)=e\int_{-\infty}^{+\infty}\frac{dx^{\mu}}{ds}\delta(\{x^{\mu}-z^{\mu}\}\{x_{\mu}-z_{\mu}\})\epsilon ds. \tag{9}$$

δ is Dirac's symbolic function, s' is the value of s corresponding to a point on the particle's world line such that

$$\{x^{\mu}(s')-z^{\mu}\}\{x_{\mu}(s')-z_{\mu}\}<0. \tag{10}$$

Equation (9) shows that the attached field does not change when the sign of the elementary interval ds is changed.

We shall call radiated field of a point particle the difference between the total field created by the particle and its attached field:

$$F^{\mu\nu}{}_{\text{rad}}=F^{\mu\nu}{}_{\text{part}}-F^{\mu\nu}{}_{\text{at}}. \tag{11}$$

The Equations of Motion

4. Our theory of the non-stationary motions of a charged particle will be derived from two basic postulates:

I) $$F^{\mu\nu}{}_{\text{act}}=F^{\mu\nu}{}_{\text{ext}}+F^{\mu\nu}{}_{\text{rad}}, \tag{12}$$

II) $$F^{\mu\nu}{}_{\text{rad}}=\frac{\epsilon}{2}(F^{\mu\nu}{}_{\text{ret}}-F^{\mu\nu}{}_{\text{adv}}). \tag{13}$$

$F^{\mu\nu}{}_{\text{ext}}$ is the tensor of the external field.

It results from Eq. (13) that, in a non-stationary motion with positive kinetic energy, a particle generates a retarded field and, in a non-stationary motion with negative kinetic energy, it generates an advanced field:

$$F^{\mu\nu}{}_{\text{part}}=\left.\begin{matrix}F^{\mu\nu}{}_{\text{ret}}\ \text{when}\ \epsilon>0\\ F^{\mu\nu}{}_{\text{adv}}\ \text{when}\ \epsilon<0\end{matrix}\right\}. \tag{14}$$

Dirac[1] has found that, on the world line of a particle, the difference between its retarded and advanced fields is given by the formula:

$$F^{\mu\nu}{}_{\text{ret}}-F^{\mu\nu}{}_{\text{adv}}=\frac{4}{3}e\left(\frac{d^3x^{\mu}}{ds^3}\frac{dx^{\nu}}{ds}-\frac{d^3x^{\nu}}{ds^3}\frac{dx^{\mu}}{ds}\right). \tag{15}$$

Therefore on the particle's world line

$$F^{\mu\nu}{}_{\rm rad}=\tfrac{2}{3}\epsilon e\left(\frac{d^3x^\mu}{ds^3}\frac{dx^\nu}{ds}-\frac{d^3x^\nu}{ds^3}\frac{dx^\mu}{ds}\right). \tag{16}$$

But $F^{\mu\nu}{}_{\rm rad}$ is the part of the particle's field which reacts on it and, so, the self-force is

$$K^\mu{}_{\rm self}=\frac{e}{c}F^{\mu\nu}{}_{\rm rad}\frac{dx_\nu}{ds}. \tag{17}$$

Taking in account Eq. (16), it is easily seen that

$$K^\mu{}_{\rm self}=\frac{2}{3}\epsilon\frac{e^2}{c}\left(\frac{d^3x^\mu}{ds^3}+\frac{dx^\mu}{ds}\frac{d^2x_\nu}{ds^2}\frac{d^2x^\nu}{ds^2}\right). \tag{18}$$

Equation (13) shows that the radiated field of a particle in non-stationary motion is a solution of the Maxwell equations corresponding to a field of pure waves; Eq. (11) shows that both $F^{\mu\nu}{}_{\rm part}$ and $F^{\mu\nu}{}_{\rm at}$ are solutions of the Maxwell equations which correspond to the current due to the particle.

5. The equations of motion of the particle are

$$mc\frac{d^2x^\mu}{ds^2}=\frac{e}{c}F^{\mu\nu}{}_{\rm act}\frac{dx_\nu}{ds}. \tag{19}$$

Taking in account the expression (18) of the self-force Eq. (19) becomes

$$\frac{d}{ds}\left(mc\frac{dx^\mu}{ds}-\frac{2}{3}\epsilon\frac{e^2}{c}\frac{d^2x^\mu}{ds^2}\right)-\frac{2}{3}\epsilon\frac{e^2}{c}\frac{dx^\mu}{ds}\frac{d^2x_\nu}{ds^2}\frac{d^2x^\nu}{ds^2}=\frac{e}{c}F^{\mu\nu}{}_{\rm ext}\frac{dx_\nu}{ds}. \tag{20}$$

Hence when $\epsilon=1$, Eq. (19) coincides with the Lorentz-Dirac equations. The electromagnetic momentum of the particle is

$$G^\mu{}_{\rm el}=\frac{e}{c}A^\mu{}_{\rm ext}-\frac{2}{3}\frac{\epsilon e^2}{c}\frac{d^2x^\mu}{ds^2}=\frac{e}{c}A^\mu{}_{\rm ext}+G^\mu{}_{\rm ac}, \tag{21}$$

$$G^\mu{}_{\rm ac}=-\frac{2}{3}\frac{\epsilon e^2}{c}\frac{d^2x^\mu}{ds^2}. \tag{22}$$

$G^\mu{}_{\rm ac}$ is the four vector of the acceleration momentum, introduced by Schott[11] in the case of positive kinetic energies. We see that $G^\mu{}_{\rm ac}$ is the part of the electromagnetic momentum of the particle due to its radiated field.

The third term in the left-hand side of the equation of motion (20) corresponds to the Larmor loss when $\epsilon=1$. In the case of a negative kinetic energy this term corresponds to a Larmor "gain," but it leads to a damping of the particle's motion because, in order to stop a particle with negative kinetic energy, it is necessary to supply it with positive energy.

Equation (20) does not change if we change simultaneously the signs of both the charge e and the interval ds. Hence:

A particle with charge e has the same equations of motion as a particle with charge $-e$ and opposite kinetic momentum $-G^\mu{}_{\rm kin}$.

The preceding theorem explains the reason why it is necessary to assume that a particle in a state of negative kinetic energy generates an advanced field: otherwise it would not move as a particle with positive kinetic energy and opposite charge. It is important to keep in mind that, though the particle with negative kinetic energy moves as if it had positive kinetic energy and opposite charge, it generates a field corresponding to the real sign of its charge.

Proper Time in Motions with Negative Kinetic Energy

6. The basic postulates I and II can be expressed in a more elegant form by introducing a generalized conception of the flow of proper time.

In relativistic theories the time appears in two different forms: (*a*) as one of the coordinates in the four-dimensional universe; (*b*) as the proper time of the particles, measured by the length of the arcs described on their world lines. Of course the real signification of time must be attached to the length of arc described on the world line. This is seen clearly in general relativity because, in a curved universe, it is not possible to choose the coordinates in such a way that the differentials dx^0 along the world lines of the particles coincide always with the elementary intervals ds.

The usual ideas about the flow of time impose the choice of a preferred orientation of the world lines of the particles, in such a way that

$$dx^0/ds>0. \tag{23}$$

There is no strong argument against the assumption that there are two possible orientations of

[11] G. A. Schott, Phil. Mag. 29, 49 (1915).

the world lines of the particles, corresponding to two different types of motion: motions with positive kinetic energy, with $dx^0/ds>0$, and motions with negative kinetic energy, with $dx^0/ds<0$. Thus, our purely mathematical convention (5) would correspond to the two possible orientations of the particles world lines.

Each observer chooses the coordinate χ^0 according to its proper time flow and, thus, the particles whose proper times do not flow in the same direction as the observer's time appear as having negative kinetic energies. Now it becomes intuitive why particles with negative kinetic energy generate advanced fields:

The field generated by a particle in non-stationary motion is always a retarded field for an observer whose proper time flows in the same sense as the particle's proper time.

This proposition can be verified easily by observing that a change of sign of ds in Eq. (8) transforms the advanced potentials of a particle with charge e into the retarded potentials of a particle with the same charge, moving on the same world line.

Now the apparent asymmetry of the treatment of the states with positive and negative kinetic energies disappears. We may formulate postulate II as:

The radiated field of a particle is half the difference between its retarded and advanced fields measured by an observer whose time flows in the same direction as the particle's proper time.

From the preceding considerations it results that the direction of time flow for a charged particle in non-stationary motion is always such that there is an emission of energy for an observer attached to the particle, when there are no incoming external waves. It is not necessary to introduce statistical quantities such as the entropy in order to define the direction of time flow for an observer, the radiation of electromagnetic waves already selects a direction of time flow.

Equations of Motion for Systems of Particles

7. The postulates I and II allow us to write the equations of motion of a system of particles. The field $F^{\mu\nu}{}_{i,\,\mathrm{act}}$ which acts on the ith particle is the sum of the total fields $F^{\mu\nu}{}_{j,\,\mathrm{part}}$ due to the other particles, of the external field $F_{\mathrm{ext}}{}^{\mu\nu}$ in which the system moves and of the particle's radiated field $F^{\mu\nu}{}_{i,\,\mathrm{rad}}$

$$F^{\mu\nu}{}_{i,\,\mathrm{act}}=\sum_{j\neq i}F^{\mu\nu}{}_{j,\,\mathrm{part}}+F^{\mu\nu}{}_{\mathrm{ext}}+F^{\mu\nu}{}_{i,\,\mathrm{rad}},\qquad(24)$$

because the fields $F^{\mu\nu}{}_{j,\,\mathrm{part}}$ are external fields for the ith particle. Therefore the equations of motion of the system are

$$m_ic\frac{d^2x_i{}^\mu}{ds_i{}^2}=\frac{e_i}{c}\left(\sum_{j\neq i}F^{\mu\nu}{}_{j,\,\mathrm{part}}+F^{\mu\nu}{}_{\mathrm{ext}}+F^{\mu\nu}{}_{i,\,\mathrm{rad}}\right)\frac{dx_{i,\,\nu}}{ds_i}.\qquad(25)$$

Formula (24) leads to an expression of the force $K_i{}^\mu$ which generalizes the electrostatic formula (1)

$$K_i{}^\mu=\frac{e_i}{c}\left(\sum_{j\neq i}F^{\mu\nu}{}_{j,\,\mathrm{part}}+F^{\mu\nu}{}_{\mathrm{ext}}\right)+\frac{2}{3}\frac{\epsilon_ie_i{}^2}{c}\times\left(\frac{d^3x_i{}^\mu}{ds_i{}^3}+\frac{dx_i{}^\mu}{ds_i}\frac{d^2x_{i,\,\nu}}{ds_i{}^2}\frac{d^2x_i{}^\nu}{ds_i{}^2}\right).\qquad(26)$$

It is convenient to define the radiation field of a system as the sum of the radiated fields of its particles

$$F^{\mu\nu}{}_{\mathrm{rad}}=\sum{}_i\,F^{\mu\nu}{}_{i,\,\mathrm{rad}}.\qquad(27)$$

The radiative losses of a system are due to the action of $F^{\mu\nu}{}_{\mathrm{rad}}$ on the particles of the system, as we shall see later.

PART II. STATIONARY MOTIONS

8. Postulate I is a generalization of the law of inertia which is presumably valid for any linear field theory, provided we replace the tensor $F^{\mu\nu}{}_{\mathrm{act}}$ by the adequate quantities of the considered field. It corresponds to the physical idea that the radiated field gets detached from the particle and behaves as an external field, so that the self-force arising from it has really the character of an external action.

Postulate II can be easily generalized. The natural condition we must impose on a radiated field is that it be a solution of the field equations corresponding to a wave field. In the electromagnetic case this condition is satisfied if we take

$$\text{(IIa)}\qquad F^{\mu\nu}{}_{\mathrm{rad}}=\frac{\eta}{2}(F^{\mu\nu}{}_{\mathrm{ret}}-F^{\mu\nu}{}_{\mathrm{adv}}),\qquad(28)$$

η being an arbitrary constant. The non-stationary motions correspond to $\eta=\epsilon$. Let us examine what results by assuming that

$$\text{(IIb)}\qquad \eta=0. \tag{29}$$

From Eqs. (12), (28), and (29) it results that

$$F^{\mu\nu}{}_{\text{act}}=F^{\mu\nu}{}_{\text{ext}}. \tag{30}$$

Taking into account the definition (11) of the radiated field we get

$$F^{\mu\nu}{}_{\text{part}}=F^{\mu\nu}{}_{\text{at}}. \tag{31}$$

Therefore condition II b corresponds to a vanishing self-force and to a total field created by the particle equal to the attached field. This is an acceptable value of $F^{\mu\nu}{}_{\text{part}}$ because it is a solution of the Maxwell equations corresponding to the charge and current generated by the particle (any value of η would give an $F^{\mu\nu}{}_{\text{part}}$ which satisfies this Maxwell system). Since there is no self-force, there is no radiative damping of the particle's motion and, presumably, no emission of radiation. The value (31) of $F^{\mu\nu}{}_{\text{part}}$ is precisely the value of the Fokker theory[9] in which there are no radiation losses.

Postulate II b characterizes the stationary motions. Since there is now no self-force the equations of motion are

$$mc\frac{d^2x^\mu}{ds^2}=\frac{e}{c}F^{\mu\nu}{}_{\text{ext}}\frac{dx_\nu}{ds}. \tag{32}$$

It is very remarkable that the order of the equations of motion of a charged particle is not always the same: in non-stationary motions the equations are of the third order but they are of the second order in stationary motions. This is a consequence of the fact that the third-order derivatives are introduced by Schott's acceleration momentum $G^\mu{}_{\text{ac}}$ which does not exist in stationary motions.

Let us consider now a system of particles in stationary motion. It is easily seen that

$$\begin{aligned}F^{\mu\nu}{}_{i,\,\text{act}}&=\sum_{j\neq i}F^{\mu\nu}{}_{j,\,\text{part}}+F^{\mu\nu}{}_{\text{ext}}\\&=\sum_{j\neq i}F^{\mu\nu}{}_{j,\,\text{at}}+F^{\mu\nu}{}_{\text{ext}}.\end{aligned} \tag{33}$$

Therefore the equations of motion are

$$m_ic\frac{d^2x_i{}^\mu}{ds_i{}^2}=\frac{e_i}{c}\bigg(\sum_{j\neq i}F^{\mu\nu}{}_{j,\,\text{at}}+F^{\mu\nu}{}_{\text{ext}}\bigg)\frac{dx_{i,\,\nu}}{ds_i}. \tag{34}$$

The most interesting type of stationary motion is that of a system free from external actions. This is the type originally considered by Fokker,[9] it is a relativistic generalization of the n-body problem of Newtonian mechanics.

Conservation of Energy, Momentum, and Angular Momentum

9. The total momentum of the ith particle of an isolated system in stationary motion is

$$G_i{}^\mu=m_ic\frac{dx_i{}^\mu}{ds_i}+\frac{e_i}{c}\sum_{j\neq i}A^\mu{}_{j,\,\text{at}}. \tag{35}$$

Taking into account the expressions of the fields in terms of the potentials, the equations of motion can be written in the following form

$$\frac{dG_i{}^\mu}{ds_i}=\frac{e_i}{c}\sum_{j\neq i}\frac{\partial A^\nu{}_{j,\,\text{at}}}{\partial x_{i,\,\mu}}\frac{dx_{i,\,\nu}}{ds_i}. \tag{36}$$

The rate of variation of the total momentum of the system, for an observer in a Lorentz frame of reference whose time is t, is in general not nil

$$\frac{d}{dt}\sum\nolimits_i G_i{}^\mu=\sum_i\bigg(\frac{dG_i{}^\mu}{ds_i}\frac{ds_i}{dt}\bigg)_{x_i{}^0=ct}\neq0, \tag{37}$$

but its time integral vanishes

$$\int_{-\infty}^{+\infty}\frac{d}{dt}\sum_i G_i{}^\mu dt=\sum_i\int_{-\infty}^{+\infty}\frac{dG_i{}^\mu}{ds_i}ds_i=0. \tag{38}$$

Equation (38) may be considered as the law of conservation of energy and momentum in stationary motions. It shows that there is no permanent loss of energy and momentum by radiative processes. Equation (38) is a consequence of the relativistic generalization of the principle of action and reaction:

$$e_i\int_{-\infty}^{+\infty}\frac{\partial A^\nu{}_{j,\,\text{at}}}{\partial x_{i,\,\mu}}\frac{dx_{i,\,\nu}}{ds_i}ds_i=-e_j\int_{-\infty}^{+\infty}\frac{\partial A^\nu{}_{i,\,\text{at}}}{\partial x_{j,\,\mu}}\frac{dx_{j,\,\nu}}{ds_j}ds_j. \tag{39}$$

The principle of action and reaction (39) is valid even in non-stationary motions, because it results immediately from the expression (9) of the attached potentials of a particle.

The total angular momentum of the spinless particles which we consider is

$$M_i^{\mu\nu}=x_i^{\mu}G_i^{\nu}-x_i^{\nu}G_i^{\mu}. \tag{40}$$

There is a law of average conservation of $\sum_i M_i^{\mu\nu}$ in stationary motions of isolated systems

$$\sum_i \int_{-\infty}^{+\infty} \frac{dM_i^{\mu\nu}}{ds_i} ds_i = 0. \tag{41}$$

Equation (41) can be easily derived from Eqs. (36) by taking into account the expression (9) of the attached potentials of a particle.

Some Stationary Motions of a Particle

10. The simplest stationary motion is that of an isolated particle. It results from the equations of motion that it is a uniform rectilinear motion. Dirac[1] has shown that the Lorentz-Dirac equation for a free particle has two types of solutions: the uniform rectilinear and the self-accelerated motions. We can get rid of the self-accelerated motions by assuming as relativistic law of inertia that: *The motion of an isolated point charge is stationary.*

Another very simple motion of a particle is its stationary motion in a Coulomb field. In this case Eqs. (32) are the same studied by Sommerfeld in the old theory of the atom. So we see that the orbits of the old quantum theory correspond to classical stationary motions; the quantization rules of Bohr and Sommerfeld select those orbits which correspond to the quantum stationary motions.

Eliezer[10] has found difficulties in the application of the Lorentz-Dirac equations to the theory of the electromagnetic Kepler problem and to the electromagnetic two-body problem. These difficulties disappear if we treat those motions as stationary motions. As we have already indicated, there is an essential difference between stationary and non-stationary motions due to the acceleration energy. In stationary motions there is no acceleration energy, which behaves as an invisible source of energy and accounts for the self-accelerated motions and other physically meaningless situations. So, for instance, in the self-accelerated, motions of a free particle the acceleration energy tends to $-\infty$ when time tends to ∞ and, thus, allows the kinetic energy to increase indefinitely, supplying at the same time the radiated energy. It is necessary, in the cases in which the Lorentz-Dirac equations are valid, to introduce suitable time boundary conditions in order to avoid the difficulties due to the acceleration energy. This was already done by Dirac[1] in the case of a particle which receives a light pulse. We will discuss later these time boundary conditions in a general form. In Eliezer's analysis the initial conditions in the recessive motion of the electron are taken in such a way that this motion is of the self-accelerated type; the decrease of the acceleration energy allows the particles to get apart with steadily increasing relative velocity. The time boundary condition that we introduce in Section 18 excludes such motions.

11. Synge[7] has discussed the electromagnetic two-body problem, assuming that each particle generates a retarded field which acts on the others, but not on itself. He found that, with these assumptions, there are no stable circular orbits, though the rates of shrinking of the radii are small. He found also an emission of radiation at a rate much smaller than the Larmor rate. The considerations we developed in Part I show that the basic assumptions of Synge's theory are not entirely satisfactory, because they exclude the action of the radiated part of the field of a particle on it, but not on the other particles. It can be seen that the small emission of radiation and the correlated shrinkage of the radii of the circular orbits are precisely due to the interactions between each of the particles and the radiated field of the other. If it is assumed, as we did in our theory of the stationary motions, that the two particles do not generate radiated fields then there will be permanent circular orbits. (See Appendix I.)

It is noteworthy that, even in Synge's theory, there are permanent circular orbits when one of the particles has infinite mass. In this case there are no radiation losses because the radiated field of the heavy particle vanishes and, thus, there are no dissipative forces acting on the light particle. This shows that there is a radiative loss due to the action of the radiated field of one particle on the other.

PART III

Electromagnetic Fields Generated by Point Particles

12. In Parts I and II we have shown how it is possible to derive unique equations of motion for charged point particles, avoiding the difficulties due to infinite self-forces. The whole theory of the equations of motion was built on two basic postulates

$$\begin{aligned} &\text{(I)} \qquad F^{\mu\nu}{}_{\text{act}} = F^{\mu\nu}{}_{\text{ext}} + F^{\mu\nu}{}_{\text{rad}}, \\ &\text{(II)} \qquad F^{\mu\nu}{}_{\text{rad}} = \frac{\eta}{2}(F^{\mu\nu}{}_{\text{ret}} - F^{\mu\nu}{}_{\text{adv}}). \end{aligned} \tag{42}$$

We considered three types of motions characterized by the values -1, 0, and 1 of the parameter η. These three types include all the known motions of electric point particles, but the theory of the equations of motion can be developed for any value of η. In any case there is a self-force $K^{\mu}{}_{\text{self}}$

$$K^{\mu}{}_{\text{self}} = \frac{e}{c} F^{\mu\nu}{}_{\text{rad}} \frac{dx_\nu}{ds} = \frac{2}{3}\frac{e^2}{c}\eta\left(\frac{d^3x^\mu}{ds^3} + \frac{dx^\mu}{ds}\frac{d^2x_\nu}{ds^2}\frac{d^2x^\nu}{ds^2}\right), \tag{43}$$

and the corresponding equations of motion are

$$mc\frac{d^2x^\mu}{ds^2} - \frac{2}{3}\frac{e^2}{c}\eta\left(\frac{d^3x^\mu}{ds^3} + \frac{dx^\mu}{ds}\frac{d^2x_\nu}{ds^2}\frac{d^2x^\nu}{ds^2}\right) = \frac{e}{c}F^{\mu\nu}{}_{\text{ext}}\frac{dx_\nu}{ds}. \tag{44}$$

The total field generated by the particle $F^{\mu\nu}{}_{\text{part}}$ is

$$F^{\mu\nu}{}_{\text{part}} = F^{\mu\nu}{}_{\text{at}} + F^{\mu\nu}{}_{\text{rad}} = \frac{1+\eta}{2}F^{\mu\nu}{}_{\text{ret}} + \frac{1-\eta}{2}F^{\mu\nu}{}_{\text{adv}}, \tag{45}$$

and satisfies the Maxwell equations

$$\begin{aligned} &\frac{\partial F^{\mu\nu}{}_{\text{part}}}{\partial x^\nu} = -4\pi\frac{e}{c}\frac{dx^\mu}{dt}\delta(x^1 - x^1(t))\delta(x^2 - x^2(t)) \\ &\frac{\partial F^{\mu\nu}{}_{\text{part}}}{\partial x^\lambda} + \frac{\partial F^{\lambda\mu}{}_{\text{part}}}{\partial x^\nu} + \frac{\partial F^{\nu\lambda}{}_{\text{part}}}{\partial x^\mu} = 0. \end{aligned} \tag{46}$$

Until now we have not investigated whether there is conservation of energy and momentum of the system field+particles. In order to investigate this point let us introduce the stress tensor of the field $T_\nu{}^\mu$

$$4\pi T_\nu{}^\mu = F^{\mu\rho}{}_{\text{tot}}F_{\text{tot},\,\rho\nu} + \tfrac{1}{4}\delta_\nu{}^\mu(F^{\rho\sigma}{}_{\text{tot}}F_{\text{tot},\,\rho\sigma}) - \sum_i[F^{\mu\rho}{}_{i,\,\text{at}}F_{i,\,\text{at};\,\rho\nu} + \tfrac{1}{4}\delta_\nu{}^\mu(F^{\rho\sigma}{}_{i,\,\text{at}}F_{i,\,\text{at};\,\rho\sigma})]. \tag{47}$$

$F^{\mu\nu}{}_{\text{tot}}$ is the total field created by the particles plus the field of the incoming external waves $F^{\mu\nu}{}_{\text{wav}}$

$$F^{\mu\nu}{}_{\text{tot}} = \sum_i F^{\mu\nu}{}_{i,\,\text{part}} + F^{\mu\nu}{}_{\text{wav}}. \tag{48}$$

Our choice of the stress tensor is justified, because there is a conservation equation

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = -\sum_i \frac{e_i}{c}F^{\mu\rho}{}_{i,\,\text{act}}\frac{dx_{i,\,\rho}}{dt}\delta(x^1 - x_i{}^1) \times\delta(x^2 - x_i{}^2)\delta(x^3 - x_i{}^3). \tag{49}$$

We may write, instead of (47),

$$\begin{aligned} 4\pi T_\nu{}^\mu = {} & (F^{\mu\rho}{}_{\text{wav}} + \textstyle\sum_i F^{\mu\rho}{}_{i,\,\text{rad}})(F_{\text{tot};\,\rho\nu} + \sum_j F_{j,\,\text{at};\,\rho\nu}) \\ & + \tfrac{1}{4}\delta_\nu{}^\mu(F^{\rho\sigma}{}_{\text{wav}} + \textstyle\sum_i F^{\rho\sigma}{}_{i,\,\text{rad}})(F_{\text{tot};\,\rho\sigma} + \sum_j F_{j,\,\text{at};\,\rho\sigma}) \\ & + \sum_i\sum_{j\neq i}(F^{\mu\rho}{}_{i,\,\text{at}}F_{j,\,\text{at};\,\rho\nu} + \tfrac{1}{4}\delta_\nu{}^\mu F^{\rho\sigma}{}_{i,\,\text{at}}F_{j,\,\text{at};\,\rho\sigma}). \end{aligned} \tag{50}$$

Equation (50) shows that $T^{\mu\nu}$ diverges only as r^{-2} at the particles' positions, because the radiated fields are finite and the attached fields diverge as r^{-2} at the positions of the corresponding particles. Therefore, the space integrals of the components of the stress tensor over any region are finite and they are not differences of infinite quantities, as it might appear from Eq. (47).

The momentum of the field is $G^\mu{}_{\text{field}}$

$$G^\mu{}_{\text{field}} = \int T^{0\mu}d\tau. \tag{51}$$

The components of $G^\mu{}_{\text{field}}$ are finite; thus we have been able to get a finite energy and finite momenta for the field, without introducing unknown quantities of non-electromagnetic nature and without taking differences of infinite quantities.

13. The momentum $G^\mu{}_{\text{field}}$ has a remarkable property: it is a four vector. Let us represent by $G^\mu{}_f(V)$ the momentum of the field contained in a volume V of space. If we take as volume V the entire space minus n small spheres of radii α with centers at the point particles, we shall have at any point of V

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}}=0. \tag{52}$$

According to a well-known theorem, it results from (52) that the $G^{\mu}{}_{f}(V)$ are the components of a four vector. If we make α tend to zero, the $G^{\mu}{}_{f}(V)$ will go over continuously into the $G^{\mu}{}_{\text{field}}$ because the components of the stress tensor diverge only as r^{-2} at the particles' positions. Therefore the $G^{\mu}{}_{\text{field}}$ are the components of a four vector.

It is noteworthy that we have been able to obtain a four vector of momentum for the field without excluding the Coulomb forces, but only the self-forces due to the attached fields. It is well known that the classical objection against the theories which consider the mass as being of electromagnetic origin was based on the non-vectorial nature of the field's momentum $G^{\mu}{}_{\text{field}}$. But this objection disappears when we take for the field's stress tensor the expression (47). However, our theory does not lead, at least in a straightforward way, to an electromagnetic theory of the mass. So, for instance, the energy of the field created by a particle in a free stationary motion is *nihil.*

The Poynting Vector

14. The components P^{a} of the Poynting vector of the field, in which there are n point particles, are

$$P^{a}=cT^{0a}. \tag{53}$$

Hence

$$\mathbf{P}=\frac{c}{4\pi}[\mathbf{E}_{\text{tot}}\times\mathbf{H}_{\text{tot}}]-\frac{c}{4\pi}\sum_{i}[\mathbf{E}_{i,\,\text{at}}\times\mathbf{H}_{i,\,\text{at}}]. \tag{54}$$

From Eq. (49) it follows that

$$\frac{d}{dt}G^{0}{}_{\text{field}}=-\lim_{S\to\infty}\int_{S}(\mathbf{P}\cdot\mathbf{n})dS-\sum_{i}\frac{e_i}{c}F^{0\nu}{}_{i,\,\text{act}}\frac{dx_{i,\,\nu}}{dt}. \tag{55}$$

$\mathbf{n}$ is the unit vector on the outer normal of S. The surface integral in the right-hand side of Eq. (55) does not vanish automatically when $S\to\infty$ because, in general, the electromagnetic field varies inversely with the distance in the wave zone, so that the Poynting vector varies as the square of the inverse of the distance to the point which we take as origin of the coordinates. In order to make the surface integral tend to zero, we must impose suitable boundary conditions to the motion of the charged particles. We shall examine later these conditions.

In order to investigate the behavior of the field at the boundary it is convenient to replace the retarded and advanced fields by their parts which vary inversely with the distance. We have:[12]

$$\begin{aligned}\mathbf{E}_{\text{ret}}&=-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}^{\text{tr}}{}_{\text{ret}}, & \mathbf{H}_{\text{ret}}&=[\mathbf{u}\times\mathbf{E}_{\text{ret}}],\\ \mathbf{E}_{\text{adv}}&=-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}^{\text{tr}}{}_{\text{adv}}, & \mathbf{H}_{\text{adv}}&=[\mathbf{E}_{\text{adv}}\times\mathbf{u}].\end{aligned} \tag{56}$$

$\mathbf{u}$ is the unit vector in the direction of the position vector of the point in which the fields are computed, relatively to the origin of the coordinates. This origin is taken as a fixed point, inside the charge distribution. $\mathbf{A}^{\text{tr}}{}_{\text{ret}}$ and $\mathbf{A}^{\text{tr}}{}_{\text{adv}}$ are the transversal parts of the retarded and advanced vector potentials (we call transversal components those perpendicular to $\mathbf{u}$ and radial those parallel to $\mathbf{u}$). In the wave zone we may take for the transversal components of the retarded and advanced potentials of the field generated by the current density $c\mathbf{J}$ the following values:

$$\begin{aligned}\mathbf{A}^{\text{tr}}{}_{\text{ret}}&=\frac{1}{R}\int\mathbf{J}^{\text{tr}}(x'^{a},\,x^{0}-R-u_{a}x'^{a})dx'^{1}dx'^{2}dx'^{3},\\ \mathbf{A}^{\text{tr}}{}_{\text{adv}}&=\frac{1}{R}\int\mathbf{J}^{\text{tr}}(x'^{a},\,x^{0}+R+u_{a}x'^{a})dx'^{1}dx'^{2}dx'^{3}.\end{aligned} \tag{57}$$

R is the distance between the origin of the coordinates and the point in the wave zone where the potentials are computed. x^{0} corresponds to the time: $x^{0}=ct$.

It results from formulae (56) that the Poynting vector of a field, in which there are no other waves besides those of the radiated parts of the particles, is

$$\mathbf{P}=\frac{c}{16\pi}[(1+\eta)^{2}(\textstyle\sum_{i}\mathbf{E}_{i,\,\text{ret}})^{2}-(1-\eta)^{2}(\textstyle\sum_{i}\mathbf{E}_{i,\,\text{adv}})^{2}]\mathbf{u}-\frac{c}{16\pi}\textstyle\sum_{i}(\mathbf{E}^{2}{}_{i,\,\text{ret}}-\mathbf{E}^{2}{}_{i,\,\text{adv}})\mathbf{u}. \tag{58}$$

[12] W. Pauli, *Handbuch der Physik* (1933), XXIV/1, p. 203.

It can be seen that (see Appendix II):

$$\int_{-\infty}^{+\infty}(\sum_i \mathbf{E}_{i,\,\mathrm{ret}})^2dt=\int_{-\infty}^{+\infty}(\sum_i \mathbf{E}_{i,\,\mathrm{adv}})^2dt,$$
$$\int_{-\infty}^{+\infty}\mathbf{E}^2_{i,\,\mathrm{ret}}dt=\int_{-\infty}^{+\infty}\mathbf{E}^2_{i,\,\mathrm{adv}}dt. \tag{59}$$

Hence, we have:

$$\int_{-\infty}^{+\infty}\mathbf{P}dt=\frac{c}{4\pi}\eta\mathbf{u}\int_{-\infty}^{+\infty}(\sum_i \mathbf{E}_{i,\,\mathrm{ret}})^2dt$$
$$=\frac{c}{4\pi}\eta\mathbf{u}\int_{-\infty}^{+\infty}(\sum_i \mathbf{E}_{i,\,\mathrm{adv}})^2dt. \tag{60}$$

Equation (60) shows that there is a radiation loss when $\eta>0$, a gain when $\eta<0$, and an average conservation of energy in stationary motions since $\eta=0$.

The Stress Tensor of the Radiation Field

15. Equation (50) shows that the stress tensor of the total field is the sum of two parts: a part which contains the radiated fields of the particles and a part which does not contain these fields. We shall call the first prat stress tensor of the radiation field and denote it by $T^{\mu\nu}{}_{\mathrm{rad}}$

$$4\pi T^{\mu}{}_{\mathrm{rad};\,\nu}=(F^{\mu\rho}{}_{\mathrm{wav}}+\sum_i F^{\mu\rho}{}_{i,\,\mathrm{rad}})$$
$$\times(F_{\mathrm{tot};\,\rho\nu}+\sum_j F_{j,\,\mathrm{at};\,\rho\nu})$$
$$+\tfrac{1}{4}\delta_\nu{}^\mu(F^{\rho\sigma}{}_{\mathrm{wav}}+\sum_i F^{\rho\sigma}{}_{i,\,\mathrm{rad}})$$
$$\times(F_{\mathrm{tot};\,\rho\sigma}+\sum_j F_{j,\,\mathrm{at};\,\rho\sigma}). \tag{61}$$

We shall call the second part stress tensor of the interior actions and denote it by $T^{\mu\nu}{}_{\mathrm{int}}$

$$4\pi T^{\mu}{}_{\mathrm{int};\,\nu}=\sum_i\sum_{j\neq i}(F^{\mu\rho}{}_{i,\,\mathrm{at}}F_{j,\,\mathrm{at};\,\rho\nu}$$
$$+\tfrac{1}{4}\delta_\nu{}^\mu F^{\rho\sigma}{}_{i,\,\mathrm{at}}F_{j,\,\mathrm{at};\,\rho\sigma}). \tag{62}$$

Equation (49) can be split in two:

$$\frac{\partial T^{\mu\nu}{}_{\mathrm{rad}}}{\partial x^\nu}=-\sum_i\frac{e_i}{c}(F^{\mu\rho}{}_{\mathrm{wav}}+\sum_j F^{\mu\rho}{}_{j,\,\mathrm{rad}})\frac{dx_{i,\,\rho}}{dt}$$
$$\times\delta(x^1-x_i{}^1)\delta(x^2-x_i{}^2)\delta(x^3-x_i{}^3), \tag{63}$$

$$\frac{\partial T^{\mu\nu}{}_{\mathrm{int}}}{\partial x^\nu}=-\sum_i\frac{e_i}{c}\Big(\sum_{j\neq i}F^{\mu\rho}{}_{j,\,\mathrm{at}}\Big)\frac{dx_{i,\,\rho}}{dt}$$
$$\times\delta(x^1-x_i{}^1)\delta(x^2-x_i{}^2)\delta(x^3-x_i{}^3). \tag{64}$$

We may define a Poynting vector for the radiation field and a Poynting vector for the interior actions:

$$\mathbf{P}_{\mathrm{rad}}=\frac{c}{4\pi}[\mathbf{E}_{\mathrm{tot}}\times\mathbf{H}_{\mathrm{tot}}]$$
$$-\frac{c}{4\pi}[(\sum_i \mathbf{E}_{i,\,\mathrm{at}})\times(\sum_j \mathbf{H}_{j,\,\mathrm{at}})], \tag{65}$$

$$\mathbf{P}_{\mathrm{int}}=\frac{c}{4\pi}[(\sum_i \mathbf{E}_{i,\,\mathrm{at}})\times(\sum_j \mathbf{H}_{j,\,\mathrm{at}})]$$
$$-\frac{c}{4\pi}\sum_i[\mathbf{E}_{i,\,\mathrm{at}}\times\mathbf{H}_{i,\,\mathrm{at}}]. \tag{66}$$

Taking in account Eqs. (59), we get easily, in the wave zone:

$$\int_{-\infty}^{+\infty}\mathbf{P}_{\mathrm{rad}}dt=\int_{-\infty}^{+\infty}\mathbf{P}dt, \tag{67}$$

$$\int_{-\infty}^{+\infty}\mathbf{P}_{\mathrm{int}}dt=0. \tag{68}$$

Equation (68) shows the conservative character of the actions due to the attached fields: in the average they do not lead to any energy loss. The radiation losses are entirely due to the action of the radiation field on the particles, as it results from Eq. (67). Not only the particle's own radiated field, but also the radiated fields of the other particles and the field of the traveling waves lead to radiative losses of energy.

The Actions at a Distance

16. We cannot consider the interior actions to be due to a field, because the interior force which acts on each particle is due to the attached fields of all the others, so that there is no single field which accounts for all the interior forces.

The interior actions of a system are the actions at a distance of the Tetrode[13] theory; they are also the same forces considered by Fokker[9] and Wheeler. These actions at a distance are a generalization of the forces of the old electrostatical theory: it can be seen that the space integral of $T^{oo}{}_{\mathrm{int}}$ coincides with the electrostatical energy W of Eq. (2), in the case of charges at rest. Since we have now relativistic actions at a distance, we can understand why they do depend on the motions of the particles at different times: they are actions at a distance both in space and time. These interior forces do not lead to any

[13] H. Tetrode, Zeits. f. Physik **10**, 317 (1922).

loss of energy, momentum, or angular momentum as we have seen in the analysis of the stationary motions; we also got the same result from the analysis of the field.

In the same way as the interior forces, the total forces acting on the particles actually do not arise from any field, because for each particle we have a different $F^{\mu\nu}{}_{i,\,\mathrm{act}}$. This is the main point of the theory of charged point particles:

In order to exclude infinite self forces we must introduce for each particle a different acting field so that we can no longer say that the total forces acting on all the particles arise from the existence of a single field, as it happens in the Faraday-Maxwell theory.

Our theory of the point particles may be considered as a synthesis of the Faraday-Maxwell theory of actions through a field with the Tetrode conception of relativistic actions at a distance. The total radiation field is, indeed, a field in the Faraday-Maxwell sense and it accounts for the propagation of the electromagnetic waves. But, besides the field forces, there are also actions at a distance which are due to the attached fields of the particles and which do not correspond to any wave propagation. The circumstance that the attached fields are half-retarded and half-advanced shows precisely that they are not propagated, because they are symmetrical in relation to both directions of time flow.

The difference between the actions at a distance and the actions through the field becomes sharper in quantum theory: the radiation field is quantized but the actions at a distance remain unquantized. Presumably the difficulties which appear in the quantum theory of the interaction of point particles with fields are all due to the physically inacceptable identification of the actions at a distance with actions through the field. In particular, the nuclear forces should be considered as actions at a distance of non-electromagnetic nature.

PART IV

Boundary Conditions

17. The various types of motions of the particles are characterized by the kind of radiated field they create. The radiated field is a solution of the Maxwell equations with no current density which can be characterized by convenient boundary conditions. We may also characterize the motions by means of the $F^{\mu\nu}{}_{i,\,\mathrm{part}}$, which are solutions of the Maxwell equations corresponding to the current density generated by the particle, determined by suitable boundary conditions. Therefore, our whole theory is contained in the Eqs. (69)

$$m_i c\frac{d^2x_i^\mu}{ds_i^2}=\frac{e_i}{c}\Big(F^{\mu\nu}{}_{\mathrm{ext}}+\textstyle\sum_j F^{\mu\nu}{}_{j,\,\mathrm{part}} - F^{\mu\nu}{}_{i,\,\mathrm{at}}\Big)\frac{dx_{i,\,\nu}}{ds_i}, \quad (69a)$$

$$\frac{\partial^2 A^\mu{}_{i,\,\mathrm{part}}}{\partial x^\rho\partial x_\rho}=4\pi\frac{e_i}{c}\frac{dx_i^\mu}{dt}\delta(x^1-x_i^1)\times\delta(x^2-x_i^2)\delta(x^3-x_i^3), \quad (69b)$$

together with the boundary conditions for the $A^\mu{}_{i,\,\mathrm{part}}$.

Boundary Conditions for the $\mathbf{A}^\mu{}_{i,\,\mathrm{part}}$

It can be seen that the boundary condition for $A^\mu{}_{i,\,\mathrm{part}}$ is

$$(1+\eta_i)\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}+\frac{1}{c}\frac{\partial}{\partial t}\right)A^\mu{}_{i,\,\mathrm{part}}\left(\mathbf{r},\,t-\frac{r}{c}\right) = -(1-\eta_i)\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}-\frac{1}{c}\frac{\partial}{\partial t}\right)\times A^\mu{}_{i,\,\mathrm{part}}\left(\mathbf{r},\,t+\frac{r}{c}\right). \quad (70)$$

The boundary condition (70) results immediately from the well-known formulae

$$A^\mu(\mathbf{r},t)=\int_V\frac{J^\mu(\mathbf{r},\,t-r/c)}{r}d\mathbf{r} -\frac{1}{4\pi}\int_S\frac{1}{r}\frac{\partial A^\mu(\mathbf{r},\,t-r/c)}{\partial n}dS +\frac{1}{4\pi}\int_S\left[A^\mu\left(\mathbf{r},\,t-\frac{r}{c}\right)\frac{\partial(1/r)}{\partial n} +\frac{1}{cr}\frac{\partial A^\mu(\mathbf{r},\,t-r/c)}{\partial t}\right]dS, \quad (71a)$$

$$A^\mu(\mathbf{r},t)=\int_V \frac{J^\mu(\mathbf{r},t+r/c)}{r}d\mathbf{r}$$
$$-\frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial A^\mu(\mathbf{r},t+r/c)}{\partial n}dS$$
$$+\frac{1}{4\pi}\int_S\left[A^\mu\left(\mathbf{r},t+\frac{r}{c}\right)\frac{\partial(1/r)}{\partial n}-\frac{1}{cr}\frac{\partial A^\mu(\mathbf{r},t+r/c)}{\partial t}\right]dS, \quad (71\text{b})$$

which give two alternative expressions of any solution A^μ of the d'Alembert equations

$$\frac{\partial^2 A^\mu}{\partial x^\rho \partial x_\rho}=4\pi J^\mu. \quad (72)$$

$\mathbf{r}$ is the position vector of the point where the potentials are computed; V is a region of space limited by the closed surface S; n is the inner normal of S.

Until now we have considered only systems in which all the particles have the same value of η; Eq. (70) corresponds to the most general case in which each particle has its own η_i.

18. Excepting the stationary motions in which the equations of motion are of the second order, the equations of motion are in general of the third order. Therefore, it is no more sufficient to give the initial positions and velocities of the particles in order to determine their motions, as it happens in non-relativistic dynamics. Dirac[1] has shown in a particular case, that it is possible to determine completely the motion of a particle by giving its initial position and velocity and imposing a time boundary condition for $t=\infty$. This time boundary condition was introduced in order to avoid motions with indefinitely increasing self-accelerations of the particle. Since the self-accelerations are due to the possibility of an indefinite decrease of the acceleration energy, we can generalize Dirac's time boundary condition in the following way:

(III) "The acceleration energy and momentum of the particles of a system cannot decrease or increase indefinitely when t tends to $\pm\infty$, if the particles are not accelerated indefinitely by exterior forces."

It follows from this condition that, in general, the acceleration of the particles of an isolated system must tend to zero when t tends to $\pm\infty$, if the motion of the system is not stationary. Indeed, the rate of momentum loss by the ith particle of a system is

$$\frac{2}{3}\eta_i\frac{e_i^2}{c}\frac{dx_i^\mu}{ds_i}\frac{d^2x_i^\rho}{ds_i^2}\frac{d^2x_{i,\rho}}{ds_i^2}$$

so, if its acceleration does not tend to zero when t tends to $\pm\infty$, the total amount of radiated energy will be infinite and, since there is conservation of energy, this can only happen if the acceleration energy diverges, if we exclude an initial divergence of the kinetic energy. There is also the possibility of an infinite radiation of energy with a collapse of the system which liberates an infinite amount of potential energy, without an indefinite acceleration of the particles. The boundary condition III does not impose any restriction at all to the behavior of the systems in stationary motion, because in such motions there is no acceleration momentum, since $\eta_i=0$.

The boundary condition III is of a more restrictive kind than the condition introduced by Dirac, because it restricts the behavior both for $t=\infty$ and $t=-\infty$. However, in the particular case considered by Dirac it is satisfied automatically for $t=-\infty$. It is necessary to restrict the behavior at both time boundaries in order to have a vanishing flux of energy at the space boundary of the field: it is easily seen that the Poynting vector of the field created by the particles vanishes at infinity if the accelerations of the particles vanish for $t=\pm\infty$. Therefore, Eq. (55) becomes

$$\frac{d}{dt}G^0{}_{\text{field}}=-\frac{1}{c}\sum_i e_iF^{0\nu}{}_{i,\,\text{act}}\frac{dx_{i,\nu}}{dt}, \quad (73)$$

and shows that there is conservation of the energy of the system particles plus field. So we see that the *time boundary conditions for the motion of the particles are necessary in order to insure the conservation of energy, in motions which are not stationary.* In stationary motions the conservation of energy is automatically insured by the nature of the interior actions, so that it is not necessary to introduce time boundary conditions.

The time boundary condition III is compatible only with two kinds of behavior of the particles for $t=+\infty$: (a) uniform rectilinear motions of the particles infinitely separated, or (b) a collapsed configuration of a part of the system with the other particles infinitely separated with uniform rectilinear motions.

PART V. THE ACTION PRINCIPLE

19. The equations of motion of the field and the particles can be derived from an action principle

$$\delta I=0, \tag{74}$$

$$\begin{aligned} I=&-\sum_i m_i c\int_{-\infty}^{+\infty} ds_i-\sum_i \frac{e_i}{c}\int_{-\infty}^{+\infty} A^\mu{}_{\text{rad}}\frac{dx_{i,\mu}}{ds_i}ds_i \\ &-\frac{1}{2}\sum_i\sum_{j\neq i}\frac{e_ie_j}{c}\int\int_{-\infty}^{+\infty}\frac{dx_i{}^\mu}{ds_i}\frac{dx_{j,\mu}}{ds_j} \\ &\times\delta(\{\mathbf{x}_i{}^\rho-\mathbf{x}_j{}^\rho\}\{\mathbf{x}_{i,\rho}-\mathbf{x}_{j,\rho}\})ds_ids_j \\ &-\frac{1}{16\pi c}\int(F^{\rho\sigma}{}_{\text{tot}}+\textstyle\sum_i F^{\rho\sigma}{}_{i,\text{at}})F_{\text{rad},\rho\sigma} \\ &\times dx^0dx^1dx^2dx^3. \end{aligned} \tag{75}$$

$F^{\rho\sigma}{}_{\text{rad}}$ denotes the total radiation field

$$F^{\rho\sigma}{}_{\text{rad}}=F^{\rho\sigma}{}_{\text{wav}}+\textstyle\sum_i F^{\rho\sigma}{}_{i,\text{rad}}. \tag{76}$$

In the action principle (74) we variate the coordinates of the particles $x_i{}^\mu$ and the potentials $A^\mu{}_{\text{rad}}$ of the total radiation field, the Euler equations are

$$m_ic\frac{d^2x_i{}^\mu}{ds_i{}^2}=\frac{e_i}{c}\left(F^{\mu\nu}{}_{\text{rad}}+\sum_{j\neq i}F^{\mu\nu}{}_{j,\text{at}}\right)\frac{dx_{i,\nu}}{ds_i}, \tag{77}$$

$$\frac{\partial^2A^\nu{}_{\text{rad}}}{\partial x^\mu\partial x_\mu}=0, \tag{78}$$

provided we introduce the supplementary conditions

$$\frac{dx_i{}^\mu}{ds_i}\frac{dx_{i,\mu}}{ds_i}=1, \tag{79}$$

$$\frac{\partial A^\mu{}_{\text{rad}}}{\partial x^\mu}=0. \tag{80}$$

In the action principle (74) we consider the $A^\mu{}_{i,\text{at}}$ as functionals of the coordinates of the particles, taken as functions of the respective s, defined by Eq. (9). It is worthwhile to observe that even if we variate separately the $A^\mu{}_{i,\text{rad}}$ we get only the Eqs. (78). However, Eqs. (78) and (80) can always be satisfied by assuming that

$$\begin{aligned} \frac{\partial^2A^\nu{}_{i,\text{rad}}}{\partial x^\mu\partial x_\mu}=0,\quad \frac{\partial^2A^\nu{}_{\text{wav}}}{\partial x^\mu\partial x_\mu}=0, \\ \frac{\partial A^\mu{}_{i,\text{rad}}}{\partial x^\mu}=0,\quad \frac{\partial A^\mu{}_{\text{wav}}}{\partial x^\mu}=0. \end{aligned} \tag{81}$$

22. We can get an action principle for the system of the particles, from (74), by neglecting the last integral in the right-hand side of (75):

$$\delta L=0, \tag{82}$$

$$\begin{aligned} L=&-\sum_i m_ic\int_{-\infty}^{+\infty}ds_i-\sum_i\frac{e_i}{c}\int_{-\infty}^{+\infty}A^\mu{}_{\text{rad}}\frac{dx_{i,\mu}}{ds_i}ds_i \\ &-\frac{1}{2}\sum_i\sum_{j\neq i}\frac{e_ie_j}{c}\int\int_{-\infty}^{+\infty}\frac{dx_i{}^\mu}{ds_i}\frac{dx_{j,\mu}}{ds_j}\delta(\{\mathbf{x}_i{}^\rho-\mathbf{x}_j{}^\rho\} \\ &\{\mathbf{x}_{i,\rho}-\mathbf{x}_{j,\rho}\})ds_ids_j. \end{aligned} \tag{83}$$

The action principle (82) coincides with the action principle of Tetrode[13] and Fokker[9] when

$$\eta_i=0\quad(i=1,2,\cdots n), \tag{84}$$

and with Dirac's[1] action principle when

$$\eta_i=1\quad(i=1,2,\cdots\cdot n). \tag{85}$$

APPENDIX I

Circular Orbits of Two Particles

Let us consider two charged particles describing concentrical circular orbits in a plane, in such a way that the line joining both particles passes at each moment through the center of the circular orbits. It is easily seen that the electric field of each particle, at the position occupied by the other, is directed along the radius of the orbit and the magnetic field is perpendicular to the orbits' plane. Therefore, each particle will be attracted to the center of the orbits by a constant radial force, in case both have opposite signs and are, at each moment, at different sides in relation to the center. We must see whether it is possible to determine the radii R_1 and R_2 of the two orbits and the circular frequency ω of the two circular motions in such a way that the equations of stationary motion of the two particles be satisfied. Since the forces acting on both particles are radial, we must consider only two equations:

$$\frac{m_i}{\alpha_i}R_i\omega^2=e_i\left[E_{j,\,\mathrm{at}}+\frac{\omega}{c}R_iH_{j,\,\mathrm{at}}\right], \qquad (86)$$

$$\alpha_i=\left[1-\frac{\omega^2}{c^2}R_i^2\right]^{\frac{1}{2}}. \qquad (87)$$

$E_{j,\,\mathrm{at}}$ and $H_{j,\,\mathrm{at}}$ are, respectively, the moduli of $\mathbf{E}_{j,\,\mathrm{at}}$ and $\mathbf{H}_{j,\,\mathrm{at}}$. For each value of ω we have a system of two equations for the radii R_1 and R_2. It is evident from the circular symmetry of the forces that there must be circular orbits, though the equations are not simple enough to allow a direct solution.

If we take two particles describing circular orbits and assume that the retarded field of each one acts on the other, though not on itself, the circular symmetry of the forces disappears, because there are now tangential forces due to the action of the radiated fields of each particle on the other. These tangentials forces, which spoil the radial symmetry, produce a damping of the motions and a shrinkage of the radii. Assuming that the interaction between the particles is due to their attached fields, we get rid of the tangential forces, because the tangential forces due to the advanced fields cancel those due to the retarded fields.

APPENDIX II

Let us represent **J** by a Fourier integral

$$\mathbf{J}=\int_{-\infty}^{+\infty}\mathbf{J}(k)e^{-ickt}dk. \qquad (88)$$

Taking in account formulae (57), we get the Fourier integrals of the transversal components of $\mathbf{A}_{\mathrm{ret}}$ and $\mathbf{A}_{\mathrm{adv}}$ in the wave zone

$$\mathbf{A}^{\mathrm{tr}}{}_{\mathrm{ret}}=\frac{1}{R}\int_{-\infty}^{+\infty}\mathbf{J}^{\mathrm{tr}}(k)\exp\left[-ik(ct-R-u_ax'^a)\right]\times dkdx'^1dx'^2dx'^3, \qquad (89)$$

$$\mathbf{A}^{\mathrm{tr}}{}_{\mathrm{adv}}=\frac{1}{R}\int_{-\infty}^{+\infty}\mathbf{J}^{\mathrm{tr}}(k)\exp\left[-ik(ct+R+u_ax'^a\right]\times dkdx'^1dx'^2dx'^3. \qquad (90)$$

Therefore, we have, in the wave zone

$$\mathbf{E}_{\mathrm{ret}}=\frac{i}{R}\int_{-\infty}^{+\infty}\mathbf{J}^{\mathrm{tr}}(k)k\exp\left[-ik(ct-R-u_ax'^a)\right]\times dkdx'^1dx'^2dx'^3, \qquad (91)$$

$$\mathbf{E}_{\mathrm{adv}}=\frac{i}{R}\int_{-\infty}^{+\infty}\mathbf{J}^{\mathrm{tr}}(k)k\exp\left[-ik(ct+R+u_ax'^a)\right]\times dkdx'^1dx'^2dx'^3. \qquad (92)$$

It is well known that the time integral of the square of a quantity M represented by the Fourier integral

$$M=\int_{-\infty}^{+\infty}e^{-ickt}\mathfrak{M}(k)dk, \qquad (93)$$

is given by the formula

$$\int_{-\infty}^{+\infty}M^2dt=\frac{2\pi}{c}\int_{-\infty}^{\infty}\mathfrak{M}(k)\mathfrak{M}(-k)dk. \qquad (94)$$

Taking in account this formula and the Fourier integrals of $\mathbf{E}_{\mathrm{ret}}$ and $\mathbf{E}_{\mathrm{adv}}$ in the wave zone, we see that

$$\int_{-\infty}^{+\infty}\mathbf{E}^2{}_{\mathrm{ret}}dt=\int_{-\infty}^{+\infty}\mathbf{E}^2{}_{\mathrm{adv}}dt. \qquad (95)$$

Formulae (59) are special cases of this general relation.