Relativistic Interaction of Electrons on Podolsky's Generalized Electrodynamics

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The wave equation for a system of particles is derived on the basis of Podolsky's generalized electrodynamics. An extension of some work of Fock leads to a representation in terms of a series of functionals. With this formalism the matrix element for the relativistic interaction of two electrons is determined, and is seen to be a generalization of Møller's formula.

1. INTRODUCTION

IN a series of papers Podolsky[1] has formulated the basis of a generalized electrodynamics involving higher derivatives in the field equations, and Podolsky and Kikuchi[2] have developed the theory to include quantum electrodynamics. Here we extend the formalism, basing the treatment on the work of Fock,[3] and apply the results to the determination of the relativistic interaction of two electrons.

2. WAVE EQUATION FOR A SYSTEM OF PARTICLES

Derivation of Wave Equation for a System of Particles

According to Dirac, Fock, and Podolsky,[4] and GE II and GE III, the Dirac wave equation for a system of particles and field, together with their interaction, is

$$(H_p+\bar{H}_f+H_{pf})\Psi=i\hbar\partial\Psi/\partial T, \tag{2.1}$$

with

$$\Psi=\Psi(\mathbf{r}_1\cdots\mathbf{r}_n;\mathbf{A}(\mathbf{k}),\mathbf{A}^*(\mathbf{k}),\phi(\mathbf{k}),\phi^*(\mathbf{k});\tilde{\mathbf{A}}(\mathbf{k}),\tilde{\mathbf{A}}^*(\mathbf{k}),\tilde{\phi}(\mathbf{k}),\tilde{\phi}^*(\mathbf{k});T), \tag{2.2}$$

where

$$H_p\equiv\sum_{s=1}^{n}(c\boldsymbol{\alpha}_s\cdot\mathbf{p}_s+m_sc^2\beta_s); \tag{2.3}$$

$$\bar{H}_f\equiv\int[\mathbf{A}^*(\mathbf{k})\cdot\mathbf{A}(\mathbf{k})-\phi^*(\mathbf{k})\phi(\mathbf{k})+\mathbf{A}(\mathbf{k})\cdot\mathbf{A}^*(\mathbf{k})-\phi(\mathbf{k})\phi^*(\mathbf{k})]k^2d\mathbf{k}$$
$$-\int[\tilde{\mathbf{A}}^*(\mathbf{k})\cdot\tilde{\mathbf{A}}(\mathbf{k})-\tilde{\phi}^*(\mathbf{k})\tilde{\phi}(\mathbf{k})+\tilde{\mathbf{A}}(\mathbf{k})\cdot\tilde{\mathbf{A}}^*(\mathbf{k})-\tilde{\phi}(\mathbf{k})\tilde{\phi}^*(\mathbf{k})]\tilde{k}^2d\mathbf{k}; \tag{2.4}$$

$$H_{pf}\equiv\sum_{s=1}^{n}\epsilon_s[\phi(\mathbf{r}_s,T)-\boldsymbol{\alpha}_s\cdot\mathbf{A}(\mathbf{r}_s,T)]. \tag{2.5}$$

When the single Eq. (2.1) with common time T is replaced by the set of equations with separate times t_s in accordance with DFP, and after several transformations and the use of auxiliary conditions, it is shown in GE III that the equations to be solved are

$$(c\boldsymbol{\alpha}_s\cdot\mathbf{P}_s'+m_sc^2\beta_s)\Omega=T_s'\Omega, \tag{2.6}$$

where

$$\mathbf{P}_s'=\mathbf{p}_s-(\epsilon_s/c)\mathbf{D}(\mathbf{r}_s,t_s)-(\epsilon_s/2c)\nabla_sU_s, \tag{2.7}$$

$$T_s'=i\hbar\partial/\partial t_s-(\epsilon_s/2c)\partial U_s/\partial t_s-(\epsilon_s^2/8\pi a). \tag{2.8}$$

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[1] B. Podolsky, Phys. Rev. 62, 68 (1942). This paper has been called GE I.
[2] B. Podolsky and C. Kikuchi, Phys. Rev. 65, 228 (1944); 67, 184 (1945). These papers will be called GE II and GE III, respectively.
[3] V. Fock, Physik. Zeits. Sowjetunion 6, 425 (1934).
[4] P. Dirac, V. Fock, and B. Podolsky, Physik. Zeits. Sowjetunion 2, 468 (1932). This paper will be called DFP.

The last two equations are GE III (4.2) and (4.3), with obvious minor changes in notation. The definition of U_s is given by GE III (4.1):

$$U_s \equiv \sum_u' (\epsilon_u/8\pi^3)\cdot \int [(1/k^3)\sin(\varphi_s-\varphi_u)-(1/\tilde{k}^3)\sin(\tilde{\varphi}_s-\tilde{\varphi}_u)]d\mathbf{k},$$

where

$$\varphi_s = ckt_s - \mathbf{k}\cdot\mathbf{r}_s, \quad \tilde{\varphi}_s = c\tilde{k}t_s - \mathbf{k}\cdot\mathbf{r}_s.$$

In order to obtain the wave equation for the system, DFP shows that the equations for the individual particles are to be added, and the times are to be set equal to the common time T. Now it is readily shown that

$$\partial/\partial T = \partial/\partial t + \sum_s \partial/\partial t_s;$$

and inasmuch as Ω is independent of t, the effect of adding the equations and setting the times equal is to replace $\partial/\partial t_s$ in the individual equations by $\partial/\partial T$ in the combined equation. The resultant equation is further simplified by showing that $\nabla_s U_s = 0$ when the times are set equal, and making use of GE III (4.3) and (4.12), namely:

$$(1/c)\partial U_s/\partial t_s = \sum_u'(\epsilon_u/4\pi|\mathbf{r}_s-\mathbf{r}_u|)[1-\exp(-|\mathbf{r}_s-\mathbf{r}_u|/a)].$$

The resulting wave equation for a system of particles in the generalized quantum electrodynamics is thus[5]

$$\sum_s \{\boldsymbol{\alpha}_s\cdot[c\mathbf{p}_s-\epsilon_s\mathbf{D}(\mathbf{r}_s,t)]+m_sc^2\beta_s\}\Omega = \{i\hbar\partial/\partial t-(1/8\pi a)\sum_s \epsilon_s^2 - \sum_{s,u}'(\epsilon_s\epsilon_u/8\pi|\mathbf{r}_s-\mathbf{r}_u|)[1-\exp(-|\mathbf{r}_s-\mathbf{r}_u|/a)]\}\Omega. \quad (2.9)$$

Representation of Wave Equation in Functional Formalism[6]

It will be convenient to transform the field variables to their Fourier amplitudes, by means of GE II (3.3):

$$\mathbf{D}(\mathbf{r}_s,t) = (1/2\pi)^{\frac{3}{2}}\int \{\mathbf{D}(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}_s-kct)]+\mathbf{D}^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}_s-kct)]\}d\mathbf{k} + (1/2\pi)^{\frac{3}{2}}\int \{\tilde{\mathbf{D}}(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}_s-\tilde{k}ct)]+\tilde{\mathbf{D}}^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}_s-\tilde{k}ct)]\}d\mathbf{k}.$$

From the commutation rules in GE III (3.7),[7] it follows that $\mathbf{D}(\mathbf{k})$ may be represented by

$$\mathbf{D}(\mathbf{k}) = (c\hbar/2k)^{\frac{1}{2}}\sum_{j=1}^{3}\beta_j(1/k)\mathbf{k}\times\mathbf{e}_j b(\mathbf{k},j), \quad (2.10)$$

where $\beta_j^2=1$, $\mathbf{e}_j$ are a set of Cartesian unit base vectors, and the $b(\mathbf{k},j)$ are operators satisfying

$$[b(\mathbf{k},j), b^*(\mathbf{k}',j')] = \delta_{jj'}\delta(\mathbf{k}-\mathbf{k}'); \quad (2.11)$$

and $\tilde{\mathbf{D}}(\mathbf{k})$ by

$$\tilde{\mathbf{D}}(\mathbf{k}) = (c\hbar/2\tilde{k})^{\frac{1}{2}}\sum_{j=1}^{3}\tilde{\beta}_j(1/a\tilde{k})(a\mathbf{k}\times\mathbf{e}_j+\mathbf{e}_j)\tilde{b}(\mathbf{k},j), \quad (2.12)$$

where $\tilde{\beta}_j^2=1$, $\mathbf{e}_j$ are defined previously, and $\tilde{b}(\mathbf{k},j)$ are operators satisfying

$$[\tilde{b}(\mathbf{k},j), \tilde{b}^*(\mathbf{k}',j')] = -\delta_{jj'}\delta(\mathbf{k}-\mathbf{k}'). \quad (2.13)$$

[5] This equation was derived earlier by C. Kikuchi in different manner, but has not been previously published.
[6] This is the formalism developed by Fock in the reference of footnote 3.
[7] Note the typographical error in the second of the equations mentioned; the tilde over the k^2 in the right-hand side has been omitted.

For, according to (2.10) and (2.11),

$$[D_l(\mathbf{k}), D_m^*(\mathbf{k}')] = (c\hbar/2k)^{\frac{1}{2}}(c\hbar/2k')^{\frac{1}{2}} \sum_{j,j'} \beta_j\beta_{j'}(1/k)(1/k')(\mathbf{k}\times\mathbf{e}_j)\cdot\mathbf{e}_l(\mathbf{k}'\times\mathbf{e}_{j'})\cdot\mathbf{e}_m[b(\mathbf{k},j), b^*(\mathbf{k}',j')]$$
$$= (c\hbar/2k^3)\sum_{j,j'}\beta_j\beta_{j'}(\mathbf{k}\times\mathbf{e}_l\cdot\mathbf{e}_j)(\mathbf{k}\times\mathbf{e}_m\cdot\mathbf{e}_j)\delta_{jj'}\delta(\mathbf{k}-\mathbf{k}')$$
$$= (c\hbar/2k^3)(\mathbf{k}\times\mathbf{e}_l)\cdot(\mathbf{k}\times\mathbf{e}_m)\delta(\mathbf{k}-\mathbf{k}')$$
$$= (c\hbar/2k^3)(\mathbf{k}\cdot\mathbf{k}\mathbf{e}_l\cdot e_m - \mathbf{k}\cdot\mathbf{e}_l\mathbf{k}\cdot\mathbf{e}_m)\delta(\mathbf{k}-\mathbf{k}')$$
$$= (c\hbar/2k)(\delta_{lm}-k_lk_m/k^2)\delta(\mathbf{k}-\mathbf{k}'),$$

which is the same as GE III (3.7). Analogously, using (2.12) and (2.13), we obtain

$$[\tilde{D}_l(\mathbf{k}), \tilde{D}_m^*(\mathbf{k}')] = -(c\hbar/2\tilde{k})(\delta_{lm}-k_lk_m/\tilde{k}^2)\delta(\mathbf{k}-\mathbf{k}'),$$

the second of Eqs. GE III (3.7).

Upon definition of

$$G^*(\mathbf{k},j) \equiv (1/2\pi)^{\frac{3}{2}}(c\hbar/2k)^{\frac{1}{2}}\sum_s \epsilon_s\beta_j(1/k)\boldsymbol{\alpha}_s\cdot\mathbf{k}\times\mathbf{e}_j \exp[i(\mathbf{k}\cdot\mathbf{r}_s - kct)], \tag{2.14}$$

$$\tilde{G}^*(\mathbf{k},j) \equiv (1/2\pi)^{\frac{3}{2}}(c\hbar/2\tilde{k})^{\frac{1}{2}}\sum_s \epsilon_s\tilde{\beta}_j(1/a\tilde{k})(\boldsymbol{\alpha}_s\cdot a\mathbf{k}\times\mathbf{e}_j + \boldsymbol{\alpha}_s\cdot\mathbf{e}_j)\cdot\exp[i(\mathbf{k}\cdot\mathbf{r}_s-\tilde{k}ct)], \tag{2.15}$$

and

$$H \equiv \sum_s[\boldsymbol{\alpha}_s\cdot c\mathbf{p}_s + m_sc^2\beta_s] + (1/8\pi a)\sum_s \epsilon_s^2 + {\sum_{s,u}}'(\epsilon_s\epsilon_u/4\pi|\mathbf{r}_s-\mathbf{r}_u|)[1-\exp(-|\mathbf{r}_s-\mathbf{r}_u|/a)], \tag{2.16}$$

the wave equation (2.9) becomes

$$H\Omega - i\hbar\partial\Omega/\partial t = \left\{\sum_{j=1}^{3}\int d\mathbf{k}[G^*(\mathbf{k},j)b(\mathbf{k},j) + G(\mathbf{k},j)b^*(\mathbf{k},j) + \tilde{G}^*(\mathbf{k},j)\tilde{b}(\mathbf{k},j) + \tilde{G}(\mathbf{k},j)\tilde{b}^*(\mathbf{k},j)]\right\}\Omega. \tag{2.17}$$

The time factors in the exponentials in the G's may be eliminated through transformations of type

$$e^{-iwt}b(\mathbf{k},j)e^{iwt},$$

where

$$w = c\sum_{j'=1}^{3}\int d\mathbf{k}'[k'b^*(\mathbf{k}',j')b(\mathbf{k}',j') - \tilde{k}'\tilde{b}^*(\mathbf{k}',j')\tilde{b}(\mathbf{k}',j')]. \tag{2.18}$$

From the commutation rules for b and $\tilde{b}$, it follows that

$$e^{-iwt}b(\mathbf{k},j)e^{iwt} = b(\mathbf{k},j)e^{+ickt}, \qquad e^{-iwt}\tilde{b}(\mathbf{k},j)e^{iwt} = \tilde{b}(\mathbf{k},j)e^{+ickt},$$
$$e^{-iwt}b^*(\mathbf{k},j)e^{iwt} = b^*(\mathbf{k},j)e^{-ickt}, \quad e^{-iwt}\tilde{b}^*(\mathbf{k},j)e^{iwt} = \tilde{b}^*(\mathbf{k},j)e^{-ickt}.$$

There is also the relationship

$$e^{-iwt}(i\hbar\partial/\partial t)e^{iwt} = -i\hbar\partial/\partial t + \hbar w. \tag{2.19}$$

Upon definition of

$$G_0^*(\mathbf{k},j) \equiv G^*(\mathbf{k},j)e^{ickt}, \text{ and so on,}$$

the transformed wave equation becomes (where the transformed functional is designated by the same symbol as the original functional)

$$(H - i\hbar\partial/\partial t)\Omega + \hbar c\left\{\sum_{j=1}^{3}\int d\mathbf{k}[kb^*(\mathbf{k},j)b(\mathbf{k},j) - \tilde{k}\tilde{b}^*(\mathbf{k},j)\tilde{b}(\mathbf{k},j)]\right\}\Omega$$
$$= \left\{\sum_{j=1}^{3}\int d\mathbf{k}[G_0^*(\mathbf{k},j)b(\mathbf{k},j) + G_0(\mathbf{k},j)b^*(\mathbf{k},j) + \tilde{G}_0^*(\mathbf{k},j)\tilde{b}(\mathbf{k},j) + \tilde{G}_0(\mathbf{k},j)\tilde{b}^*(\mathbf{k},j)]\right\}\Omega. \tag{2.20}$$

Explicit Representation for Field Operators and Functional

A sufficiently general form for the functional is

$$\Omega \equiv \sum_{r,s} \Omega_{rs}$$

where

$$\Omega_{rs} \equiv \sum_{i_1\cdots i_r} \sum_{j_1\cdots j_s} \int \cdots \int d\mathbf{k}_1 \cdots d\mathbf{k}_r d\mathbf{l}_1 \cdot \quad d\mathbf{l}_s \cdot \psi_{rs}(\mathbf{k}_1, i_1 \cdots \mathbf{k}_r, i_r; \mathbf{l}_1, j_1 \cdots \mathbf{l}_s, j_s) \cdot \bar{b}(\mathbf{k}_1, i_1) \cdots \bar{b}(\mathbf{k}_r, i_r) \bar{\bar{b}}(\mathbf{l}_1, j_1) \cdots \bar{\bar{b}}(\mathbf{l}_s, j_s). \quad (2.21)$$

Each sum is to be taken from 1 to 3 over the values of i and j, and each integral over the entire momentum spaces of $\mathbf{k}$ and $\mathbf{l}$.

The functional derivatives may be defined by

$$\delta\Omega[\bar{b}(\mathbf{k}, j)]/\delta\bar{b}(\mathbf{k}', j') \equiv \lim_{\eta\to 0} (1/\eta) \{\Omega[\bar{b}(\mathbf{k}, j) + \eta\delta_{jj'}\delta(\mathbf{k}'-\mathbf{k})] - \Omega[\bar{b}(\mathbf{k}, j)]\}, \quad (2.22)$$

where $\mathbf{k}$ represents the variables of integration and j the indices of summation in the functional.

Definitions (2.21) and (2.22) permit the association

$$b(\mathbf{k}, i) \sim \delta/\delta\bar{b}(\mathbf{k}, i); \quad b^*(\mathbf{k}, i) \sim \bar{b}(\mathbf{k}, i); \quad \bar{b}(\mathbf{l}, j) \sim \delta/\delta\bar{\bar{b}}(\mathbf{l}, j); \quad b^*(\mathbf{l}, j) \sim -\bar{\bar{b}}(\mathbf{l}, j), \quad (2.23)$$

for it may readily be shown that

$$(\delta/\delta\bar{b}(\mathbf{k}, i))\bar{b}(\mathbf{k}', i')\Omega - \bar{b}(\mathbf{k}', i')(\delta/\delta\bar{b}(\mathbf{k}, i))\Omega = \delta_{ii'}\delta(\mathbf{k}-\mathbf{k}')\Omega$$

and

$$(-\delta/\delta\bar{\bar{b}}(\mathbf{l}, j))\bar{\bar{b}}(\mathbf{l}', j')\Omega - \bar{\bar{b}}(\mathbf{l}', j')(-\delta/\delta\bar{\bar{b}}(\mathbf{l}, j))\Omega = -\delta_{jj'}\delta(\mathbf{l}-\mathbf{l}')\Omega. \quad (2.24)$$

These two equations are to be compared with the commutation rules (2.11) and (2.13).

Application to Wave Equation

The explicit representation developed in the preceding section may be applied to the wave equation (2.20) in order to obtain an ordinary wave equation in $\mathbf{k}$-space. For the immediate purpose of this paper, we are interested in the case where the series of functionals is to be broken off after only the first three terms. (The technique which is to be used is, however, generalized readily to an arbitrary number of terms.) Then

$$\Omega = \Omega_{00} + \Omega_{10} + \Omega_{01},$$

where

$$\Omega_{00} \equiv \psi_{00},$$

$$\Omega_{10} \equiv \sum_{i_1} \int d\mathbf{k}_1 \psi_{10}(\mathbf{k}_1, i_1)\bar{b}(\mathbf{k}_1, i_1),$$

$$\Omega_{01} \equiv \sum_{j_1} \int d\mathbf{l}_1 \psi_{01}(\mathbf{l}_1, j_1)\bar{\bar{b}}(\mathbf{l}_1, j_1).$$

The substitution of these expressions into the wave equation gives after some computation the following equations:

$$(H - i\hbar\partial/\partial t)\psi_{00} = \sum_j \int d\mathbf{k}[G_0^*(\mathbf{k}, j)\psi_{10}(\mathbf{k}, j) + \tilde{G}_0^*(\mathbf{k}, j)\psi_{01}(\mathbf{k}, j)], \quad (2.25)$$

$$(H + \hbar ck - i\hbar\partial/\partial t)\psi_{10}(\mathbf{k}, j) = G_0(\mathbf{k}, j)\psi_{00}, \quad (2.26)$$

$$(H + \hbar c\tilde{k} - i\hbar\partial/\partial t)\psi_{01}(\mathbf{k}, j) = -\tilde{G}_0(\mathbf{k}, j)\psi_{00}. \quad (2.27)$$

3. RELATIVISTIC INTERACTION OF TWO ELECTRONS

Suppose the system consists only of two electrons and their field. To obtain a first-order approximation (i.e., matrix elements proportional to the square of the electronic charge), it is possible to treat the last two terms in the definition of H (2.16), and the entire right-hand side of (2.25), as perturbations on an unperturbed Hamiltonian consisting of the first two terms of H. In order to eliminate ψ_{01} and ψ_{10} from the right-hand side of (2.25), the two succeeding equations are solved for these two functions, with ψ_{00} approximated by the wave function for the unperturbed system. The substitution of these results into (2.25) provides an equation amenable to standard methods of perturbation theory.

Wave Function for the Unperturbed System

The representative for two free electrons with momenta $\mathbf{p}_1^0$ and $\mathbf{p}_2^0$, and signs of energy and direction of spin designated by s_1^0 and s_2^0, is given in $\mathbf{r}_1$, $\mathbf{r}_2$, ζ_1, ζ_2 space by

$$(\mathbf{r}_1, \zeta_1; \mathbf{r}_2, \zeta_2 | \mathbf{p}_1^0, s_1^0; \mathbf{p}_2^0, s_2^0) = \exp(-iW^0t/\hbar)\varphi_0^0 \tag{3.1}$$

where

$$\varphi_0^0 = (1/2\pi\hbar)^3 \exp[-i(\mathbf{p}_1^0\cdot\mathbf{r}_1 + \mathbf{p}_2^0\cdot\mathbf{r}_2)/\hbar] u_{\zeta_1\zeta_2}(\mathbf{p}_1^0, s_1^0; \mathbf{p}_2^0, s_2^0). \tag{3.2}$$

Here u is antisymmetric, and has sixteen components corresponding to variables ζ_1, ζ_2, each of which has four values. The representative (3.1) is a solution of the wave equation

$$(F_1+F_2)\psi = W^0\psi,$$

where

$$F_s \equiv \boldsymbol{\alpha}_s\cdot c\mathbf{p}_s + m_s c^2\beta_s, \tag{3.3}$$

and

$$W^0 = W_1^0 + W_2^0, \tag{3.4}$$

with

$$F_1\varphi_0^0 = W_1^0\varphi_0^0, \quad F_2\varphi_0^0 = W_2^0\varphi_0^0. \tag{3.5}$$

Elimination of ψ_{10} and ψ_{01}

Let us define

$$\psi_{10} \equiv f \equiv f^0 \exp(-iW^0t/\hbar), \quad \psi_{01} \equiv g \equiv g^0 \exp(-iW^0t/\hbar). \tag{3.6}$$

It is clear from (2.26) and (2.27) that f and g have the same time dependence as (3.1) when ψ_{00} has been approximated by this wave function. Hence f^0 and g^0 are independent of time, and (2.26) and (2.27) lead to

$$(F_1+F_2+\hbar ck - W^0)f^0 = G_0(\mathbf{k}, j)\varphi_0^0, \tag{3.7}$$

$$(F_1+F_2+\hbar c\tilde{k} - W^0)g^0 = -G_0(\mathbf{k}, j)\varphi_0^0, \tag{3.8}$$

where the static interaction and the self-energy have been neglected, as they are proportional to the square of the charge, and would give terms proportional to powers higher than second in the final interaction matrix element.

The solutions for f^0 and g^0 are

$$f^0 = \theta\varphi_0^0, \quad g^0 = \tilde{\theta}\varphi_0^0, \tag{3.9}$$

where

$$\theta \equiv (1/2\pi)^{3/2}(\hbar c/2k)^{1/2}\cdot\{(F_1+\hbar ck - W_0^1)^{-1}\epsilon_1\beta_j(1/k)(\boldsymbol{\alpha}_1\cdot\mathbf{k}\times\mathbf{e}_j)\exp(-i\mathbf{k}\cdot\mathbf{r}_1)$$
$$+(F_2+\hbar ck - W_0^2)^{-1}\epsilon_2\beta_j(1/k)(\boldsymbol{\alpha}_2\cdot\mathbf{k}\times\mathbf{e}_j)\exp(-i\mathbf{k}\cdot\mathbf{r}_2)\}, \tag{3.10}$$

$$\tilde{\theta} \equiv (1/2\pi)^{3/2}(\hbar c/2\tilde{k})^{1/2}\cdot\{(F_1+\hbar c\tilde{k} - W_0^1)^{-1}\epsilon_1\beta_j(1/a\tilde{k})(\boldsymbol{\alpha}_1\cdot a\mathbf{k}\times\mathbf{e}_j + \boldsymbol{\alpha}_1\cdot\mathbf{e}_j)\exp(-i\mathbf{k}\cdot\mathbf{r}_1)$$
$$+(F_2+\hbar c\tilde{k} - W_0^2)^{-1}\epsilon_2\beta_j(1/a\tilde{k})(\boldsymbol{\alpha}_2\cdot a\mathbf{k}\times\mathbf{e}_j + \boldsymbol{\alpha}_2\cdot\mathbf{e}_j)\exp(-i\mathbf{k}\cdot\mathbf{r}_2)\}. \tag{3.11}$$

Here we have made use of the fact that $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ commute, and that φ_0^0 satisfies (3.5).

The wave equation (2.25) becomes

$$(H-i\hbar\partial/\partial t)\psi_{00}=\sum\int d\mathbf{k}[G_0{}^*\theta-\tilde{G}_0{}^*\tilde{\theta}]\psi. \tag{3.12}$$

Now ψ differs from ψ_{00} only by a perturbation contribution; and the operator preceding ψ is itself a perturbation operator. Hence we may replace ψ by ψ_{00}, and the equation is in standard form for application of perturbation theory.

Calculation of Interaction Matrix Element

The perturbing energies are, for two particles of charge ϵ_1 and ϵ_2,

$$\epsilon_1\epsilon_2[1-\exp(-|\mathbf{r}_1-\mathbf{r}_2|/a)]/4\pi|\mathbf{r}_1-\mathbf{r}_2|\equiv U_1, \tag{3.13}$$

$$(\epsilon_1{}^2+\epsilon_2{}^2)/8\pi a\equiv V_1, \tag{3.14}$$

$$-\int d\mathbf{k}\sum(G_0{}^*\theta-\tilde{G}_0{}^*\tilde{\theta})\equiv U_2+V_2, \tag{3.15}$$

where U_2 represents the part of the left-hand side of (3.15) containing terms in $\epsilon_1\epsilon_2$, and V_2 represents the part containing terms in $\epsilon_1{}^2$ and $\epsilon_2{}^2$. Since we are interested in the interaction only, we shall calculate only the matrix elements for U_1 and U_2. Further we assume conservation of energy: $W_1+W_2=W_1{}^0+W_2{}^0$. It is well known that the matrix element

$$(\mathbf{p}_1,s_1;\mathbf{p}_2,s_2|\epsilon_1\epsilon_2/4\pi|\mathbf{r}_1-\mathbf{r}_2||\mathbf{p}_1{}^0,s_2{}^0;\mathbf{p}_2{}^0,s_2{}^0)=(1/2\pi)^3\epsilon_1\epsilon_2\delta(\mathbf{P}_1+\mathbf{P}_2)(1/\hbar P_1{}^2)(u^*,u^0), \tag{3.16}$$

where $\mathbf{P}_s\equiv\mathbf{p}_s-\mathbf{p}_s{}^0$, and

$$(u^*,u^0)\equiv\sum_{\zeta_1\zeta_2}u_{\zeta_1\zeta_2}{}^*(\mathbf{p}_1,s_1;\mathbf{p}_2,s_2)u_{\zeta_1\zeta_2}(\mathbf{p}_1{}^0,s_1{}^0;\mathbf{p}_2{}^0,s_2{}^0).$$

From the $-\epsilon_1\epsilon_2\exp(-|\mathbf{r}_1-\mathbf{r}_2|/a)/4\pi|\mathbf{r}_1-\mathbf{r}_2|$ term, the contribution to the interaction matrix element is calculated to be the negative of (3.16), with $P_1{}^2$ replaced by $P_1{}^2+\hbar^2/a^2$. That is,

$$\begin{aligned}(\mathbf{p}_1,s_1;\mathbf{p}_2,s_2|&-\epsilon_1\epsilon_2\exp(-|\mathbf{r}_1-\mathbf{r}_2|/a)/4\pi|\mathbf{r}_1-\mathbf{r}_2||\mathbf{p}_1{}^0,s_1{}^0;\mathbf{p}_2{}^0,s_2{}^0)\\ &=-\sum_{\zeta_1\zeta_2}\int d\mathbf{r}_1\int d\mathbf{r}_2\varphi_0{}^*\{\epsilon_1\epsilon_2\exp(-|\mathbf{r}_1-\mathbf{r}_2|/a)/4\pi|\mathbf{r}_1-\mathbf{r}_2|\}\varphi_0{}^0\\ &=-\epsilon_1\epsilon_2(u^*,u^0)(1/2\pi\hbar)^6\int\int d\mathbf{r}_1d\mathbf{r}_2\exp(-|\mathbf{r}_1-\mathbf{r}_2|/a)\cdot\exp\{-i[(\mathbf{p}_1-\mathbf{p}_1{}^0)\cdot(\mathbf{r}_1-\mathbf{r}_2+\mathbf{r}_2)\\ &\qquad+(\mathbf{p}_2-\mathbf{p}_2{}^0)\cdot\mathbf{r}_2]/\hbar\}/|\mathbf{r}_1-\mathbf{r}_2|.\end{aligned}$$

Define $\mathbf{P}_1\equiv\mathbf{p}_1-\mathbf{p}_1{}^0$, $\mathbf{P}_2\equiv\mathbf{p}_2-\mathbf{p}_2{}^0$, $\mathbf{R}\equiv\mathbf{r}_1-\mathbf{r}_2$; the integral immediately above becomes

$$\begin{aligned}\int\int d\mathbf{r}_2d\mathbf{R}\exp\{-i[\mathbf{P}_1\cdot\mathbf{R}+(\mathbf{P}_1+\mathbf{P}_2)\cdot\mathbf{r}_2]/\hbar\}(e^{-R/a}/R)\\ =(2\pi\hbar)^3\delta(\mathbf{P}_1+\mathbf{P}_2)(4\pi\hbar/P_1)(P_1/\hbar)/[(1/a^2)+(P_1{}^2/\hbar^2)].\end{aligned}$$

Then the matrix element for U_1 is finally

$$-(1/2\pi)^3\epsilon_1\epsilon_2\delta(\mathbf{P}_1+\mathbf{P}_2)(1/\hbar)(P_1{}^2+\hbar^2/a^2)^{-1}(u^*,u^0).$$

For the part of $-\int d\mathbf{k}\sum G_0{}^*\theta$ which contains $\epsilon_1\epsilon_2$, the contribution is

$$-(1/2\pi)^3\epsilon_1\epsilon_2\delta(\mathbf{P}_1+\mathbf{P}_2)\{\hbar[P_1{}^2-(W_1-W_1{}^0)^2/c^2]\}^{-1}\cdot(u^*,(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1\cdot\mathbf{P}_1\boldsymbol{\alpha}_2\cdot\mathbf{P}_1/P_1{}^2)u^0); \tag{3.17}$$

for the part of $+\int d\mathbf{k}\sum \tilde{G}_0{}^*\tilde{\theta}$ which contains $\epsilon_1\epsilon_2$, the contribution turns out to be the negative of (3.17) with $P_1{}^2$ replaced by $P_1{}^2+\hbar^2/a^2$. The proof follows.

From the appropriate definitions ((2.14), following (2.19), and (3.10)), we have

$$\sum G_0{}^*\theta=\sum (1/2\pi)^3(\hbar c/2k^3)\cdot\{\epsilon_1{}^2\boldsymbol{\alpha}_1\cdot\mathbf{k}\times\mathbf{e}_j\exp(-i\mathbf{k}\cdot\mathbf{r}_1)(F_1+\hbar ck-W_1{}^0)^{-1}\boldsymbol{\alpha}_1\cdot\mathbf{k}\times\mathbf{e}_j\exp(-i\mathbf{k}\cdot\mathbf{r}_1)$$
$$+\epsilon_2{}^2\boldsymbol{\alpha}_2\cdot\mathbf{k}\times\mathbf{e}_j\exp(-i\mathbf{k}\cdot\mathbf{r}_1)(F_2+\hbar ck-W_2{}^0)^{-1}\boldsymbol{\alpha}_2\cdot\mathbf{k}\times\mathbf{e}_j\exp(-i\mathbf{k}\cdot\mathbf{r}_2)$$
$$+\epsilon_1\epsilon_2\boldsymbol{\alpha}_2\cdot\mathbf{k}\times\mathbf{e}_j(F_1+\hbar ck-W_1{}^0)^{-}\ \boldsymbol{\alpha}_1\cdot\mathbf{k}\times\mathbf{e}_j\exp[i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)]$$
$$+\epsilon_1\epsilon_2\boldsymbol{\alpha}_1\cdot\mathbf{k}\times\mathbf{e}_j(F_2+\hbar ck-W_2{}^0)^{-1}\boldsymbol{\alpha}_2\cdot\mathbf{k}\times\mathbf{e}_j\exp[i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)]\}.$$

We are interested only in the terms in $\epsilon_1\epsilon_2$; the matrix element for the first of these is, since F_1 is Hermitian,

$$\sum_j\sum\int\epsilon_1\epsilon_2\langle(F_1+\hbar ck-W_1{}^0)^{-1}\rangle\varphi_0{}^*\boldsymbol{\alpha}_2\cdot\mathbf{k}\times\mathbf{e}_j\boldsymbol{\alpha}_1\cdot\mathbf{k}\times\mathbf{e}_j\exp[-i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)]\varphi_0{}^0$$

where the second sum is to be taken over the spin variables, and the integral over the $\mathbf{r}_1$, $\mathbf{r}_2$ variables. The angular brackets indicate, of course, the Hermitian conjugate. With the help of (3.5), the expression becomes

$$\epsilon_1\epsilon_2\sum\int(W_1+\hbar ck-W_1{}^0)^{-1}\varphi_0{}^*(\boldsymbol{\alpha}_1\times\mathbf{k})\cdot(\boldsymbol{\alpha}_2\times\mathbf{k})\exp[i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)]\varphi_0{}^0$$
$$=\epsilon_1\epsilon_2\int(W_1+\hbar ck-W_1{}^0)^{-1}\exp[-i(\mathbf{p}_1\cdot\mathbf{r}_1+\mathbf{p}_2\cdot\mathbf{r}_2)/\hbar]$$
$$(u^*,(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2k^2-\boldsymbol{\alpha}_1\cdot\mathbf{k}\boldsymbol{\alpha}_2\cdot\mathbf{k})\exp[i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)]\exp[i(\mathbf{p}_1{}^0\cdot\mathbf{r}_1+\mathbf{p}_2{}^0\cdot\mathbf{r}_2)/\hbar]u^0)$$

The complete expression for the term without tildes is

$$-(1/2\pi)^3(\hbar c/2k)\epsilon_1\epsilon_2\int d\mathbf{k}\int d\mathbf{r}_1d\mathbf{r}_2(W_1+\hbar ck-W_1{}^0)^{-1}\cdot\exp[-i(\mathbf{P}_1-\hbar\mathbf{k})\cdot\mathbf{r}_1/\hbar]$$
$$\cdot\exp[-i(\mathbf{P}_2+\hbar\mathbf{k})\cdot\mathbf{r}_2/\hbar](u^*,(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1\cdot\mathbf{k}\boldsymbol{\alpha}_2\cdot\mathbf{k}/k^2)u^0).$$

Integration with respect to $\mathbf{r}_1$ produces a $\delta(\mathbf{P}_1-\hbar\mathbf{k})$ factor, and integration with respect to $\hbar\mathbf{k}$ replaces the $\hbar\mathbf{k}$ by $\mathbf{P}_1$. The final integration with respect to $\mathbf{r}_2$ gives a $\delta(\mathbf{P}_2+\mathbf{P}_1)$ factor. The result is

$$(1/2\pi)^3\epsilon_1\epsilon_2(c/\hbar)\delta(\mathbf{P}_1+\mathbf{P}_2)(W_1-W_1{}^0+c|\mathbf{p}_1-\mathbf{p}_1{}^0|)^{-1}\cdot(2|\mathbf{p}_1-\mathbf{p}_1{}^0|^3)^{-1}$$
$$\times(u^*,(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1\cdot\mathbf{P}_1\boldsymbol{\alpha}_2\cdot\mathbf{P}_1/P_1{}^2)u^0).$$

For the second term containing $\epsilon_1\epsilon_2$, the subscripts 1 and 2 are interchanged. Upon use of the expression for conservation of energy, we find the sum of the two to be (3.17).

The calculations for $+\int d\mathbf{k}\sum\tilde{G}_0{}^*\tilde{\theta}$ are of the same type.

The part containing the Dirac matrices can be simplified further; for by (3.2) and (3.3)

$$F_1\varphi_0{}^0=(1/2\pi\hbar)^3\exp(i\mathbf{p}_2{}^0\cdot\mathbf{r}_2/\hbar)\cdot[\boldsymbol{\alpha}_1\cdot c\mathbf{p}_1\exp(i\mathbf{p}_1{}^0\cdot\mathbf{r}_1/\hbar)+m_1c^2\beta_1\exp(i\mathbf{p}_1{}^0\cdot\mathbf{r}_1/\hbar)]u^0$$
$$=(1/2\pi\hbar)^3\exp(i\mathbf{p}_2{}^0\cdot\mathbf{r}_2/\hbar)\cdot[\exp(i\mathbf{p}_1{}^0\cdot\mathbf{r}_1/\hbar)\boldsymbol{\alpha}_1\cdot c\mathbf{p}_1u^0$$
$$+\boldsymbol{\alpha}_1\cdot c\mathbf{p}_1{}^0\exp(i\mathbf{p}_1{}^0\cdot\mathbf{r}_1/\hbar)u^0+m_1c^2\beta_1\exp(i\mathbf{p}_1{}^0\cdot\mathbf{r}_1/\hbar)u^0]$$
$$=(1/2\pi\hbar)^3\exp[i(\mathbf{p}_1{}^0\cdot\mathbf{r}_1+\mathbf{p}_2{}^0\cdot\mathbf{r}_2)/\hbar](F_1+\boldsymbol{\alpha}_1\cdot c\mathbf{p}_1{}^0)u^0$$
$$=W_1{}^0\varphi_0{}^0,$$

whence

$$(F_1+\boldsymbol{\alpha}_1\cdot c\mathbf{p}_1{}^0)u^0=W_1{}^0u^0;$$

and since, in the expression (3.3) for F_1,

$$\boldsymbol{\alpha}_1\cdot c\mathbf{p}_1u^0\equiv\boldsymbol{\alpha}_1\cdot(\hbar/i)\partial u^0/\partial\mathbf{r}_1=0,$$

it follows that $\boldsymbol{\alpha}_1\cdot\mathbf{p}_1{}^0u^0=(1/c)(W_1{}^0-m_1c^2\beta_1)u^0$. Analogous relations hold for $\boldsymbol{\alpha}_1\cdot\mathbf{p}_1$, $\boldsymbol{\alpha}_2\cdot\mathbf{p}_2{}^0$, $\boldsymbol{\alpha}_2\cdot\mathbf{p}_2$. Now

$$(u^*,\ -\boldsymbol{\alpha}_1\cdot\mathbf{P}_1\boldsymbol{\alpha}_2\cdot\mathbf{P}_1u^0)=(u^*,\ [\boldsymbol{\alpha}_1\cdot(\mathbf{p}_1-\mathbf{p}_1{}^0)\boldsymbol{\alpha}_2\cdot(\mathbf{p}_2-\mathbf{p}_2{}^0)]u^0),$$

by definition of $\mathbf{P}_1$ and application of the principle of conservation of momentum in accordance with the $\delta(\mathbf{P}_1+\mathbf{P}_2)$ factor in the matrix element. The expansion of the quantity in brackets gives

$$\boldsymbol{\alpha}_1\cdot\mathbf{p}_1\boldsymbol{\alpha}_2\cdot\mathbf{p}_2-\boldsymbol{\alpha}_1\cdot\mathbf{p}_1{}^0\boldsymbol{\alpha}_2\cdot\mathbf{p}_2-\boldsymbol{\alpha}_1\cdot\mathbf{p}_1\boldsymbol{\alpha}_2\cdot\mathbf{p}_2{}^0+\boldsymbol{\alpha}_1\cdot\mathbf{p}_1{}^0\boldsymbol{\alpha}_2\cdot\mathbf{p}_2{}^0.$$

Then

$$\begin{aligned}c^2(u^*,\ \boldsymbol{\alpha}_1\cdot\mathbf{p}_1\boldsymbol{\alpha}_2\cdot\mathbf{p}_2u^0)&=([\boldsymbol{\alpha}_2\cdot\mathbf{p}_2\boldsymbol{\alpha}_1\cdot\mathbf{p}_1u]^*,\ u^0)c^2\\&=([\boldsymbol{\alpha}_2\cdot\mathbf{p}_2(W_1-m_1c^2\beta_1)u]^*,\ u^0)c,\\&=([(W_2-m_2c^2\beta_2)(W_1-m_1c^2\beta_1)u]^*,\ u^0),\\&=(u^*,\ (W_2-m_2c^2\beta_2)(W_1-m_1c^2\beta_1)u^0);\end{aligned}$$

and

$$\begin{aligned}-c^2(u^*,\ \boldsymbol{\alpha}_1\cdot\mathbf{p}_1{}^0\boldsymbol{\alpha}_2\cdot\mathbf{p}_2u^0)&=-([\boldsymbol{\alpha}_2\cdot\mathbf{p}_2u]^*,\ \boldsymbol{\alpha}_1\cdot\mathbf{p}_1{}^0u^0)c^2,\\&=-([(W_2-m_2c^2\beta_2)u]^*,\ (W_1{}^0-m_1c^2\beta_1)u^0),\\&=-(u^*,\ (W_2-m_2c^2\beta_2)(W_1{}^0-m_1c^2\beta_1)u^0).\end{aligned}$$

Analogous expressions are computed in similar fashion for the other two terms in the expansion. Upon combining the four terms, the β's disappear, and the result is

$$(u^*,\ (W_1-W_1{}^0)(W_2-W_2{}^0)u).$$

By conservation of energy, $W_1-W_1{}^0=-(W_2-W_2{}^0)$, whence

$$(u^*,\ -\boldsymbol{\alpha}_1\cdot\mathbf{P}_1\boldsymbol{\alpha}_2\cdot\mathbf{P}_1u^0)=-(1/c^2)(u^*,\ (W_1-W_1{}^0)^2u^0).$$

Upon combining our various results, we get for the interaction matrix element the following:

$$\begin{aligned}&(1/2\pi)^3(1/\hbar)\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1{}^0-\mathbf{p}_2{}^0)\{|\mathbf{p}_1-\mathbf{p}_1{}^0|^2-(W_1-W_1{}^0)^2/c^2\}^{-1}\\&\qquad\cdot\{1+(a^2/\hbar^2)[|\mathbf{p}_1-\mathbf{p}_1{}^0|^2-(W_1-W_1{}^0)^2/c^2]\}^{-1}(u^*,\ (1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)u^0).\end{aligned}\tag{3.18}$$

This result is a generalization of Møller's formula.[8] It will be noticed that it is relativistically invariant, and reduces to Møller's expression as $a\to 0$.

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[8] C. Møller, Zeits. f. Physik **70**, 786 (1931).