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Note on Magnetic Energy

E. A. GUGGENHEIM
*Imperial College, London**
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1. OBJECT

THE object of this note is to correlate the results obtained in a previous paper[1] concerned with magnetic energy with the contents of a recent paper[2] on this subject by Livens.

I shall show that Livens' formulae are particular examples of my formulae valid for two restricted types of magnetic substances. The assertion by Livens that these are the only types of magnetic substance for which formulae for the energy are obtainable is false, since my formulae apply also to other types.

Livens gives no references to previous treatments of magnetic energy, and I agree with him that most of these are extremely unsatisfactory, but an outstanding exception is that by Cohn[3] already referred to in my earlier paper.

2. NOTATION

The following are the most important symbols used

B	magnetic induction
H	intensity of magnetic field
M	intensity of permanent magnetization
I	intensity of induced magnetization
i	electric current in linear circuit
N	total magnetic flux threading circuit
c	speed of light
$\mathcal{H}$	Hamiltonian function
$\mathcal{L}$	Lagrangian function
dV	element of volume

In order to achieve so far as possible uniformity with Livens, I shall follow him in suppressing the permeability of empty space, although I did not do so in my earlier paper, nor for reasons stated elsewhere[4] would I do so as a general practice. If, however, I followed my usual practice, there would result apparent and irrelevant differences between my formulae and those of Livens, and these differences would only obscure the issue. The only outstanding difference between my notation and that of Livens is the symbol for intensity of permanent magnetization; instead of following Cohn I use M rather than I_0. One thus has the universal relation

$$B=H+4\pi I+4\pi M. \tag{2.1}$$

The susceptibility κ and permeability μ are defined by

$$\kappa=I/H, \tag{2.2}$$

$$\mu=1+4\pi\kappa, \tag{2.3}$$

so that

$$B=\mu H+4\pi M. \tag{2.4}$$

Following Livens, I define two other coefficients κ' and μ' by

$$\kappa'=I/B, \tag{2.5}$$

$$\mu'=1-4\pi\kappa', \tag{2.6}$$

so that

$$H=\mu' B-4\pi M. \tag{2.7}$$

3. GENERAL CASE

Consider now a system consisting of linear circuits and magnetic substances. In order to obtain compact formulae valid in the presence of substances with permanent magnetization, it is expedient to postulate that each portion of such substance is surrounded by suitable auxiliary circuits so disposed that it is possible for currents to flow in these circuits such that the magnetic induction due to them is everywhere equal and opposite to that due to the permanent magnetization. When the currents in these auxiliary circuits are adjusted to achieve this object and the currents in all other circuits are zero, the magnetic induction B will vanish everywhere, while H will vanish everywhere except inside the permanent magnets. I shall for the sake of brevity refer to any such state of $B=0$ throughout as a "zero state." The importance of "zero states" is due to the following property: when the system is in a zero state, there are no forces of magnetic origin acting on any piece of magnetic matter complete with its auxiliary circuits, and so each

* Temporarily at Montreal Laboratory, National Research Council of Canada.
[1] E. A. Guggenheim, Proc. Roy. Soc. **A155**, 49 (1936).
[2] G. H. Livens, Phil. Mag. **36**, 1 (1945).
[3] E. Cohn, *Das Elekromagnetische Feld* (1927).
[4] E. A. Guggenheim, Phil. Mag. **33**, 479 (1942).

of these, together with its auxiliary circuits, can be moved relatively to the others without any change of the magnetic energy of the system.

In my previous paper, I showed that if the current i_s in the circuit s is regarded as a generalized velocity, then the corresponding generalized momentum is N_s/c, where N_s is the total magnetic flux threading this circuit, and c is the speed of light. I further showed that for each configuration of circuit and magnetic matter, provided all the N's are uniquely determined by all the i's, the Lagrangian $\mathcal{L}$ is of the form

$$\mathcal{L}=\mathcal{L}_0+\sum_s \int_{N_s=0}^{i_s} \frac{N_s}{c} di_s, \qquad (3.1)$$

where $\mathcal{L}_0$ is the Lagrangian *in the zero state of the same configuration*, and the integration has to be performed *at constant configuration;* $\mathcal{L}_0$ depends on the configuration and on the velocities, but is independent of the currents and so also of the state of magnetization of the substances in the system. The corresponding formula for the Hamiltonian $\mathcal{H}$ is

$$\mathcal{H}=\mathcal{H}_0+\sum_s \int_0^{N_s} \frac{i_s}{c} dN_s, \qquad (3.2)$$

where $\mathcal{H}_0$ is the Hamiltonian *in the zero state of the same configuration*, and the integration has to be performed *at constant configuration.*

By use of Maxwell's relations it was shown that (3.1) and (3.2) can be transformed to

$$\mathcal{L}=\mathcal{L}_0+\frac{1}{4\pi}\int dV \int_{B=0}^{H} BdH, \qquad (3.3)$$

$$\mathcal{H}=\mathcal{H}_0+\frac{1}{4\pi}\int dV \int_0^{B} HdB, \qquad (3.4)$$

where the first integration extends over the whole volume of the system (assumed to have a boundary where $B=H=0$), and the second integration is performed *at constant configuration.* I showed that the only requirement for the validity of formulae (3.3) and (3.4) is that B *and* H *should be single-valued functions of each other.* In particular it is *not necessary* that they should be *linear* functions of each other. Only hysteresis must be excluded.

For the sake of brevity, I shall henceforth denote by l the contribution per unit of volume to the magnetic term in the Lagrangian and by w the contribution per unit of volume to the magnetic term in the Hamiltonian (or energy). Thus

$$\mathcal{L}=\mathcal{L}_0+\int l dV, \qquad (3.5)$$

$$l=\frac{1}{4\pi}\int_{B=0}^{H} BdH; \qquad (3.6)$$

$$\mathcal{H}=\mathcal{H}_0+\int w dV, \qquad (3.7)$$

$$w=\frac{1}{4\pi}\int_0^{B} HdB. \qquad (3.8)$$

The integrations in (3.5) and (3.7) extend over the whole volume of the system. The integrations in (3.6) and (3.8) are to be performed at constant configuration.

4. "LINEAR" LAWS OF INDUCTION

The essential difference between my treatment and that of Livens is that mine applies to any unique relation between B and H whereas Livens confines himself to what he calls linear laws of induction. He even refers (p. 16) to "the conservation principle, which exists in fact only when the law of induction follows a linear law" and again (bottom of p. 17) to "the assumption of a linear law of induction, and it is only then that conserved energy exists." If my formulae are correct, these statements are patently untrue. Moreover, as I shall point out below, Livens uses the expression "linear law of induction" in two distinct and generally incompatible senses.

According to the definitions of μ, κ and μ', κ' given in Section 2, one has the relations

$$\frac{1}{1-\mu'}-\frac{1}{1-\mu}=\frac{1}{4\pi\kappa'}-\frac{1}{4\pi\kappa}=1+\frac{M}{I}. \qquad (4.1)$$

Now I varies with B and H, while M is a constant. It follows that, except in the trivial case $M=0$, the assumption that μ, κ are constant (that is independent of B, H) implies that μ', κ' vary with B, H and *vice versa.*

The two types of substance for which Livens gives formulae are the following:

I. μ, κ independent of B, H; μ', κ' vary with B, H.

II. μ', κ' independent of B, H; μ, κ vary with B, H.

Livens refers to both conditions indiscriminately as a linear law of induction. It is not clear from his text whether he realizes that the two assumptions are incompatible, except in the trivial case $M=0$. I shall now discuss the two cases in turn.

5. CASE OF μ, κ CONSTANT

In the special case of μ, κ independent of B, H the integration in (3.6) is readily performed, and one obtains

$$\begin{aligned} l &= \frac{1}{8\pi}BH+\frac{1}{2}MH+2\pi\frac{M^2}{\mu}, \\ &= \frac{1}{4\pi}BH-\frac{1}{8\pi}\mu H^2+2\pi\frac{M^2}{\mu}. \end{aligned} \tag{5.1}$$

The last term is a trivial constant, apart from which this is equivalent to the formulae obtained by Livens (4th and 2nd formulae displayed on p. 11).

Under the same conditions the integration in (3.8) is readily performed and one obtains

$$\begin{aligned} w &= \frac{1}{8\pi}BH-\frac{1}{2}MH-2\pi\frac{M^2}{\mu} \\ &= \frac{1}{8\pi}\mu H^2-2\pi\frac{M^2}{\mu}, \end{aligned} \tag{5.2}$$

which is equivalent to formula (11.1) of my earlier paper. The last term in (5.2) is a trivial constant, apart from which this is equivalent to Livens formula (top of p. 13) and to formula (11.3) of my earlier paper.

It is noteworthy that in the absence of permanent magnetization ($M=0$), there is no difference between l and w. This is in accordance with expectation, since it is permissible to regard the energy of electric currents as purely kinetic.

6. CASE OF μ', κ' CONSTANT

In the different case of μ', κ' independent of B, H the integration of (3.6) is again readily performed, and one obtains

$$\begin{aligned} l &= \frac{1}{8\pi}BH+\frac{1}{2}MB \\ &= \frac{1}{8\pi}\mu' B^2, \end{aligned} \tag{6.1}$$

again in agreement with Livens (last formula on p. 17).

Under the same conditions integration of (3.6) gives

$$\begin{aligned} w &= \frac{1}{8\pi}BH-\frac{1}{2}MB \\ &= \frac{1}{4\pi}BH-\frac{1}{8\pi}\mu' B^2. \end{aligned} \tag{6.2}$$

which agrees with Livens (last but one formula on p. 12).

Again in the absence of permanent magnetization ($M=0$), there is no difference between l and w.

7. DISCUSSION OF ALTERNATIVE ASSUMPTIONS

In the absence of permanent magnetization (4.1) reduces to

$$\mu\mu'=1, \tag{7.1}$$

and the alternative assumptions constant μ and constant μ' become equivalent. It is immediately evident that when $M=0$, each formula of Section 6 becomes identical with the corresponding formula of Section 5.

When $M\neq 0$ the two assumptions of constant μ and of constant μ' are incompatible. The former is the more usual, but there is no ground for this other than habit. Livens suggests (Sections 12 and 13) that the latter should be a more realistic assumption. Actually I doubt if either assumption should be regarded as anything more than a rough empirical approximation.

Experimentally it is difficult to obtain sufficiently accurate data completely free from hysteresis to distinguish between the two assumptions and, as already mentioned, none of the formulae hold rigorously when there is hysteresis.

According to the only available theory[5] of an idealized ferromagnetic without hysteresis, the

[5] E. C. Stoner, Proc. Roy. Soc. A165, 372 (1938).

relation between B and H must be much more complicated than corresponds to either of the assumptions constant μ or constant μ'.

8. NATURE OF B AND H

The concluding Sections (12, 14 and 15) of Livens' paper are largely devoted to the question which of the two vectors B and H is the "fundamental (aethereal) force vector" and Livens considers that it is B, without however explaining what he means by "fundamental (aethereal) force vector." The following remarks concerning B and H may perhaps be relevant.

Just as the force on an elementary static charge is determined by the electric field intensity E, so the force on an element of electric current is determined by B. Moreover, just as the electric displacement D is closely related by one of Maxwell's equations to the distribution of electric charge, so is H related by another of Maxwell's equations to the distribution of electric current. In this respect it may be said that B is the analog of E, while H is the analog of D. This analogy has been pointed out in many other places and in particular by Sommerfeld,[6] but is unfortunately obscured in many of the best known textbooks. Moreover, in special relativity theory E and B are parts of the same 6-component antisymmetric tensor while D and H are parts of another such tensor. It seems unprofitable to discuss which of these tensors is the more "fundamental" or the more "aethereal."

The fact that the roles of B and H in the Lagrangian and the Hamiltonian are *not* analogous to those of E and D is consistent with the treatment of magnetic energy as kinetic and the electrostatic energy as potential. This is discussed in detail in my earlier paper (p. 63).

[6] A. J. W. Sommerfeld, Zeits. f. tech. Physik **16**, 420 (1935).