PHYSICAL REVIEW VOLUME 68, NUMBERS 11 AND 12 DECEMBER 1 AND 15, 1945

A New Approach to Kinematic Cosmology

L. INFELD AND A. SCHILD
Department of Mathematics, University of Toronto, Toronto, Canada
(Received September 16, 1945)

The kinematical aspect of relativistic cosmology is examined on the basis of three postulated requirements: The constancy of the velocity of light, spatial isotropy, and homogeneity. Three distinct types of cosmological models are obtained, characterized by different motions of nebulae. The metric of any universe is conformal to Minkowski space and Maxwell's equations are the same for all possible universes. In Part II, it is shown that the cosmological models are metrically, though not topologically, equivalent to those of H. P. Robertson. Next, special models are examined and their line elements brought into the conformal-Minkowskian form. The problem of the displacement of the lines of nebular spectra is discussed; formulas are obtained and applied to some special cosmological models. Finally, idealized experiments are described which indicate the physical content of the cosmological coordinates.

I. General Theory

INTRODUCTION

BY kinematical cosmology is usually understood that part of relativistic cosmology which deals with the metric form of our universe, characterized by a four-dimensional space-time manifold, and with the motion of free particles and light rays in this universe. In this sense the present paper deals with kinematical cosmology and ignores its dynamical aspect, i.e., the connection between the Riemannian curvature tensor on the one hand and the energy-momentum tensor on the other.

The possible mathematical models of the universe are usually deduced from a few simple and convincing principles. In such models the nebulae are represented by freely moving particles of a fluid and we imagine such a particle at every point in space. Each of these particles moves along a geodesic line. Such a particle, representing a nebula, will be called a *fundamental particle* and an observer moving with it a *fundamental observer*.[1] The most important concept by which cosmological models will be described and characterized is that of fundamental particles and their motion. Next, light rays or geodesic null lines must be considered; they are the messengers of world events between fundamental particles. Finally, the universe is characterized by a Riemannian metric. The metric form and the motion of fundamental particles fully determine the kinematical behavior of the universe. It might seem that the motion of fundamental particles can be deduced immediately from the Riemannian metric as the particles move along geodesics. But this is not so. A geodesic line is, at each world-point, determined by its direction there, whereas the world-line of a fundamental particle is completely determined by the choice of a world-point. The world-line of a fundamental particle is a geodesic; but the converse statement is not necessarily true.

The questions which this paper tries to answer are: What are the possible metric forms describing a universe? What are the possible motions of fundamental particles? The guiding principle, commonly accepted and leading to a solution of these problems, is the *principle of homogeneity*, sometimes called the uniformity or the cosmological principle. It states that every fundamental observer sees and describes the world in the same way.

Traditionally, the problem of relativistic cosmology[2] is attacked by choosing a coordinate system in which all fundamental particles are at rest. The fact that such a coordinate system exists is by no means obvious and is closely

[1] These terms seem to be originally due to E. A. Milne, *Relativity, Gravitation, and World-Structure* (Clarendon Press, Oxford, 1935).

[2] An excellent account of "Relativistic Cosmology," including a bibliography complete up to 1932, is to be found in H. P. Robertson's article, Rev. Mod. Phys. **5**, 62–90 (1933); also R. C. Tolman, *Relativity Thermodynamics and Cosmology* (Clarendon Press, Oxford, 1934).

related to the principle of homogeneity. Thus the line elements discussed in cosmology are usually of the form

$$ds^2 = d\tau^2 - R^2(\tau)d\sigma^2. \tag{0.1}$$

Here $R(\tau)$ is an arbitrary function of time and $d\sigma^2$ is the metric of a three-dimensional space of constant curvature $k=1$, -1, or 0. Thus the universe is characterized by an arbitrary function $R(\tau)$ and by the choice of one among three possible spaces. The problem of the motion of fundamental particles disappears from such a presentation because these particles are always at rest. It is the *space structure*, i.e. the curvature $kR^{-2}(\tau)$ of the 3-space $\tau=$constant, which characterizes the cosmological model. It should be noted that, in the coordinate system of (0.1), the speed of light in any fixed direction is a function of time and depends on both k and $R(\tau)$.

Obviously, a discussion in which the four-dimensional universe is characterized chiefly by the curvature of a three-dimensional space is contrary to the spirit of relativity theory in which the world is represented by a four-dimensional space-time continuum. Historically, this approach goes back to Einstein's first cosmological paper,[3] to the Einstein universe of which all others seem to be natural generalizations. This point of view, based on dynamic considerations and on the generalization of the gravitational equations, was long ago abandoned by its originator.

We believe that a deeper insight into cosmological problems is gained by a new approach. Relativistic cosmology, at least in its kinematical aspect, should form a link between the restricted and general theories of relativity. In restricted relativity the world is represented by a Minkowski continuum. This is one of many cosmological backgrounds which satisfy the principle of homogeneity. We shall see that an approach to cosmology is possible in which the structure of a three-dimensional space does not enter the picture. We believe that this new approach puts into the foreground the more essential concepts of kinematical cosmology, i.e., the type of motion of fundamental particles, rather than the space structure.

The ideas which have been sketched above will become clearer if we now summarize some of our results.[4]

Every cosmological background is a Riemannian manifold with a metric of the form

$$ds^2 = \gamma(t, r)(dt^2 - dx^2 - dy^2 - dz^2), \quad r^2 = x^2 + y^2 + z^2. \tag{0.2}$$

We shall see that γ is not an arbitrary function of t and r; but for the conclusions which we shall now draw the particular form of γ does not enter the argument. A coordinate system in which the Riemannian metric has the form (0.2) will be called a *cosmological coordinate system*, or, briefly, a c.c.s.

Thus every cosmological background is conformal to a Minkowski background. Physically, the light geometry is that of a flat Minkowski continuum. The line element (0.2) appears as a natural generalization of the Minkowski space in restricted relativity.

We may, however, interpret (0.2) in a different way and say that every cosmological background differs from a Minkowski background only by a gauging function determining the behavior of clocks and measuring rods. This statement requires some amplification. Starting from (0.2), and without transforming the coordinate system, we introduce new clocks and measuring rods by means of the gauge transformation,[5]

$$ds' = \lambda ds; \quad \lambda = \gamma^{-\frac{1}{2}}.$$

We then have $\gamma' = \gamma^{-1}\gamma = 1$, and the vector field, characterizing the gauging (i.e., the metrical connection), changes from $\chi_i = 0$ to

$$\chi_i' = \partial(\log \lambda)/\partial x^i = -\tfrac{1}{2}\partial(\log \gamma)/\partial x^i.$$

We can, therefore, characterize a cosmological space by

$$ds'^2 = dt^2 - dx^2 - dy^2 - dz^2, \quad \chi_i' = -\tfrac{1}{2}\partial(\log \gamma)/\partial x^i, \tag{0.3}$$

i.e., by a Minkowski metric and an integrable

[3] A. Einstein, "Kosmologische Betrachtungen zur Allgemeinen Relativitätstheorie," Sitz. Preuss. Akad. Wiss. 142–152 (1917).

[4] L. Infeld, "A New Approach to Relativistic Cosmology," Nature **156**, 114 (1945).

[5] H. Weyl, *Space, Time, Matter* (Methuen & Company, Ltd., London, 1922), Chapter II, Section 16.

gauging field χ_i'. This is a new geometric picture and, though (0.3) can always be replaced by (0.2), this new interpretation is very suggestive as it abandons all discussion of curved, expanding universes and shifts the responsibility for cosmological phenomena to the gauging field, the gradient of a scalar field, which determines the behavior of clocks and measuring rods.

It should be added that this interpretation, though based on Weyl's famous work, has nothing in common with Weyl's unified field theory. No connection is assumed between the vector χ_i and the vector potential of an electromagnetic field.

The next conclusion which may be drawn from (0.2) and (0.3) is that Maxwell's equations are the same for flat space as for any cosmological space.[6] This immediately follows from (0.3) and from the fact that Maxwell's equations are gauge invariant. But for the sake of clarity this simple conclusion will be deduced from (0.2). We write Maxwell's equations for empty space in the usual form

$$\begin{gathered}\partial F_{ij}/\partial x^k+\partial F_{jk}/\partial x^i+\partial F_{ki}/\partial x^j=0,\\ \partial((-g)^{\frac{1}{2}}F^{ij})/\partial x^j=0.\end{gathered} \tag{0.4}$$

We have

$$\begin{aligned}(-g)^{\frac{1}{2}}F^{ij}&=(-g)^{\frac{1}{2}}g^{ik}g^{jl}F_{kl}=\gamma^2(1/\gamma^2)\eta^{ik}\eta^{jl}F_{kl}\\ &=\eta^{ik}\eta^{jl}F_{kl},\end{aligned} \tag{0.5}$$

where

$$\eta_{ij}=\eta^{ij}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

Therefore Maxwell's equations, which are differential equations for F_{ij}, have the following form both for a Minkowski and a cosmological background:

$$\begin{gathered}\partial F_{ij}/\partial x^k+\partial F_{jk}/\partial x^i+\partial F_{ki}/\partial x^j=0,\\ \eta^{ik}\eta^{jl}\partial F_{kl}/\partial x^j=0.\end{gathered} \tag{0.6}$$

Another problem which suggests itself here is that of Dirac's relativistic equations for an electron. Indeed, much work was done in solving Dirac's equations for cosmological spaces. From our point of view, the problem, properly formulated, is the following: Are Dirac's equations, like Maxwell's equations, insensitive to the choice of the γ function or not? The answer to this question is not as simple as in the case of Maxwell's equations and requires a special investigation which will be given elsewhere.

The most obvious conclusions resulting from the fact that the cosmological background can be represented by (0.2) or (0.3) have been formulated above. It has been seen how the problem of space structure disappears if the c.c.s. is used and it will be shown how the problem of the motion of fundamental particles appears instead. In a c.c.s. the fundamental particles are, in general, not at rest. Three types of motion are possible, namely, the oscillating motion, the converging-diverging motion, and the simple case of rest. To each permissible form of the function γ there belongs at least (and, in general, exactly) one kind of motion. This motion, and not the space structure, characterizes the universe. It is the study of the permissible functions γ and the associated motions which forms the chief content of this paper.

1. DERIVATION OF THE LINE-ELEMENTS

The derivation of the metrical forms which describe the universe as a whole will be based on three distinct postulates. All three assumptions have immediate physical significance. The last two are well justified by recent astronomical observations. It is not claimed that the first postulate is completely independent of the other two; however, as it has a simple physical content, we prefer to introduce it as a separate requirement.

Postulate I *on light-geometry*.—There exists a coordinate system such that the geometry of light rays is the same as in flat Minkowski space; i.e., light rays travel along straight lines with constant velocity $c=1$, say.

Mathematically, this implies that the four-space must be conformal to Minkowski space, i.e., the line element may be written in the form

$$ds^2=\gamma(dt^2-dx^2-dy^2-dz^2), \tag{1.1}$$

$$=\gamma\eta_{ij}dx^idx^j, \tag{1.11}$$

[6] This conclusion seems to be unknown. Compare E. Schrödinger, "Maxwell's and Dirac's Equations in the Expanding Universe," Proc. Roy. Irish Acad. 46A, 25–47 (1940).

where γ is a function of t, x, y, z. In (1.11) the usual summation convention applies to double indices,[7] x^i $(i=0, 1, 2, 3)\equiv(t, x, y, z)$.

Postulate II *on isotropy*.—The universe, characterized by the line element (1.1), is spatially isotropic.

Mathematically, the cosmological line element must be invariant in form under a 3-parameter continuous group of rotations which leaves every point on the t axis fixed. An immediate consequence is that γ is a function of t and $x^2+y^2+z^2$ only:

$$\gamma=\gamma(t, r), \quad r^2=x^2+y^2+z^2. \tag{1.2}$$

The assumption of isotropy is well justified by nebular counts,[8] which indicate that the distribution of nebulae, as observed from the earth, exhibits spherical symmetry when averaged on a sufficiently wide scale.

If suitable assumptions are made as to the absolute average magnitude and the luminosity of nebulae, and the density distribution of internebular matter, as yet unobservable, recent astronomical research[8] indicates that, on a scale large compared to the mean distance between nebular clusters, matter is uniformly distributed throughout the universe. Thus, neglecting local irregularities, the homogeneity of the material universe leads, by what is usually known as Mach's principle, to the following:

Postulate III *on homogeneity*.—The view of the universe is the same for every fundamental observer.

In our model of the universe we have fundamental particles, each moving along a geodesic line. Our original coordinate system is related to one fundamental particle; by this is meant that the geodesic world-line of this particle is the t axis $x=y=z=0$. If we now change the coordinate system, relating it to any other fundamental particle, our view of the universe remains unchanged; in particular, ds^2 is invariant in form.

Mathematically, the line element ds^2 must be invariant in form under a continuous group of transformations, moving the t axis into world-lines different from it. The existence of such a group of coordinate transformations restricts the possible functions $\gamma(t, r)$. In the following we find line elements which satisfy our homogeneity postulate. However, the treatment adopted here gives no indication as to whether the admissible forms have been exhausted or not. In the appendix, the powerful methods of the theory of continuous groups are applied to the problem and it is there shown that the simpler arguments of this section do, indeed, yield essentially all possible line elements.

Consider the hyperquadric

$$z_0{}^2-z_1{}^2-z_2{}^2-z_3{}^2-(1/K)z_4{}^2=-(1/K), \quad K=\text{constant} \tag{1.3}$$

in five-space and project stereographically from the pole $(0, 0, 0, 0, -1)$ onto the tangential hyperplane $z_4=1$. If the point $(z_0, z_1, z_2, z_3, z_4)$ on the hyperquadric is projected into $(x^0, x^1, x^2, x^3, 1)$, a simple calculation yields the equations

$$z_i=\frac{x^i}{1-Ka/4}, \quad z_4=\frac{1+Ka/4}{1-Ka/4}, \quad i=0, 1, 2, 3, \tag{1.31}$$

where

$$a=\eta_{ij}x^ix^j=t^2-r^2. \tag{1.311}$$

It is well known that, for a suitable metric, the stereographic projection is conformal. Thus we obtain, by differentiating (1.31), the following identity:

$$f(z_0, z_4)(dz_0{}^2-dz_1{}^2-dz_2{}^2-dz_3{}^2-(1/K)dz_4{}^2)=\gamma(t, r)(dt^2-dx^2-dy^2-dz^2)=ds^2, \tag{1.32}$$

where

$$\gamma(t, r)=f(z_0, z_4)/(1-Ka/4)^2, \tag{1.33}$$

z_0, z_4 being expressed in terms of t, r by (1.31). The problem of finding transformations leaving (1.1) invariant is obviously equivalent to that of finding transformations of the five-dimensional z space which leave (1.3) and the left-hand side of (1.32) invariant in form.

One such transformation group, if f is a function of z_0 only, is that of linear transformations of the four variables z_1, z_2, z_3, z_4 leaving $z_1{}^2+z_2{}^2+z_3{}^2+(1/K)z_4{}^2$ invariant, with $z_0\rightarrow z_0$. The group is, for positive K, that of real orthogonal transformations, i.e., Euclidean rotations, of the

[7] Latin indices will consistently range over 0, 1, 2, 3, while Greek indices will range over 1, 2, 3.

[8] A comprehensive account of astronomical observations of cosmological interest is to be found in E. P. Hubble, *The Observational Approach to Cosmology* (Clarendon Press, Oxford, 1937).

variables $z_1, z_2, z_3, K^{-\frac{1}{2}}z_4$; for negative K, that of Lorentz transformations of the "time-like" variable $(-K)^{-\frac{1}{2}}z_4$ and the "space-like" variables z_1, z_2, z_3. Thus $f=f(z_0)$ yields, by (1.33), the line elements

$$ds^2=(1-Ka/4)^{-2}f(t/(1-Ka/4))\eta_{ij}dx^idx^j, \quad (1.4)$$

which satisfy our three postulates. The complicated transformations (see appendix) leaving this form invariant appear, by the above, simply as "rotations" of a four-dimensional subspace about a line (the z_0 axis) in a five-dimensional manifold.

The metric form (1.4) is admissible for all non-zero values of K. It is natural to consider the limiting case when $K=0$, and

$$ds^2=f(t)\eta_{ij}dx^idx^j. \quad (1.41)$$

The limiting process $K\rightarrow 0$ is difficult to carry out on the five-dimensional linear transformations of the z considered above. However, it is immediately obvious that the line element (1.41) conforms to the homogeneity requirement as it is invariant under spatial translations

$$(t, x, y, z)\rightarrow(t, x+\xi, y+\eta, z+\zeta). \quad (1.42)$$

We shall formally include (1.41) in the metric forms (1.4) by permitting K to become zero. Then, as will be shown in the following and in the appendix, the metric forms (1.4) describe all possible universes which satisfy our three postulates. The line elements (1.4) can be written in the form

$$ds^2=f(t/(1-Ka/4))ds_0^2, \quad (1.43)$$

where ds_0^2 is the line element of an indefinite 4-space of constant curvature $-K$. This may be contrasted with the forms (0.1) where a 3-space of constant curvature is multiplied by an arbitrary function (of time).

We shall now obtain two further metric forms which are admissible. However, as they are obtained from some of the line elements (1.4), for $K\leq 0$, by coordinate transformations, they do not yield new universes and are essentially equivalent to those included above.

If $K<0$, then z_0^2 and $(-1/K)z_4^2$ have the same signature in (1.3) and in (1.32). Thus we may interchange those two variables. Then f is a function of z_4 only, i.e., of a; (1.3) and the left-hand side of (1.32) are invariant under Lorentz transformations of the variables z_0, z_1, z_2, z_3 with $z_4\rightarrow z_4$, i.e., with $a\rightarrow a$. By (1.31), we see that such transformations of the five-space are simply the Lorentz transformations in our original space of variables x^i. The new line elements are, by (1.33), of the form

$$ds^2=\gamma(a)\eta_{ij}dx^idx^j, \quad (1.5)$$

where

$$\gamma(a)=(1-Ka/4)^{-2}\times f\left(-(-K)^{-\frac{1}{2}}\frac{1+Ka/4}{1-Ka/4}\right). \quad (1.51)$$

From the way in which it was obtained, it is clear that (1.5) is equivalent to (1.4; $K<0$). However, (1.5) is much simpler than the previous forms (1.4), and this is also true of the group of coordinate transformations which leave the line element invariant and hence of the motion of fundamental particles. Thus we shall, whenever possible, prefer to characterize universes by the metric (1.5) rather than by (1.4) if $K<0$.

The last cosmological line element which we derive is equivalent to (1.41), where $K=0$. It is more cumbersome than (1.41) and need not be discussed in detail. However, we include it in this section for the sake of completeness. The inversion

$$(t, x^\sigma)\rightarrow(-t/a, x^\sigma/a), \quad \sigma=1, 2, 3, \quad (1.61)$$

is a conformal transformation of Minkowski space and changes (1.41) into the form[9]

$$ds^2=(1/a^2)f(-t/a)\eta_{ij}dx^idx^j. \quad (1.6)$$

The minus-sign in the inversion (1.61) is necessary to preserve the sense of time. The group of transformations which leave (1.6) invariant are obtained from the spatial translations (1.42) by subjecting them to the inversion (1.61).

This completes our survey of the cosmological line elements satisfying our three requirements. The forms (1.4), (1.5), (1.6), are shown, in the appendix, to exhaust all possibilities, except for a trivial change of the temporal origin, i.e., except

[9] It may be noted that if (1.4) be written in the form $ds^2=\phi(Kt/4(1-Ka/4))(K/4)^2(1-Ka/4)^{-2}\eta_{ij}dx^idx^j$, then, if the function ϕ does not involve the parameter K implicitly, we obtain (1.6) from (1.4) by the limiting process $K\rightarrow\infty$.

TABLE I. The cosmological line elements.

Case	K ($\alpha>0$)	γ	Transformations preserving ds
I	$1/\alpha^2$	$(1-a/4\alpha^2)^{-2}f(t/(1-a/4\alpha^2))$	Rotations in $(z_1, \alpha z_4)$ plane
II	$-1/\alpha^2$	$(1+a/4\alpha^2)^{-2}f(t/(1+a/4\alpha^2))$	Lorentz transf. in $(z_1, \alpha z_4)$ plane
II′		$\gamma(a)$	Lorentz transformations of x^i
III	0	$f(t)$	Spatial translations of x^i
III′		$(1/a^2)f(-t/a)$	From III by inversion

for the forms obtained from them by the transformation $t\rightarrow t+\text{constant}$.

We summarize the results of this section in Table I.

2. THE MOTION OF FUNDAMENTAL PARTICLES

The postulate of homogeneity ensures the existence of coordinate changes which leave the metric form invariant and transform the t axis into other world-lines which are the t axes of the new coordinate systems. It follows from the isotropy of the line element, which must hold in all equivalent coordinate systems, that these world-lines are geodesics. Thus it is natural to identify such geodesics with the world-lines of the fundamental particles. If $x^i\rightarrow x'^i(x)$ is a transformation leaving (1.1) invariant, then the equations of motion of the fundamental particle, which is related to the new coordinate system, are

$$x'^{\sigma}(t, x, y, z)=0, \quad \sigma=1, 2, 3. \tag{2.1}$$

The spatial isotropy of the universe implies that the world-line of a fundamental particle must, in cosmological coordinates, lie in a plane through the t axis; i.e., the fundamental particles move radially. Thus we may, without loss of generality, consider fundamental world-lines in the tx plane only, and, throughout this section, assume the two equations of motion

$$y=z=0. \tag{2.2}$$

The third equation, describing the motion along the x axis, will be obtained and discussed in each of the five cases enumerated in Table I.

Case I—$K=1/\alpha^2>0$.—We need consider only Euclidean rotations in the $(z_1, \alpha z_4)$ plane, since, by (1.31), $y=z=0$ implies $z_2=z_3=0$. Thus, suppressing two dimensions, the rotations about the z_0 axis and the paths of fundamental particles can easily be visualized by means of a diagram.

The t axis $x=0$ has, in the z space, the parametric equations

$$z_0=t/(1-t^2/4\alpha^2), \quad z_1=0, \\ z_4=(1+t^2/4\alpha^2)/(1-t^2/4\alpha^2), \tag{2.3}$$

by (1.31). The equations are those of a rectangular hyperbola in the $(z_0, \alpha z_4)$ plane, as shown in Fig. 1a. The path of a fundamental particle in the z space is obtained by rotating the hyperbola about the z_0 axis through some angle ρ. In Fig. 1b, this is the hyperbola in the plane p. We obtain the world-line of the fundamental particle in the tx plane by projection from the pole $S\equiv(0, 0, -1)$ onto the plane $P\cdot\equiv\cdot z_4=1$. In the figure, the points of the hyperbolas in the planes p and P, which are related by projection from S, are denoted by the same letter, lower case letters being used for points in p and capital letters for points in P. It should be noticed that the branch abc of the hyperbola in p is projected into the finite arc ABC in P.

Before obtaining the algebraic expressions for the motion of fundamental particles, the im-

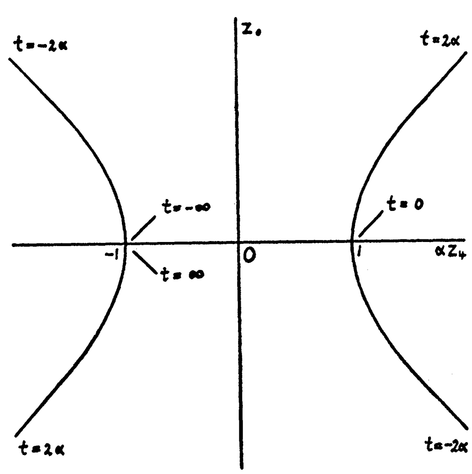

FIG. 1a. t axis in z space.

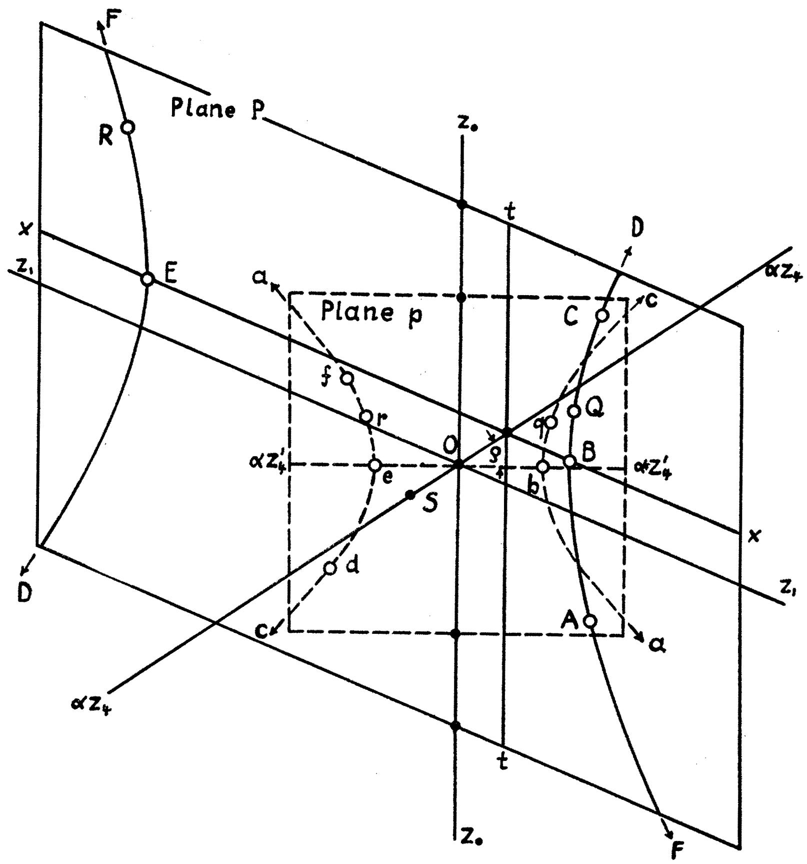

FIG. 1b. The hyperbola in plane p is the world-line of a fundamental particle in z space. The world-line in the tx plane is obtained by projection from S onto the tx plane P. Points related by this projection are denoted by the same letter (lower case letters for points in p, capitals for points in P).

portant problem of connectivity has to be dealt with. It follows immediately from the fact that an operation of projection has been used that the "ends at infinity" of any straight line must be identified. Thus both the z and x spaces have the connectivity of projective spaces. We shall, however, postulate a further connectivity. Returning for the moment to the full five-dimensional space, the points $(z_0, z_1, z_2, z_3, z_4)$ and $(z_0, -z_1, -z_2, -z_3, -z_4)$ on the hyperquadric (1.3) will be identified. This is permissible as the transformation $(z_0, z_1, z_2, z_3, z_4)\rightarrow(z_0, -z_1, -z_2, -z_3, -z_4)$ leaves the form (1.32) invariant, f being a function of z_0 only. Thus the three-dimensional section

$$z_0=\text{constant},\qquad z_1{}^2+z_2{}^2+z_3{}^2+\alpha^2z_4{}^2=z_0{}^2+\alpha^2 \tag{2.31}$$

is a hypersphere on which antipodal points are identified, i.e., it is an elliptic three-space. In Fig. 1b, the points A, B, Q (and a, b, q) are identified with the points C, E, R (and c, e, r), respectively. The path of a fundamental particle is the branch abc $(c\equiv a)$ of the hyperbola in p or the arc ABC $(C\equiv A)$ in P. The set of all fundamental particles is obtained by letting ρ assume all values from $-\pi/2$ to $\pi/2$.

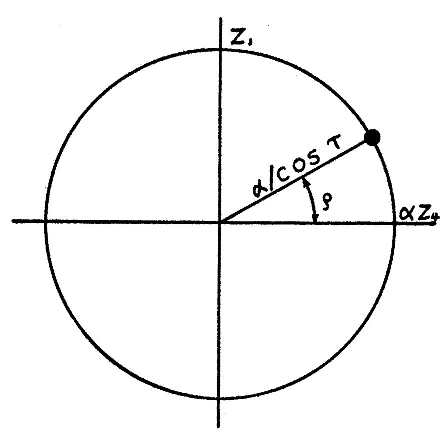

FIG. 1c. Section z_0=constant of hyperboloid $-z_0{}^2+z_1{}^2+\alpha^2z_4{}^2=\alpha^2$.

Proceeding now to the analytic treatment of the motion of fundamental particles, we put

$$X=(1/2\alpha)(t+x),\quad Y=(1/2\alpha)(t-x) \tag{2.32}$$

and have, by (1.31),

$$z_0/\alpha=(X+Y)/(1-XY),$$
$$z_1/\alpha=(X-Y)/(1-XY), \tag{2.33}$$
$$z_4=(1+XY)/(1-XY).$$

Let

$$X=\tan\xi,\quad Y=\tan\eta, \tag{2.34}$$

and

$$\xi+\eta=\tau,\quad \xi-\eta=\rho; \tag{2.35}$$

then

$$z_0/\alpha=\tan\tau,\quad z_1/\alpha=\sin\rho/\cos\tau,$$
$$z_4=\cos\rho/\cos\tau. \tag{2.36}$$

τ and ρ are parameters on the hyperboloid obtained by rotation of the hyperbola in Fig. 1a about the z_0 axis, ρ being, as before, the angle of rotation; $\alpha/\cos\tau$ is the distance of a point on the hyperboloid from the z_0 axis. Fig. 1c shows a section $z_0=$constant, and the geometric content of the parameters τ and ρ is clearly indicated. The elliptic connectivity identifies the pairs of points $(\tau,\rho)\equiv(\tau+\pi,\rho)$ and $(\tau,\rho)\equiv(\tau,\rho+\pi)$, as is immediately verified by (2.36). Thus we may restrict both parameters to the interval $-\pi/2$ to $\pi/2$. A straightforward calculation shows that the pair of points (X, Y) and $(-1/X, -1/Y)$ must be identified, i.e.,

$$(t, x)\equiv(-4\alpha^2t/a,\ 4\alpha^2x/a), \tag{2.37}$$

points connected by an inversion which leaves the line element (1.4, $K=1/\alpha^2$) invariant, and which changes (z_0, z_1, z_4) into $(z_0, -z_1, -z_4)$.

The equation of motion of a fundamental particle is simply $\rho=$constant, or

$$(1/2\alpha)(t+x)=\tan\tfrac{1}{2}(\tau+\rho),$$
$$(1/2\alpha)(t-x)=\tan\tfrac{1}{2}(\tau-\rho), \tag{2.4}$$

where τ is a variable parameter. Eliminating τ, these equations become

$$t^2-x^2-(4\alpha/v)x+4\alpha^2=0,\quad v=\tan\rho. \tag{2.41}$$

Thus the world-lines of fundamental particles are rectangular hyperbolas which do not meet the t axis; their common transverse axis is the x axis and their asymptotes are the null lines $t=\pm(x+2\alpha/v)$.

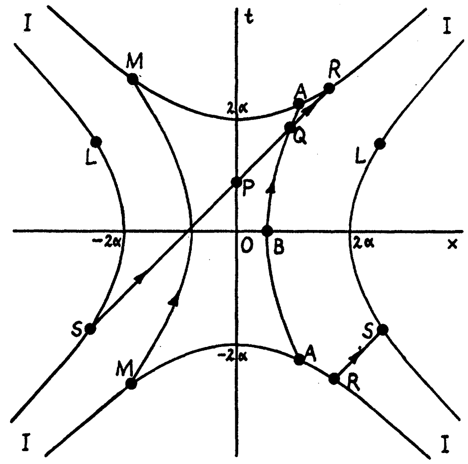

FIG. 1d. The world-lines of a fundamental particle ($ABQA$) and of a light ray (PQR, RS, SP) in elliptic universes of type I.

Putting $\rho=\pm\frac{1}{2}\pi$ in (2.4) while τ varies and $\tau=\pm\frac{1}{2}\pi$ while ρ varies, we obtain the two rectangular hyperbolas

$$x^2-t^2=\pm4\alpha^2, \tag{2.42}$$

which bound our elliptic universe, as shown in Fig. 1d. The points outside the concave quadrangle formed by the four branches of the hyperbolas (2.42) are each identified with an interior point by the elliptic connectivity (2.37). In Fig. 1d the same letter is used to denote a pair of points which are thus identified. The topology of the tx plane is that of a torus which can be obtained by first folding the plane along the t axis, say, and connecting the branches of the hyperbolas (2.42) which meet the x axis; then folding along the x axis and connecting the two branches of the other hyperbola with each other without crossing.

Returning to the motion of a fundamental particle, characterized by the world-line $ABQA$, say, in Fig. 1d, the following simple conclusions are immediate: The fundamental particle, starting from A, moves toward the spatial origin $x=0$ with decreasing velocity until it is at instantaneous rest at B, i.e., when $t=0$ and

$$x=x_{\min}=(2\alpha/v)[+(1+v^2)^{\frac{1}{2}}-1]. \tag{2.43}$$

It then recedes with increasing velocity, first to Q, then to A, and the cycle is repeated. Thus, the observer at the spatial origin sees the funda-

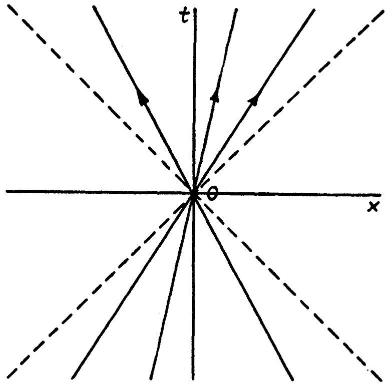

FIG. 2. Fundamental particles in universes of type II′.

mental particles move back and forth; and we may well typify this motion by the adjective *oscillating*.

The first bounding hyperbola $x^2-t^2=4\alpha^2$, whose two branches are identified, is the world-line of the fundamental particle at the greatest distance from the spatial origin. For this particle $x_{\min}=2\alpha$. As is clear from Fig. 1d, the coordinate distance 4α may be described as the "perimeter of the universe" at time $t=0$; it is the finite length of the x axis.

The second bounding hyperbola $x^2-t^2=-4\alpha^2$, whose two branches are identified, is the locus of the points at the greatest distance from the spatial origin on each world-line of a fundamental particle. To put it differently, it is the locus of the events where the fundamental particles reach the amplitudes of their oscillations, stop receding, and start moving toward the spatial origin. The coordinate time interval 4α is the period of the oscillating universe as determined by the fundamental observer $x=0$.

The periodicity of the oscillating universe is further exemplified by following the path of a light ray emitted at some event P in the direction of the positive x axis. Without loss of generality, we may take P on the t axis of our coordinate system. Keeping the elliptic connectivity in mind, we see that the world-line is PQR, RS, SP, as shown in Fig. 1d. Thus the light emitted by a fundamental observer returns to him after a finite coordinate time 4α.

It is important to note that the *physical behavior* of a cosmological model of type I is not necessarily of a periodic nature. The finite t interval $(-2\alpha, 2\alpha)$ of one period's duration may correspond to an infinite proper time. Then, since all experiences of any observer are within this finite t interval, the motion of fundamental particles is no longer oscillating, and the light-signal emitted by a fundamental observer does not return throughout his entire proper life. By Table I (case I), the proper time corresponding to the t interval $(-2\alpha, 2\alpha)$ is

$$S=\int_{-2\alpha}^{2\alpha} ds=\int_{-2\alpha}^{2\alpha}(1-t^2/4\alpha^2)^{-1} \times f^{\frac{1}{2}}(t/(1-t^2/4\alpha^2))dt. \quad (2.44)$$

We have the following criterion: A universe of type I is periodic or non-periodic in its physical behavior according as the integral S converges or diverges. Keeping this in mind, we shall nevertheless, for reasons of economy, retain the term *oscillating* to typify the general universe I and the motion of fundamental particles.

Two simple examples follow, the first of a periodic and the second of a non-periodic universe:

$$ds^2=\{(1-a/4\alpha^2)^2+t^2/\alpha^2\}^{-1}\eta_{ij}dx^i dx^j \quad (2.45)$$

is the line element of a cosmological model of type I, which will later (in part II) be identified with the Einstein universe. We immediately have

$$S=\int_{-2\alpha}^{2\alpha} dt/(1+t^2/4\alpha^2)=\pi\alpha. \quad (2.46)$$

Thus S is finite and the universe is periodic.

Next, we consider the De Sitter universe of type I, sometimes referred to as the De Sitter-Lanczos universe (see Part II). Its line element is

$$ds^2=(1-a/4\alpha^2)^{-2}\eta_{ij}dx^i dx^j. \quad (2.47)$$

In this case

$$S=\int_{-2\alpha}^{2\alpha} dt/(1-t^2/4\alpha^2) \quad (2.48)$$

TABLE II. The motion of fundamental particles.

Case	Equation of motion	Type of motion
I	$t^2-x^2-(4\alpha/v)x+4\alpha^2=0$	Oscillating
II	$t^2-x^2+(4\alpha/v)x-4\alpha^2=0$	Converging-diverging
II′	$x=vt$	
III	$x=\text{constant}$	Rest
III′	$t^2-x^2-x/v=0$	

converges logarithmically at both limits. Thus this universe is non-periodic in its physical behavior.

In conclusion, we may note that the *spherical* universe in which antipodal points of the section (2.31) are not identified is similar in its behavior to the *elliptic* universe described above, and need not be discussed in detail.

Case II—$K=-1/\alpha^2<0$.—We shall discuss this case briefly, mainly for the sake of completeness; in most applications, II′ rather than II will be appealed to. The equations of motion of fundamental particles are immediately obtained from those in Case I by applying the transformation $\alpha\rightarrow -i\alpha$. However, the variables X, Y, ξ, η, τ, ρ, introduced above, are now imaginary. In order to avoid this, the definitions will be slightly changed and these variables, in Eqs. (2.32) to (2.36), replaced by iX, iY, $i\xi$, $i\eta$, $i\tau$, $i\rho$. We then have

$$X=(1/2\alpha)(t+x),\quad Y=(1/2\alpha)(t-x), \tag{2.5}$$

$$z_0/\alpha=(X+Y)/(1+XY),$$
$$z_1/\alpha=(X-Y)/(1+XY), \tag{2.51}$$
$$z_4=(1-XY)/(1+XY),$$

$$X=\tanh\xi,\qquad Y=\tanh\eta, \tag{2.52}$$

$$\xi+\eta=\tau,\qquad \xi-\eta=\rho, \tag{2.53}$$

$$z_0/\alpha=\tanh\tau,\quad z_1/\alpha=\sinh\rho/\cosh\tau,\quad z_4=\cosh\rho/\cosh\tau, \tag{2.54}$$

where all variables are now real.

The equation of motion of a fundamental particle is $\rho=$constant, or,

$$t^2-x^2+(4\alpha/v)x-4\alpha^2=0,\quad v=\tanh\rho. \tag{2.55}$$

It should be noted that the parameter v is now limited to the interval -1 to 1. The world-lines (2.55) are rectangular hyperbolas meeting the t axis in the two fixed points $t=\pm 2\alpha$.

In contrast to the previous case, the only connectivity of the tx plane in the present case is that of a projective plane. All finite points represent distinct events and no two are identified.

Rather than continue the examination of this cosmological model in the c.c.s. II we shall find it more convenient now to introduce the c.c.s. II′.

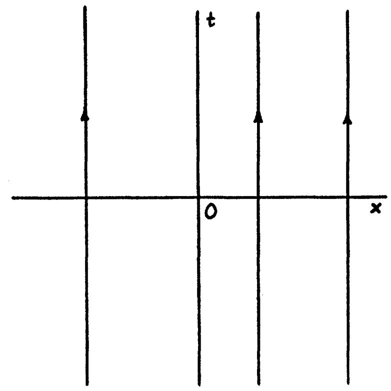

FIG. 3. Fundamental particles in universes of type III.

Case II′.—From the discussion immediately preceding equation (1.5) we know that it is the group of Lorentz transformations which leaves the line element II′ invariant. Under such transformations the t axis is carried into the line

$$x=vt,\quad |v|<1, \tag{2.6}$$

which is the world-line of a fundamental particle.

The fundamental particles all move radially, each with constant velocity v. They *converge*, meet at the origin $t=x=0$, and then *diverge*. This simple behavior of the particles is shown in Fig. 2. It should be noted that the whole observable universe, into which particles can penetrate, is confined to the interior and the surface of the double light-cone with vertex at the origin $t=x=0$.

The explicit transformation which leads from the coordinate system II to II′ is easily obtained in terms of the variables X, Y. As we saw in the first section, the transformation, in the z space, is

$$(z_0,\, z_1,\, \alpha z_4)\rightarrow(-\alpha z_4,\, z_1,\, z_0). \tag{2.61}$$

The minus sign has to be introduced in order that the sense of time be preserved. Using (2.51), we obtain, by a short calculation,

$$(X,\, Y)\rightarrow((X-1)/(X+1),\; (Y-1)/(Y+1)). \tag{2.62}$$

If this transformation is applied to the equation of motion (2.55), the equation of motion (2.6) results. Thus the parameter v is actually the same in both (2.55) and (2.6). It might be noted that (2.62) transforms the time interval $(-2\alpha, 2\alpha)$ in Case II into the interval $(0, \infty)$ of the t axis in II′.

Case III—K=0.—It follows from the transformation (1.42) that the equation of motion of fundamental particles is

$$x=\text{constant}. \tag{2.7}$$

The fundamental particles are at *rest*, as shown in Fig. 3.

Case III′.—Subjecting equation (2.7) to the inversion (1.61), we obtain the equation of motion

$$t^2-x^2-x/v=0, \tag{2.8}$$

where v is a constant. Thus the motion of fundamental particles is along rectangular hyperbolas, all touching the t axis at the origin.

The results of this section are summarized in Table II.

APPENDIX

In this section the theory of continuous groups will be applied to the problem of finding all functions $\gamma(t, r)$ in the line element (1.1) which satisfy the postulate of homogeneity. This principle states that the space with the metric form (1.1) admits a continuous group G of motions which do not move the t axis into itself. The contravariant components ξ^i of the infinitesimal motion, which generates the group, satisfy the equations of Killing[10]

$$\xi^k g_{ij,k}+g_{ik}\xi^k{}_{,j}+g_{jk}\xi^k{}_{,i}=0, \tag{A1}$$

where the comma denotes partial differentiation; thus $g_{ij,k}=\partial g_{ij}/\partial x^k$, etc.

After solving Killing's equations for the ξ^i, the finite transformations of a one-parameter group are obtained by integrating the system of differential equations[11]

$$d\bar{x}^i/dp=\xi^i(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3), \tag{A11}$$

with the initial conditions $\bar{x}^i=x^i$ when $p=0$; p is the parameter of the group.

It follows from the isotropy of the space that it is sufficient to find a one-parameter subgroup G_1 of the complete group G, such that G_1 moves a plane through the t axis into itself. The transformations of G_1 may be combined with suitable spatial rotations to yield the required three-parameter group G. The invariant plane of G_1, the tx plane, say, determines a preferred spatial direction. Thus it is convenient to introduce cylindrical spatial coordinates (x, ρ, φ) in which the line element (1.1) becomes

$$ds^2=\gamma(t, r)(dt^2-dx^2-d\rho^2-\rho^2d\varphi^2);\quad r^2=x^2+\rho^2. \tag{A2}$$

As we are not interested in spatial rotations, we may at the outset assume

$$\bar{\varphi}=\varphi, \quad \xi^\varphi=0. \tag{A21}$$

The equations of Killing are now, for $i=j$,

$$\begin{aligned}
&\gamma_{,t}\xi^t+\gamma_{,r}\xi^x x/r+\gamma_{,r}\xi^\rho\rho/r+2\gamma\xi^t{}_{,t}=0,\\
&\gamma_{,t}\xi^t+\gamma_{,r}\xi^x x/r+\gamma_{,r}\xi^\rho\rho/r+2\gamma\xi^x{}_{,x}=0,\\
&\gamma_{,t}\xi^t+\gamma_{,r}\xi^x x/r+\gamma_{,r}\xi^\rho\rho/r+2\gamma\xi^\rho{}_{,\rho}=0,\\
&\gamma_{,t}\xi^t+\gamma_{,r}\xi^x x/r+\gamma_{,r}\xi^\rho\rho/r+2\gamma\xi^\rho/\rho=0,
\end{aligned} \tag{A22}$$

and for $i\neq j$,

$$\begin{aligned}
&\xi^t{}_{,x}-\xi^x{}_{,t}=0, \quad \xi^t{}_{,\rho}-\xi^\rho{}_{,t}=0,\\
&\xi^x{}_{,\rho}+\xi^\rho{}_{,x}=0, \quad \xi^t{}_{,\varphi}=\xi^x{}_{,\varphi}=\xi^\rho{}_{,\varphi}=0.
\end{aligned} \tag{A23}$$

The last three equations of (A23) show that all ξ^i are independent of φ. From the Eqs. (A22), we have

$$\xi^t{}_{,t}=\xi^x{}_{,x}=\xi^\rho{}_{,\rho}=\xi^\rho/\rho. \tag{A24}$$

We shall now proceed as follows: The first three equations of (A23) and Eqs. (A24) will be solved and the functions ξ^i obtained. Substituting in any of the Eqs. (A22) we next obtain a differential equation for γ. This equation will be solved under different assumptions which exhaust all possibilities. Finally, substituting for ξ^i in Eqs. (A11), the finite transformations of the group G_1 may be obtained. However, as the main purpose of this appendix is to determine all admissible functions γ, this last step will not be discussed in detail and only the results will be given.

From the last of the relations (A24), we have

$$\xi^\rho=\rho f(t, x); \quad \xi^\rho{}_{,\rho\rho}=0.$$

[10] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1926), Eq. (70.1).
[11] Reference 10, Eq. (66.3).

But, by (A24) and (A23),

$$\xi^{\rho}{}_{,\rho\rho}=\xi^{t}{}_{,t\rho}=\xi^{t}{}_{,\rho t}=\xi^{\rho}{}_{,tt}.$$

Thus $\xi^{\rho}{}_{,tt}=0$ and, similarly, $\xi^{\rho}{}_{,xx}=0$. Therefore $f_{,tt}=f_{,xx}=0$ and

$$\xi^{\rho}=\rho(axt+2bx+2ct+d),$$

where a, b, c, d, are constants. From this and (A23), (A24), we have

$$\xi^{t}{}_{,t}=\xi^{\rho}{}_{,\rho}=axt+2bx+2ct+d,$$

$$\xi^{t}{}_{,\rho}=\xi^{\rho}{}_{,t}=ax\rho+2c\rho.$$

Therefore

$$\xi^{t}=\tfrac{1}{2}ax(t^2+\rho^2)+c(t^2+\rho^2)+2btx+dt+F(x),$$

and, similarly,

$$\xi^{x}=\tfrac{1}{2}at(x^2-\rho^2)+b(x^2-\rho^2)+2ctx+dx+\Phi(t).$$

But $\xi^{t}{}_{,x}=\xi^{x}{}_{,t}$, by (A23), and thus

$$\tfrac{1}{2}a(t^2+\rho^2)+2bt+F'(x)=\tfrac{1}{2}a(x^2-\rho^2)+2cx+\Phi'(t).$$

Hence $a=0$, $F=cx^2+ex+f$, $\Phi=bt^2+et+g$, e, f, g, being constants. Finally, we obtain

$$\xi^{t}=c(t^2+x^2+\rho^2)+2btx+dt+ex+f, \quad \text{(A25)}$$

$$\xi^{x}=b(t^2+x^2-\rho^2)+2ctx+et+dx+g, \quad \text{(A26)}$$

$$\xi^{\rho}=\rho(2bx+2ct+d). \quad \text{(A27)}$$

Substituting in any of the Eqs. (A22), we have, on multiplication by r,

$$r\gamma_{,t}(ct^2+cr^2+dt+f)+r^2\gamma_{,r}(2ct+d)$$
$$+2r\gamma(2ct+d)=-x\{r\gamma_{,t}(2bt+e)$$
$$+\gamma_{,r}(bt^2+br^2+et+g)+4br\gamma\}. \quad \text{(A28)}$$

As γ is a function of t and r only, we have the alternative $c=d=f=0$ or $b=e=g=0$. The latter possibility does not lead to suitable transformations: Using (A11), a short calculation yields the explicit form of the finite transformations and it is seen that $r=0$ implies $\bar{r}=0$, i.e., the t axis is moved into itself. We therefore put $c=d=f=0$ and have

$$r\gamma_{,t}(2bt+e)+\gamma_{,r}(bt^2+br^2+et+g)$$
$$+4br\gamma=0. \quad \text{(A3)}$$

This is the differential equation for γ. The transformations of the group G_1 are obtained by solving the system of equations

$$d\bar{t}/dp=\bar{x}(2b\bar{t}+e), \quad \text{(A31)}$$

$$d\bar{x}/dp=b(\bar{t}^2+\bar{x}^2-\bar{\rho}^2)+e\bar{t}+g, \quad \text{(A32)}$$

$$d\bar{\rho}/dp=2b\bar{x}\bar{\rho}, \quad \text{(A33)}$$

with the initial conditions $(\bar{t}, \bar{x}, \bar{\rho})=(t, x, \rho)$ when $p=0$.

We now have the following mutually exclusive possibilities:

1. $b=e=0$, $g\neq 0$.—Then $\gamma_{,r}=0$; thus

$$\gamma=\gamma(t). \quad \text{(A4)}$$

This is the universe III, Table I. Solving (A31) to (A33), we immediately find that the transformations of the group G_1 are translations along the x axis.

2. $b=0$, $e\neq 0$.—Making a suitable choice of the temporal origin, we replace $t+g/e$ by t. Then (A3) simplifies to

$$r\gamma_{,t}+t\gamma_{,r}=0.$$

Thus

$$\gamma=\gamma(t^2-r^2). \quad \text{(A5)}$$

This is the universe II′, Table I. The finite transformations of G_1 are easily obtained and seen to be Lorentz transformations in the variables t and x.

3. $b\neq 0$, $g/b-e^2/4b^2\neq 0$.—Changing the temporal origin, we replace $t+e/2b$ by t and also write $K/4$ for $(g/b-e^2/4b^2)^{-1}$. Then (A3) becomes

$$2tr\gamma_{,t}+\gamma_{,r}(t^2+r^2+4/K)+4r\gamma=0. \quad \text{(A61)}$$

Introducing

$$v\ =\ t/(1-(t^2-r^2)K/4),$$

and putting

$$\gamma=f(t,v)v^2/t^2,$$

a short computation shows that (A61) simplifies to $f_{,t}=0$. Therefore $f=f(v)$, and we have

$$\gamma=(1-(t^2-r^2)K/4)^{-2}$$
$$\times f(t/(1-(t^2-r^2)K/4)). \quad \text{(A6)}$$

This is the universe I or II, Table I, according as $K>0$ or $K<0$. The transformations which

TABLE III. Line elements and motion.

Type	K	γ	Equation of motion	Type of motion
I	4	$(1-a)^{-2}f(t/(1-a))$	$t^2-r^2-(2/v)r+1=0$	Oscillating
II	-4	$(1+a)^{-2}f(t/(1+a))$	$t^2-r^2+(2/v)r-1=0$	
II′		$\gamma(a)$	$r=vt$	Converging-diverging
III	0	$\gamma(t)$	$r=\text{constant}$	Rest
III′		$(1/a^2)f(-t/a)$	$t^2-r^2-(1/v)r=0$	

leave the form (A6) invariant are

$$\bar{x}\pm(\bar{t}^2-\bar{\rho}^2)^{\frac{1}{2}}=2K^{-\frac{1}{2}}\tan\{\tan^{-1}\tfrac{1}{2}K^{\frac{1}{2}}\times(x\pm(t^2-\rho^2)^{\frac{1}{2}})+2K^{-\frac{1}{2}}bp\},\quad \text{(A62)}$$
$$\bar{\rho}/\bar{t}=\rho/t.$$

4. $b\neq 0$, $g/b-e^2/4b^2=0$.—This is the last possibility and concludes our examination of spaces which admit a group G of motions. From Eq. (A61), we now have

$$2rt\gamma_{,t}+\gamma_{,r}(t^2+r^2)+4r\gamma=0. \quad \text{(A71)}$$

Putting

$$v=-t/(t^2-r^2),\quad \gamma=f(t,v)v^2/t^2,$$

we obtain $f_{,t}=0$. Thus $f=f(v)$, and

$$\gamma=(t^2-r^2)^{-2}f(-t/(t^2-r^2)). \quad \text{(A7)}$$

This is the universe III′, Table I. The finite transformations which leave the line element (A7) invariant can be shown to be

$$\bar{x}\pm(\bar{t}^2-\bar{\rho}^2)^{\frac{1}{2}}=(x\pm(t^2-\rho^2)^{\frac{1}{2}}-bp)^{-1},\qquad \bar{\rho}/\bar{t}=\rho/t. \quad \text{(A72)}$$

In each of the cases examined here the transformations of the group G can easily be shown to agree with those discussed in Section 1.

II. Special Problems and Applications

3. INTRODUCTION

We give here a short summary of the main results of Part I[12] of this paper before proceeding to applications.

In Part I, the metric forms and the associated motion of fundamental particles (nebulae), suitable for the kinematical description of our universe at large, are examined on the basis of three postulates, namely, the constancy of the velocity of light, spatial isotropy, and homogeneity. The permissible line elements are of the form

$$ds^2=\gamma(t,r)(dt^2-dx^2-dy^2-dz^2)=\gamma(t,r)\eta_{ij}dx^idx^j,\quad r^2=x^2+y^2+z^2. \quad (3.1)$$

The function γ is restricted by the homogeneity requirement, and three distinct types of cosmological models are obtained. The permissible functions γ and the radial equations of motion of fundamental particles are listed in Tables I and II.

[12] The numbering of sections, equations, and footnotes is carried over from Part I. References to Eqs. (0.1)–(2.8) and to footnotes 1–11 are to those of Part I.

We introduce two changes of notation. The equations of motion of fundamental particles, which move radially, will be written in terms of t and r; it will be understood that they are to be supplemented by the equations θ, $\varphi=\text{constant}$. Next, in cases I and II, we change the coordinate unit, measuring t and r in terms of the *natural cosmological unit* 2α. This is achieved by the coordinate transformation

$$x^i/2\alpha\rightarrow x^i. \quad (3.2)$$

The coordinates, being measured in natural units, now have no physical dimensions. This introduces a considerable simplification in most formulas. Table III is a summary of the main results of Sections 1 and 2.

In Table III, v is a parameter characterizing the individual fundamental particle, and

$$a=t^2-r^2. \quad (3.3)$$

In universes of type I, the points

$$(t,r)\equiv(-t/a,r/a) \quad (3.4)$$

are identified by virtue of the elliptic connectivity.

The forms II′ are obtained from II by the coordinate transformation

$$(X, Y)\to((X-1)/(X+1), (Y-1)/(Y+1)), \quad (3.5)$$

where

$$X=t+r, \quad Y=t-r. \quad (3.6)$$

The forms III′ are obtained from III by the inversion

$$(t, r)\to(-t/a, r/a). \quad (3.7)$$

Using spherical spatial coordinates (r, θ, φ), the transformation equations (3.5) and (3.7) must be supplemented by

$$(\theta, \varphi)\to(\theta, \varphi). \quad (3.8)$$

4. THE COSMOLOGICAL AND THE ROBERTSON COORDINATE SYSTEMS

The coordinate system by which cosmological models have usually been described in the past is, in its general form, due to H. P. Robertson.[13] An *a priori* separation of space-time into space and time is ensured by writing the line element in the form

$$ds^2=d\tau'^2-R_1^2d\sigma^2, \quad (4.1)$$

where $d\sigma^2$ is the positive definite metric of a 3-space. The assumptions of isotropy and homogeneity of space impose two restrictions: $R_1=R_1(\tau')$ is a function of time only, and $d\sigma^2$ is the differential form of a 3-space of constant curvature $k=1, -1$, or 0.

Let

$$\tau=\int d\tau'/R_1(\tau'), \quad R(\tau)=R_1(\tau'). \quad (4.11)$$

Then the line element (4.1) assumes the form

$$ds^2=R^2(\tau)(d\tau^2-d\sigma^2), \quad (4.2)$$

where, with suitable spatial coordinates, we may write

$$d\sigma^2=d\rho^2+\sin^2\rho(d\theta^2+\sin^2\theta d\varphi^2), \quad \text{if } k=1, \quad (4.21)$$

$$d\sigma^2=d\rho^2+\sinh^2\rho(d\theta^2+\sin^2\theta d\varphi^2), \quad \text{if } k=-1, \quad (4.22)$$

$$d\sigma^2=d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\varphi^2), \quad \text{if } k=0. \quad (4.23)$$

[13] H. P. Robertson "On the Foundations of Relativistic Cosmology," Proc. Nat. Acad. Sci. **15**, 822–829 (1929).

The cosmological line element (3.1), in spatial polar coordinates, is

$$ds^2=\gamma(t, r)(dt^2-dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2), \quad (4.3)$$

where all permissible functions γ are listed in Table III. It will be shown that the line elements (4.2) and (4.3) are equivalent and that they can be obtained from each other by a coordinate transformation.

We introduce the variables:

$$X=(t+r), \quad Y=(t-r), \quad (4.31)$$

and

$$\xi=\tfrac{1}{2}(\tau+\rho), \quad \eta=\tfrac{1}{2}(\tau-\rho). \quad (4.32)$$

It may be noted that X, Y, and, as will be seen shortly, ξ, η, τ, ρ, are essentially the same quantities as were introduced in Section 2 (where y and z were both zero and therefore x appeared instead of r).

Case I—K=4.—In terms of the variables X and Y, we have, by (4.3) and Table III,

$$ds^2=f\left(\frac{1}{2}\frac{X+Y}{1-XY}\right)\cdot\frac{1}{4}\left\{1+\left(\frac{X+Y}{1-XY}\right)^2\right\}\cdot\frac{4dXdY-(X-Y)^2(d\theta^2+\sin^2\theta d\varphi^2)}{(1+X^2)(1+Y^2)}.$$

The transformation to the Robertson coordinate system is

$$X=\tan\xi, \quad Y=\tan\eta. \quad (4.4)$$

We immediately obtain

$$ds^2=f(\tfrac{1}{2}\tan(\xi+\eta))\cdot\tfrac{1}{4}\sec^2(\xi+\eta)\cdot\{4d\xi d\eta-\sin^2(\xi-\eta)(d\theta^2+\sin^2\theta d\varphi^2)\},$$

which, by (4.32), assumes the form (4.2–4.21), where

$$R^2(\tau)=\tfrac{1}{4}\sec^2\tau f(\tfrac{1}{2}\tan\tau), \quad \text{and} \quad k=1. \quad (4.41)$$

From this last relation we note that not all Robertson universes, with $k=1$, can actually be brought into the cosmological form I. Only if the function $R(\tau)$ is periodic and of period π is this possible. This can also be seen, and perhaps better, from the transformations (4.4). An increase of π in τ changes, by (4.32), both ξ and η by an amount $\tfrac{1}{2}\pi$. The effect of this is to transform X, Y into $-1/X$, $-1/Y$; and events (X, Y) and $(-1/X, -1/Y)$ are identified in the elliptic connectivity, given by Eq. (3.4).

Thus the forms (4.2–4.21) seem to include a more extensive class of cosmological models than the corresponding forms I, Table III. We shall show, however, that this limitation is not very serious.

First, the physical significance of the τ-interval π will be examined. If (4.21) is interpreted as the line element of an elliptic 3-space, the finite length of a straight line is π, the linear unit being, as usually in elliptic geometry, the radius of curvature $1/k$. Equations (4.2–4.21) show that the coordinate velocity of light rays traveling radially is 1. Thus the interval π on the τ scale is, physically, the time interval in which light circumnavigates the universe and returns to the fundamental particle which emitted it. This is, obviously, a considerable time interval and so, even in a case where $R(\tau)$ is not periodic, a large portion of the Robertson universe can always be represented by our model I.

Next, it should be noted that the periodicity of $R(\tau)$ does not necessarily imply the periodicity of $R_1(\tau')$. In fact, in some important special models the infinite proper interval $(-\infty, \infty)$ on the τ' scale transforms into a finite interval π on the τ-scale, in which case the universe of type I is non-periodic in its physical behavior (see discussion in Section 2).

To illustrate this, we examine the De Sitter-Lanczos universe,[14] where $k=1$, and

$$R_1^2(\tau') = \tfrac{1}{4}\cosh^2 2(\tau'-\tau_0'), \qquad (4.42)$$

τ_0' being a constant. $R_1(\tau')$ is not periodic. We have

$$\tau = 2\int \operatorname{sech} 2(\tau'-\tau_0')d\tau' = 2\tan^{-1} e^{2(\tau'-\tau_0')} - \tfrac{1}{2}\pi. \qquad (4.43)$$

The interval $-\infty<\tau'<\infty$ corresponds to $-\frac{1}{2}\pi<\tau<\frac{1}{2}\pi$ and, by (4.4), to $-1<t<1$. A short calculation yields

$$R^2(\tau) = \tfrac{1}{4}\sec^2\tau. \qquad (4.44)$$

Therefore, by (4.41), $f=1$, and, by I, Table III, the De Sitter-Lanczos line element has the form

$$ds^2 = (1-a)^{-2}\eta_{ij}dx^i dx^j \qquad (4.45)$$

in the c.c.s.

We have shown that there is no essential *metrical* difference between Robertson's universes and ours. However, there is an essential *topological* difference. Suppressing θ and φ, we have in the Robertson case universes represented by a (τ, ρ) plane in which the events (τ, ρ) and $(\tau, \rho+\pi)$ are identified; i.e., the universes are topologically equivalent to a cylinder, whose τ dimension goes to infinity. The topology of the universe I is different; the events (τ, ρ) and $(\tau+\pi, \rho)$ must also be identified, because in the c.c.s. they are connected by the inversion (3.4). Thus we now have the topology, *not* of a *cylinder*, but of a *torus*. Even in the case of a "spherical universe," the topology remains that of a torus as the event (τ, ρ) must now be identified with $(\tau, \rho+2\pi)$ and with $(\tau+2\pi, \rho)$. Though the topological difference between Robertson's universes and ours does not reveal itself in the problems considered in this paper, it will become decisive in the treatment of Maxwell's and Dirac's equations on a cosmological background.

Case II—K = −4.—The transformation to the Robertson coordinate system is obtained from (4.4) by replacing X, Y, ξ, η, by iX, iY, $i\xi$, $i\eta$:

$$X=\tanh\xi, \quad Y=\tanh\eta. \qquad (4.5)$$

Corresponding to (4.41), we now have

$$R^2(\tau)=\tfrac{1}{4}\operatorname{sech}^2\tau\cdot f(\tfrac{1}{2}\tanh\tau),$$

and

$$k=-1. \qquad (4.51)$$

There are in this case no limitations on the function $R(\tau)$.

Case II′.—The line element is, by (4.3) and Table III,

$$ds^2=\gamma(XY)\cdot XY \frac{dXdY-(\frac{1}{2}(X-Y))^2(d\theta^2+\sin^2\theta d\varphi^2)}{XY}.$$

The transformation to the Robertson coordinate system[15] is

$$X=e^{2\xi}, \quad Y=e^{2\eta}. \qquad (4.6)$$

[14] H. P. Robertson, "Relativistic Cosmology," Rev. Mod. Phys. **5**, 62–90 (1933), Eq. (6.6), where we put $c=1$, $a=\frac{1}{2}$.

[15] This transformation has been obtained by A. G. Walker, "On the Formal Comparison of Milne's Kinematical System with the Systems of General Relativity," M. N. R. A. S. **95**, 263–269 (1935).

We immediately obtain

$$ds^2=\gamma(e^{2(\xi+\eta)})\cdot e^{2(\xi+\eta)}$$
$$\cdot\{4d\xi d\eta-\sinh^2(\xi-\eta)(d\theta^2+\sin^2\theta d\varphi^2)\},$$

which, by (4.32), assumes the form (4.2–4.22), where

$$R^2(\tau)=e^{2\tau}\gamma(e^{2\tau}), \quad \text{and} \quad k=-1. \qquad (4.61)$$

Comparing (4.5) and (4.6), we easily regain the transformation formula (3.5) leading from Case II to Case II′.

Case III, III′—K=0.—The cosmological line element III is of the form (4.2–4.23) and no transformation is required. For the sake of completeness, we may write

$$X=2\xi, \quad Y=2\eta, \qquad (4.7)$$

and

$$R^2(\tau)=\gamma(\tau), \quad k=0. \qquad (4.71)$$

In Case III′, the transformation is the inversion (3.7) and need not be discussed further.

It has been shown that our cosmological models are the same as those considered by Robertson and his predecessors. However, the mathematical picture as well as the physical emphasis differ considerably in the two representations. The universes which, in Robertson's coordinates, are of type $k=1$, -1, 0, are, in a c.c.s., of type I, II′ (or II), III (or III′), respectively. In the Robertson coordinate system the fundamental particles are at rest; in a c.c.s. their motion is, in the three cases, the oscillating motion, the converging-diverging motion, and rest, respectively. In Robertson's coordinates the velocity of light is a function of position and direction; in a c.c.s. it is constant.

The form of the cosmological line element (4.2) in the Robertson coordinate system suggests the following consideration. The temporal translation

$$\tau\rightarrow\tau+\tau_0 \qquad (4.8)$$

changes the function $R(\tau)$, and therefore also the line element (4.2), but preserves the type of the universe, which is determined by $d\sigma^2$. The translation (4.8) induces the following transformations in the cosmological coordinates:

Case I: $(X, Y)\rightarrow((X+t_0)/(1-t_0X),$
$(Y+t_0)/(1-t_0Y)), \qquad (4.81)$

Case II′: $(t, r)\rightarrow(t_0t, t_0r), \qquad (4.82)$

Case III: $(t, r)\rightarrow(t+t_0, r). \qquad (4.83)$

These transformations change the cosmological line elements into others which are formally different but of the same type; the equations of motion of fundamental particles are preserved, and, in particular, the t axis is moved into itself. The event $(t_0, 0)$ is transformed into (0, 0), (1, 0), (0, 0), respectively. Thus, by a suitable choice of the line element and the function γ, any event on the t axis may be made to coincide with the natural origin (0, 0), in Cases I and III, and with (1, 0) in Case II′. This different behavior of models of type II′ is due to the fact that the motion of fundamental particles has a singularity at $t=0$ (when the universe shrinks to a point).

The magnification (4.82) may also be applied to a model of type III. The event $(t_0, 0)$ is transformed into (1, 0), if $t_0\neq 0$.

5. SOME SPECIAL UNIVERSES

In this section some special cosmological models will be examined and the line element obtained in a c.c.s.

Universes of Type I.—The line elements of these universes are of the form

$$ds^2=(1-a)^{-2}f(t/(1-a))\eta_{ij}dx^idx^j, \qquad (5.1)$$

and the motion of fundamental particles is, in general, the *oscillating* motion. As was seen in Section 2, some of the universes I are, physically, not of a periodic nature. In such cases the fundamental particles only approach and then recede from a fundamental observer within his infinite proper life.

The simplest model, in our representation, is the non-periodic *De Sitter-Lanczos universe*, which we denote by S_1. Its line element is

$$ds^2=(1-a)^{-2}\eta_{ij}dx^idx^j. \qquad (5.11)$$

The line element of *Einstein's cylindrical universe E* is usually obtained in the Robertson coordinate system:

$$ds^2=d\tau'^2-d\sigma^2, \qquad (5.12)$$

where $d\sigma^2$ is the metric (4.21) of a 3-space of constant curvature $k=1$. By (4.1) and (4.11),

TABLE IV. Special cosmological models.

Case	*	Universe		γ	Type	Transformation of line elements
$K=4$ ($k=1$)	Oscillating	S_1	(a)	$(1-a)^{-2}$	I	(a), $(h)\rightarrow(b)$, (i), (n), (o), $(d)\rightarrow(f)$, $(e)\rightarrow(g)$, $(j)\rightarrow(k)$, by transformation $(X, Y)\rightarrow((1+X)/(1-X), (1+Y)/(1-Y))$.
			(b)	$1/4t^2$	I	
		E	(c)	$4\{(1-a)^2+4t^2\}^{-1}$	I	
$K=-4$ ($k=-1$)	Converging-diverging	M_2	(d)	1	II′	
			(e)	$1/a^2$	II′	
			(f)	$4(1+a-2t)^{-2}$	II	
			(g)	$4(1+a+2t)^{-2}$	II	(d), $(l)\rightarrow(e)$, (m), $(n)\rightarrow(o)$, $(q)\rightarrow(r)$, by inversion $(t, r)\rightarrow(-t/a, r/a)$.
		S_2	(h)	$(1-a)^{-2}$	II′	
			(i)	$1/4t^2$	II	
		Mi	(j)	$1/a$	II′	
			(k)	$4\{(1+a)^2-4t^2\}^{-1}$	II	
$K=0$ ($k=0$)	Rest	M_3	(l)	1	III	$(f)\rightarrow(m)$, (a), $(h)\rightarrow(p)$, by temporal translation $t\rightarrow t+1$.
			(m)	$1/a^2$	III′	
		S_3	(n)	$1/4t^2$	III	
			(o)	$1/4t^2$	III′	
			(p)	$(1/a^2)(1+2t/a)^{-2}$	III′	$(g)\rightarrow(m)$, by temporal translation $t\rightarrow t-1$.
		ES	(q)	t^4	III	
			(r)	t^4/a^6	III′	

* Motion of fundamental particles.

we have $R_1^2(\tau')=R^2(\tau)=1$, and Eq. (4.41) then yields

$$f(x)=4(1+4x^2)^{-1}.$$

Thus the Einstein universe has the cosmological line element

$$ds^2=4\{(1-a)^2+4t^2\}^{-1}\eta_{ij}dx^idx^j. \quad (5.13)$$

We see that in the c.c.s. the Einstein universe does not have a simple line element, and that it seems more artificial than the De Sitter model.

Finally, we examine very briefly *Lemaître's universe*. The Lemaître line element in the Robertson coordinate system[16] is of the type $k=1$, and the function $R_1(\tau')$ is defined by

$$d\tau'=(3R_1)^{\frac{1}{2}}dR_1/(R_1+2)^{\frac{1}{2}}(R_1-1).$$

Since $d\tau'=R_1d\tau$, by (4.11), and $R_1(\tau')=R(\tau)$, we immediately have

$$d\tau=3^{\frac{1}{2}}dR/(R^2+2R)^{\frac{1}{2}}(R-1),$$

[16] Reference 14, Eq. (8.2), where we put $R_e=1$.

and, on integration,

$$R=2/(3\coth^2\tfrac{1}{2}\tau-1). \quad (5.14)$$

Thus R is not a periodic function of τ and we can, at best, represent only a portion of the Lemaître universe in a c.c.s.

Universes of Type II, II′.—The line elements of these universes are preferably taken in the form II′:

$$ds^2=\gamma(a)\eta_{ij}dx^idx^j. \quad (5.2)$$

The fundamental particles move with constant coordinate velocities and they all meet in a point at time $t=0$. We have described this motion as *converging-diverging*.

The transition from the form II′ to II is achieved by the inverse of the transformation (3.5):

$$(X, Y)\rightarrow((1+X)/(1-X), (1+Y)/(1-Y)). \quad (5.21)$$

The simplest model of Type II′ is the *Minkowski universe* ($\gamma(a)=1$) which we denote by M_2. Its metric is given by the pseudo-Euclidean line element

$$ds^2=\eta_{ij}dx^i dx^j. \tag{5.22}$$

The De Sitter line element (5.11), besides being of type I, is also quite obviously of type II′. Thus among the models II′ we have a *De Sitter universe*, which we denote by S_2, and whose line element is

$$ds^2=(1-a)^{-2}\eta_{ij}dx^i dx^j. \tag{5.23}$$

Subjecting this form to the transformation (5.21), we obtain the line element of S_2 in the form II:

$$ds^2=(1/4t^2)\eta_{ij}dx^i dx^j. \tag{5.24}$$

This is also a special case of the forms (5.1) of type I, where $f(x)=1/4x^2$. Thus (5.24) is an alternative line element of the universe S_1.[17]

Finally, we mention a model which we call *Milne's universe* and denote by *Mi*. Its line element is

$$ds^2=(1/a)\eta_{ij}dx^i dx^j. \tag{5.25}$$

In this universe, the motion of free particles along any geodesic is identical with the motion of particles in Milne's "kinematical relativity" under the influence of the "substratum"[18]; in particular, the fundamental particles, which form Milne's substratum, behave like those in all our models II′. We may add that in Robertson's coordinates Milne's universe has a form analogous to Einstein's universe; its line element is (5.12) but $d\sigma^2$ is now the metric (4.22) of a 3-space of constant curvature $k=-1$.

Universes of Type III, III′.—It is convenient to use the c.c.s. III:

$$ds^2=\gamma(t)\eta_{ij}dx^i dx^j. \tag{5.3}$$

In these models the fundamental particles are at *rest*.

The pseudo-Euclidean line element (5.22), besides being of type II′, is also of type III. Thus, among the models III, we have a *Minkowski universe* M_3 with line element

$$ds^2=\eta_{ij}dx^i dx^j. \tag{5.31}$$

The De Sitter line element (5.24) is immediately seen to be a special case of the forms (5.3). Thus we have a third *De Sitter universe* S_3 of type III and with line element

$$ds^2=(1/4t^2)\eta_{ij}dx^i dx^j. \tag{5.32}$$

S_3 is sometimes referred to as the "stationary" De Sitter universe.[19]

The *Einstein-De Sitter universe ES* has, in the Robertson coordinate system,[20] a line element of type $k=0$, where

$$R_1=(3\tau')^{\frac{2}{3}}. \tag{5.33}$$

Using (4.11), a short calculation yields the line element in the c.c.s. III:

$$ds^2=t^4\eta_{ij}dx^i dx^j. \tag{5.34}$$

This cosmological model has also been proposed by *Dirac*.[21]

It must be emphasized that the two Minkowski universes M_2 and M_3 are quite distinct cosmological models and differ in their physical behavior; e.g., M_2 exhibits a nebular red shift, M_3 does not. M_2 and M_3 are characterized by the same line element (5.22) but by different motions of fundamental particles, i.e., the converging-diverging motion and rest, respectively. A corresponding statement holds for the three De Sitter universes S_1, S_2, S_3, which may be described by the same metric form (5.24) and by the three different types of motion of fundamental particles.

We conclude this section with a short examination of *stationary universes*.[22] We may define this term as follows: A universe is stationary if its line element and the equation of motion of its fundamental particles are invariant in form under a coordinate transformation which moves the t axis into itself and which transforms any non-singular event on it into any other.

[17] The transformation (5.21) is a special case of (4.81), $t_0=1$, and therefore changes any line element of type I into another of the same type, and also preserves the equation of motion of fundamental particles. A similar statement is easily verified to hold for the inversion (3.7) applied to a line element of type II′.

[18] C. Gilbert, "On the Occurrence of Milne's Systems of Particles in General Relativity," Quart. J. Math. 9, 187, Eq. (9) (1938). See also reference 15.

[19] Reference 14, Eq. (6.1).

[20] Reference 14, Eq. (8.8), where we put $\kappa E=12$.

[21] P. A. M. Dirac, "A New Basis for Cosmology," Proc. Roy. Soc. **A165**, 199–208 (1938), Eq. (6).

[22] Reference 13, p. 824, assumption II′.

It is easily verified that the line elements of the universes E, Mi, M_3, are invariant in form under the transformations (4.81), (4.82), (4.83), respectively, and are therefore stationary. The line element of the universe S_3 is invariant under (4.82), which, however, preserves the origin (0, 0). But this line element is singular at $t=0$, and thus the definition of stationary cosmologies applies to S_3.

The different cosmological forms of the special models examined in this section are summarized in Table IV.

6. THE NEBULAR RED SHIFT

The displacement towards the red of the spectral lines of nebulae, which is roughly proportional to their distance from us, is now a well established astronomical phenomenon.[8] One of the chief advantages of many relativistic models of our universe is that this nebular red shift emerges as a natural consequence of their structure. In this section expressions for the red shift will be obtained in terms of the coordinates of the fundamental particles under observation. No comparison with observational data will be attempted as the coordinates x^i are not immediately interpretable in terms of physical time and distance as estimated by the astronomer. Such a comparison involves an examination of the apparent magnitudes and luminosities of nebulae in the model; the problem is not difficult but is outside the scope of this report. However, in most cosmological models where the red shift is not altogether absent (as it is in the Einstein universe) the first-order red shift effect is linear, and agreement with observation can be achieved by a suitable choice of some of the constants of the model, such as 2α (the cosmological unit) or the present time t_0 (which may, for practical purposes, be treated as a constant). Thus the red shift phenomenon, at least to the first order, is not a very effective criterion for narrowing down the large number of possible cosmological models. Most restrictions on the models, which are suitable for describing our universe at large, are obtained from the red shift effects of higher order and from dynamical considerations which are outside the domain of pure kinematics.

We base the derivation of the red shift formula on a practical, corpuscular theory of light. This is completely justified by the fact that Maxwell's equations have in all cosmological spaces the same form as in Minkowski space and that therefore light is propagated exactly as in flat space. Appeal is also made to a fundamental principle of relativity which states that the proper period of vibration (i.e., the period measured in proper time ds) of an atom emitting a sharp spectral line is constant, whatever the motion or position of the atom. Consider an atom, moving with a fundamental particle $P\equiv(t, r)$ of radial velocity V, emitting a light signal at time t and again, after one complete vibration, at time $t+dt$. The two light signals reach the observer O at the spatial origin at times $t_0=t+r$ and t_0+dt_0, respectively, where $dt_0=dt(1+V)$. This is immediately seen from the fact that the second light signal is emitted from the event $(t+dt,\ r+Vdt)$ and reaches O at time $t+r+dt(1+V)$. Remembering the form of the cosmological line element (3.1), we find that the proper period of vibration of the atom (which equals the proper wave-length λ, since the velocity of light is 1) is given by

$$ds=\lambda=\gamma^{\frac{1}{2}}(t, r)(1-V^2)^{\frac{1}{2}}dt.$$

The proper period of vibration as observed by O is $ds_0=\gamma_0^{\frac{1}{2}}dt_0$, or,

$$ds_0=\lambda_0=\gamma_0^{\frac{1}{2}}(1+V)dt, \quad \gamma_0=\gamma(t_0, 0).$$

Thus the ratio of observed to proper wave-length is

$$\lambda_0/\lambda=(\gamma_0/\gamma)^{\frac{1}{2}}((1+V)/(1-V))^{\frac{1}{2}}. \tag{6.1}$$

This expression for the red shift may be analysed into two independent components, the Doppler effect which contributes the factor $((1+V)/(1-V))^{\frac{1}{2}}$, and the gravitational effect which contributes the factor $(\gamma_0/\gamma)^{\frac{1}{2}}$.

An explicit formula for the red shift will now be obtained in each of the three types of universes and the result expressed in terms of the time t_0 of observation and the distance r of the fundamental particle P, determined at the time t when the observed radiation left P.

Case I.—The velocity V of a fundamental particle is obtained by differentiating the equation of motion, given in Table III.

$$V=2tr/(t^2+r^2+1). \tag{6.2}$$

TABLE V. First-order red shift effects.

Universe	Type	γ	$\Delta\lambda/\lambda$ to first order	Remarks
S_1 (a)	I	$(1-a)^{-2}$	$(4t_0/(1+t_0^2))\bar{r}$	$-1<t_0<0$: v.s.* $0<t_0<1$: r.s.
E (c)	I	$4\{(1-a)^2+4t^2\}^{-1}$	0	rigorously
M_2 (d)	II′	1	$(1/t_0)\bar{r}$	$t_0<0$: v.s. $t_0>0$: r.s.
S_2 (h)	II′	$(1-a)^{-2}$	$\frac{\lvert 1-t_0^2\rvert}{1-t_0^2}\cdot\left(\frac{1}{t_0}+t_0\right)\cdot\bar{r}$	$t_0<-1$, $0<t_0<1$: r.s. $-1<t_0<0$, $t_0>1$: v.s.
Mi (j)	II′	$1/a$	0	rigorously
M_3 (l)	III	1	0	rigorously
S_3 (n)	III	$1/4t^2$	$-2(\lvert t_0\rvert/t_0)\bar{r}$	$t_0<0$: r.s.; $t_0>0$: v.s.
ES (q)	III	t^4	$(2/t_0^3)\bar{r}$	$t_0<0$: v.s.; $t_0>0$: r.s.

* r.s. is red shift; v.s. is violet shift.

Replacing t by t_0-r, we have

$$V=2r(t_0-r)/((t_0-r)^2+r^2+1). \quad (6.21)$$

Thus the Doppler effect is given by the expression

$$((1+V)/(1-V))^{\frac{1}{2}} =((1+t_0^2)/(1+(t_0-2r)^2))^{\frac{1}{2}}. \quad (6.22)$$

Before considering the complete red shift in the general case, it is convenient to examine the special case of the Einstein universe E, whose line element is given by (5.13). Remembering that $\gamma=\gamma(t, r)=\gamma(t_0-r, r)$, we have

$$\left(\frac{\gamma_0}{\gamma}\right)^{\frac{1}{2}}=\left(\frac{(1-(t_0-r)^2-r^2)^2+4(t_0-r)^2}{(1-t_0^2)^2+4t_0^2}\right)^{\frac{1}{2}}, \quad (6.23)$$

which, after simplification, reduces to the reciprocal of (6.22). Thus $\lambda_0/\lambda=1$ in the Einstein universe and there is no red shift.

The Doppler effect is the same for all universes of Type I. Writing $\gamma=F\cdot\gamma_E$, where $\gamma_E\eta_{ij}dx^idx^j$ is the line element of the Einstein universe, we see, from (6.1), and from the fact that $(\gamma_{E0}/\gamma_E)^{\frac{1}{2}}$ just cancels the Doppler effect, that

$$\frac{\lambda_0}{\lambda}=\left(\frac{F_0}{F}\right)^{\frac{1}{2}}=\left(\frac{F(t_0/(1-t_0^2))}{F((t_0-r)/(1-t_0^2+2t_0r))}\right)^{\frac{1}{2}}. \quad (6.3)$$

In terms of the function f, introduced in Section 1, F is defined by

$$F(x)=(\tfrac{1}{4}+x^2)f(x), \quad (6.31)$$

as is easily verified by Eqs. (5.1) and (5.13).

Case II′.—The fundamental particles now move with constant velocity $V=r/t$ and $\gamma=\gamma(a)$. Thus $V=r/(t_0-r)$, and Eq. (6.1) becomes

$$\lambda_0/\lambda=(t_0\gamma(t_0^2)/(t_0-2r)\gamma(t_0^2-2t_0r))^{\frac{1}{2}}. \quad (6.4)$$

It is interesting to note that Milne's universe Mi, for which $\gamma=1/a$, exhibits no red shift.

Case III.—The fundamental particles in these models are at rest and hence the red shift is completely due to the gravitational effect. Remembering that $\gamma=\gamma(t)$, we have

$$\lambda_0/\lambda=(\gamma(t_0)/\gamma(t_0-r))^{\frac{1}{2}}. \quad (6.5)$$

The only universe of this type without red shift is the Minkowski universe M_3.

We proceed to obtain formulas for the first-order red shift effect in the special cosmological models listed in Table IV. The red shift will be given by the usual expression

$$\Delta\lambda/\lambda=(\lambda_0-\lambda)/\lambda. \quad (6.6)$$

It will be expressed in terms of the time t_0 of observation and a new distance coordinate $\bar{r}$, defined by

$$\bar{r}=\lvert\gamma_0^{\frac{1}{2}}\rvert r, \quad \gamma_0=\gamma(t_0, 0). \quad (6.61)$$

From the line element (4.3) we note that $\bar{r}$ is the proper distance at time t_0 and in the immediate neighborhood of the spatial origin. Neglecting second-order corrections, $\bar{r}$ is actually the distance estimated by the astronomer, who always assumes the rectilinear propagation of light with constant velocity.

With the formulas (6.3)–(6.5), (6.61), and the functions γ of Table IV, the calculations are quite straightforward. Therefore no computational details need be given and the first-order red shift effects are collected in Table V.

Finally, it is of interest to note that the three models E, Mi, M_3, are the only universes with no red shift.

7. THE PHYSICAL CONTENT OF THE COSMOLOGICAL COORDINATES

Suppose a physical cosmological model be given which is of the kind considered in this report. In this concluding section we deal with the problem of finding the line element of the model and of determining the coordinates of any event in it. This is achieved by means of idealized experiments. They give a clear indication of the physical content of our time and distance coordinates which so far were only convenient mathematical constructs.

We choose a fundamental particle (arbitrarily) and associate with it a coincident fundamental observer O. Without loss of generality, we may assume the particle to be at the spatial origin. The observer O is to be equipped with a theodolite and with apparatus for sending and receiving light signals. Also he is to carry a mechanical or *atomic clock* whose vibrations, according to the fundamental principles of relativity, measure proper time ds.

For the purposes of this section, the most convenient coordinate system is that introduced in Section 4, Eq. (4.2). The line element is

$$ds^2 = R^2(\tau)\{d\tau^2 - d\rho^2 - S^2(\rho)(d\theta^2 + \sin^2\theta d\varphi^2)\}, \quad (7.1)$$

where $S(\rho)$ is $\sin\rho$, $\sinh\rho$, or ρ, according as the model is of type I, II′, or III. From (7.1) simple transformations lead to the form of the line element in Robertson's coordinate system [Eq. (4.11)] or in a c.c.s. [Eqs. (4.4), (4.6), (4.7), in cases I, II′, III, respectively].

Introducing new variables

$$(\bar{\tau}, \bar{\rho}) = (\alpha\tau, \alpha\rho), \quad (7.11)$$

where α is a constant, the line element (7.1) becomes

$$ds^2 = \phi^2(\bar{\tau})\{d\bar{\tau}^2 - d\bar{\rho}^2 - \alpha^2 S^2(\bar{\rho}/\alpha)(d\theta^2 + \sin^2\theta d\varphi^2)\}, \quad (7.2)$$

where $\phi(\bar{\tau}) = (1/\alpha)R(\bar{\tau}/\alpha)$. In (7.1), τ, ρ are dimensionless numbers; R has the dimensions of ds. In (7.2), ϕ is dimensionless; $\bar{\tau}$, $\bar{\rho}$ have the dimensions of ds. By a suitable choice of α, we can always ensure that

$$\phi(0) = 1. \quad (7.21)$$

We know that in the coordinate systems of (7.1) and (7.2) the fundamental particles are at rest. Moreover, the velocity of radial light signals is 1; this is an important property which these coordinate systems share with the c.c.s.

We shall now proceed in the following order: 1. A $\bar{\tau}$-clock is constructed which measures $\bar{\tau}$-time at O; it is used to obtain the coordinates $(\bar{\tau}, \bar{\rho})$ of any event. 2. The type of the universe and the constant α are determined. 3. The line element is obtained. 4. The transition to a c.c.s. is examined.

1. The observer O sends a light signal to a fundamental particle P in his immediate neighborhood. P reflects the light ray back to O who returns it to P, and this procedure is repeated indefinitely. The fundamental observer O counts the number of times the light-pulse has reached him and this number serves as a measure of time, $\bar{\tau}$ time. This is immediately seen from the fact that, in the coordinates of (7.2), all fundamental particles are at rest and that the radial velocity of light is constant.

Since, in the general line element (7.2), there is no preferred point on the temporal axis, O may, without loss of generality, denote the initial moment at which he starts his light ray experiment by $\bar{\tau} = 0$. In accordance with (7.21), O normalizes the $\bar{\tau}$-time measure to agree initially with the proper time of his atomic clock; i.e.,

$$ds = d\bar{\tau}, \quad \text{when} \quad \bar{\tau} = 0. \quad (7.22)$$

The mechanism just described will be called a *$\bar{\tau}$-clock.*

In order to obtain the coordinates $(\bar{\tau}, \bar{\rho}, \theta, \varphi)$ of an event E, the observer O sends, at time $\bar{\tau}_1$, a light signal to E which is reflected by E and returns to O at time $\bar{\tau}_2$. The direction of the light ray, measured by O's theodolite, immediately yields the polar coordinates θ and φ. Since the light signal travels radially with unit velocity, the coordinates $\bar{\tau}$, $\bar{\rho}$ of E are given by

$$\bar{\tau} = \tfrac{1}{2}(\bar{\tau}_2 + \bar{\tau}_1), \quad \bar{\rho} = \tfrac{1}{2}(\bar{\tau}_2 - \bar{\tau}_1). \quad (7.3)$$

2. The observer O selects two fundamental particles P_1 and P_2 close together and at equal distance from him. By the procedure outlined above, O determines the coordinate distance $OP_1 = OP_2 = \bar{\rho}$ and the small angle $P_1OP_2 = d\theta$.

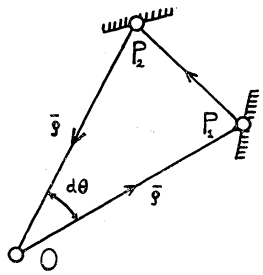

FIG. 4. Arrangement of light signals to determine the type of universe.

O sends a light signal to P_1 who reflects it to P_2, who, in turn, sends it back to O. This arrangement is sketched in Fig. 4. O measures the time interval $\Delta\bar{\tau}$ in which the light ray completes the circuit OP_1P_2O. Then the light ray travels the distance P_1P_2 in time

$$d\bar{\tau}=\Delta\bar{\tau}-2\bar{\rho},$$

and O can compute the transverse velocity of light $\bar{\rho}d\theta/d\bar{\tau}=\bar{c}_t$. From the line element (7.2) it follows that

$$\bar{c}_t=(\bar{\rho}/\alpha)/S(\bar{\rho}/\alpha). \tag{7.4}$$

$S(x)$ is $\sin x$, $\sinh x$, x, in Cases I,II′, III, respectively, and $x/\sin x>1$, $x/\sinh x<1$, $x/x=1$. Thus we have the following criterion:

The given cosmological model is of type I, II′, or III, according as $\bar{c}_t$ is greater than, less than, or equal to 1.

In Cases I and II′, the constant α can now be obtained from Eq. (7.4). In Case III, α does not enter the line element (7.2).

The experiment described above is based on the observation of *transverse effects*. These transverse effects also enter such phenomena as the distribution of nebulae in depth and the second-order red shift effect, when the estimate of nebular distances is based on the observation of apparent luminosities. These phenomena may serve as a practical means of determining the type of the universe and the order of magnitude of the constant α.

3. As described above, O normalizes his $\bar{\tau}$-clock so that, initially, its time measure $d\bar{\tau}$ coincides with the proper-time measure ds of his atomic clock. After a period of time, however, he finds that his two time measures no longer agree. O can, by prolonged observation, determine $ds/d\bar{\tau}$ as a function of $\bar{\tau}$. But, by (7.2),

$$ds/d\bar{\tau}=\phi(\bar{\tau}). \tag{7.5}$$

Thus the function ϕ, and therefore also the line element, are known.

4. Applying the transformation (7.11) to (7.2), we regain the line element (7.1). Regraduating the coordinate time scale in accordance with (7.11), O obtains a *τ-clock*.

From the line element (7.1), the coordinate transformations (4.4), (4.6), or (4.7), according as the cosmological model is of type I, II′, or III, lead to the form of the line element in cosmological coordinates.

In a c.c.s. the observer O can determine the coordinates t, r of an event E by the procedure outlined above for the coordinates $\bar{\tau}$, $\bar{\rho}$. We have

$$t=\tfrac{1}{2}(t_2+t_1), \quad r=\tfrac{1}{2}(t_2-t_1), \tag{7.6}$$

the notation corresponding to that in Eqs. (7.3). However, the structure of O's *t-clock* is different. From the transformation equations it follows, by putting $r=\rho=0$, that the t-clock is characterized by the following equations

$$\text{Case I:} \quad t=\tan\tfrac{1}{2}\tau, \tag{7.61}$$

$$\text{Case II}': \quad t=e^{\tau}, \tag{7.62}$$

$$\text{Case III:} \quad t=\tau. \tag{7.63}$$

We see that the structure of the t-clock does not depend on the particular function R, as does the τ'-clock measuring Robertson's τ'-time.

This completes the examination which we undertook in this section. However, we now have an opportunity to indicate how the transformations leading to a c.c.s., which we merely stated in Section 4, can be arrived at by logical deduction and without guesswork. In Case III, the line element (7.1) is automatically of the cosmological form, and no transformation is required. We shall limit ourselves to an examination of Case II′, the treatment of Case I being similar.

In the c.c.s. II′, the equation of motion of a fundamental particle P was (in Section 1) shown to be

$$r=vt. \tag{7.7}$$

A light signal is, at time t_1, sent by O to P who reflects it, the light signal returning to O at time t_2. By (7.6) and (7.7), we have

$$(t_2-t_1)/(t_2+t_1)=v. \tag{7.71}$$

We assume that P is close to O, i.e., that v is small. We denote the short cosmological time element t_2-t_1 by dt. The constant $(t_2-t_1)/(t_2+t_1)$, or $dt/2t$, since OP is small, is proportional to a constant τ-interval $d\tau$. Thus

$$d\tau=\beta dt/t, \tag{7.72}$$

where β is a constant. Integrating, we have

$$\tau=\beta\log(t/\beta),\quad t=\beta e^{\tau/\beta}. \tag{7.8}$$

The constant of integration has, without loss of generality, been chosen such that $d\tau=dt$ when $\tau=0$. Regraduating the τ-clock in accordance with Eq. (7.8), we obtain a *t clock* measuring cosmological time.

By use of the t-clock the coordinates of an event E may be obtained from Eq. (7.6). Similarly, the coordinates τ, ρ of E are given by

$$\tau=\tfrac{1}{2}(\tau_2+\tau_1),\quad \rho=\tfrac{1}{2}(\tau_2-\tau_1).$$

Thus

$$\begin{aligned} t_2&=(t+r), & \tau_2&=(\tau+\rho),\\ t_1&=(t-r), & \tau_1&=(\tau-\rho). \end{aligned} \tag{7.81}$$

The times t_2, t_1 are connected with τ_2, τ_1, respectively, by (7.8). Thus we obtain the equations of transformation

$$t\pm r=\beta e^{(\tau\pm\rho)/\beta}. \tag{7.9}$$

If this transformation is applied to the line element (7.1), it is seen that the resulting line element is of the cosmological form only if $\beta=1$. Thus the constant β is determined and (7.9) now agrees with (4.6).

Similarly, the transformation leading from the coordinates $(\bar{\tau},\ \bar{\rho})$ to cosmological coordinates $(\bar{t},\ \bar{r})$, with dimensions, is given by

$$\bar{t}\pm\bar{r}=\alpha e^{(\bar{\tau}\pm\bar{\rho})/\alpha};\quad (\bar{t}\pm\bar{r})/\alpha=t\pm r; \tag{7.91}$$

if we demand that $d\bar{t}=d\bar{\tau}$ when $\bar{\tau}=0$. The constant α which appeared in the line element (7.2) as the negative radius of curvature of the spatial section $\bar{\tau}=0$, appears now, in the c.c.s., as the age of the universe on the cosmological time scale.

In Case I, parallel considerations show that the transformation equations are

$$\begin{aligned} &(\bar{t}\pm\bar{r})/2\alpha=\tan((\bar{\tau}\pm\bar{\rho})/2\alpha);\\ &(\bar{t}\pm\bar{r})/2\alpha=t\pm r. \end{aligned} \tag{7.92}$$

The radius of curvature α of the spatial section $\bar{\tau}=0$ appears, in the c.c.s., as one-quarter of the coordinate period of oscillation.