PHYSICAL REVIEW VOLUME 68, NUMBERS 7 AND 8 OCTOBER 1 AND 15, 1945

Calculation of the Binding Energy of the Deuteron and the Neutron-Proton Scattering by a New Potential

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(Received March 6, 1945)

The binding energy of the deuteron and the scattering cross section of the proton by fast neutrons are calculated by using new forms of nuclear potential suggested by K. C. Wang. Results obtained are found to be in good agreement with experimental values when "zero cut-off" of the potential is employed.

I. INTRODUCTION

It has been pointed out by K. C. Wang[1] that the force between two nuclear particles may be related to the gravitational force. He considers two alternative forms of the nuclear potential:

$$V=-Ae^{k/r},\tag{1a}$$

and

$$V=-(B/r)e^{k/r},\tag{1b}$$

with

$$k=\hbar/mc=3.84\times10^{-11}\text{ cm},$$

where the constants A and B, as determined by the gravitational constants, are 4.78×10^{-45} and 1.83×10^{-55}, respectively.

The purpose of the present work has been to determine whether the potential (1a) and (1b) can be used to obtain satisfactory values for the binding energy of the deuteron and the scattering cross section of the neutron by the proton. The calculations follow closely those of Bethe and Bacher.[2]

II. THE BINDING ENERGY OF THE DEUTERON

The wave equation for the relative motion of the two nuclear particles is

$$\Delta\psi+(M/\hbar^2)(E-V)\psi=0.\tag{2}$$

The potential is spherically symmetric. Equation (2) can thus be separated in polar co-ordinates r, θ, ϕ, by putting

$$\psi(r,\theta,\phi)=(1/r)u_lP_{lm}(\theta)e^{im\phi},$$

where P_{lm} is a spherical harmonic. The wave equation for u_l is then

$$\frac{\hbar^2}{M}\left(\frac{d^2u_l}{dr^2}-\frac{l(l+1)}{r^2}u_l\right)+(E-V)u_l=0.$$

For the ground state $l=0$, we have

$$\frac{d^2u_0}{dr^2}+\frac{M}{\hbar^2}(E-V)u_0=0.\tag{3}$$

To solve (3), two forms of potential cut-off given by (1a) and (1b) are employed;

(*a*) zero cut-off: $V=0$ for $r<a$, $V=V(r)$ for $r>a$;
(*b*) straight cut-off: $V=V(a)$ for $r<a$, $V=V(r)$ for $r>a$.

(*a*) *Zero cut-off*.—For $r<a$, $V=0$; Eq. (3) takes the form,

$$(d^2u_0/dr^2)-(M\epsilon/\hbar^2)u_0=0,$$

where $\epsilon=-E$, the binding energy of the deuteron. Its solution is

$$u_0=D\{e^{\beta r}-e^{-\beta r}\},$$

where

$$\beta=(M\epsilon/\hbar^2)^{\frac{1}{2}},$$

and

$$du_0/dr=D\beta(e^{\beta r}+e^{-\beta r}).$$

Therefore,

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_{r=a}=\beta\coth\beta a.\tag{4}$$

For $r>a$, $V=V(r)$; the asymptotic solution of Eq. (3) is

$$u_0=ce^{-\beta r}.$$

Treating c as a slowly varying quantity whose second derivative with respect to r may be set

[1] K. C. Wang and H. L. Tsao, Phys. Rev. **66**, 155 (1944); Nature **155**, 512 (1945).
[2] H. A. Bethe and R. E. Bacher, Rev. Mod. Phys. **8**, 82 (1936).

equal to zero, we have

$$\frac{du_0}{dr}=\frac{dc}{dr}e^{-\beta r}-\beta c e^{-\beta r},\quad \frac{d^2u_0}{dr^2}=-2\beta\frac{dc}{dr}e^{-\beta r}+\beta^2 c e^{-\beta r}.$$

Substituting this last expression in (3), we get the differential equation for c,

$$\frac{dc}{dr}=-\frac{MV}{2\beta\hbar^2}c,$$

the solution of which is

$$c=F\exp\left(-\int\frac{MV}{2\beta\hbar^2}dr\right).$$

Thus

$$u_0=F\exp\left(-\beta r-\int\frac{MV}{2\beta\hbar^2}dr\right),$$

and

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_{r=a}=\frac{M|V(a)|}{2\beta\hbar^2}-\beta. \tag{5}$$

In order that the wave function u_0 could be joined smoothly at $r=a$ the expressions (4) and (5) must be equal; i.e.,

$$\frac{M|V(a)|}{2\beta\hbar^2}-\beta=\beta\coth\beta a. \tag{6}$$

(*b*) *Straight cut-off.*—For $r<a$, $V=V(a)$; Eq. (3) takes the form,

$$\frac{d^2u_0}{dr^2}+\frac{M}{\hbar^2}(V_0-\epsilon)u_0=0.$$

Its solution is

$$u_0=G\sin\gamma r,$$

with

$$\gamma=\left\{\frac{M}{\hbar^2}(|V(a)|-\epsilon)\right\}^{\frac{1}{2}}.$$

Thus

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_{r=a}=\gamma\cot\gamma a.$$

TABLE I. Values for Eq. (7).

ϵ, 10^{-6} erg	$\coth[1.24\times10^{15}(V_0-\epsilon)^{\frac{1}{2}}a]$	$V_0/2\epsilon-1$
3.50	1.33	1.38
3.60	1.32	1.32
3.70	1.31	1.26

For $r>a$, $[(1/u_0)(du_0/dr)]_{r=a}$ is the same as given by (5). Therefore, instead of (6), we have

$$\frac{M|V(a)|}{2\beta\hbar^2}-\beta=\gamma\cot\gamma a. \tag{6a}$$

Equations (6) and (6a) serve to determine the binding energy of the deuteron, if we use the experimental value of $-V(a)=V_0$ (the depth of the potential well) and calculate a (the range of the nuclear force) from (1a) or (1b). Conversely, if we use the experimental value of ϵ, we can calculate the value of V_0 and a by the aid of the Eqs. (1a) or (1b).

(*c*) *Numerical calculation.*—(1) $V=0$, for $r<a$; $V=V(r)=-Ae^{k/r}$, for $r>a$. We take the recent experimental value, $V_0=10.5$ Mev, then a calculated from (1a) is equal to 4.21×10^{-13} cm. Equation (6) becomes numerically

$$\coth[1.24\times10^{15}(V_0-\epsilon)^{\frac{1}{2}}a]=(V_0/2\epsilon)-1. \tag{7}$$

From Table I we see that the root of (7), the binding energy of deuteron, is 3.60×10^{-6} erg or 2.26 Mev.

(2) $V=0$, for $r<a$; $V(r)=-(B/r)\epsilon^{k/r}$, for $r>a$. Also take $V_0=10.5$ Mev, then $a=4.42\times10^{-13}$ cm. from (1b).

From Table II we get the binding energy of the deuteron, 3.66×10^{-6} erg or 2.30 Mev. The results given by (1) and (2) are in good agreement with the experimental value 2.27 Mev.

(3) $V=V(a)$, for $r<a$; $V(r)=-Ae^{k/r}$, for $r>a$. Also take $V_0=10.5$ Mev, then $a=4.20\times10^{-13}$ cm from (1a). Equation (6a) becomes numerically

$$\cot[1.24\times10^{15}(V_0-\epsilon)^{\frac{1}{2}}a]=\frac{0.5V_0/\sqrt{\epsilon}-\sqrt{\epsilon}}{(V_0-\epsilon)^{\frac{1}{2}}}.$$

TABLE II. Values for Eq. (7).

ϵ, 10^{-6} erg	$\coth[1.24\times10^{15}(V_0-\epsilon)^{\frac{1}{2}}a]$	$V_0/2\epsilon-1$
3.50	1.29	1.38
3.60	1.29⁻	1.32
3.70	1.27	1.26
3.80	1.27⁻	1.19

(4) $V=V(a)$, for $r<a$; $V(r)=-(B/r)e^{k/r}$, for $r>a$. We get the results listed in Table III.

From Tables III and IV, the binding energy of the deuteron is 8.0×10^{-6} and 8.3×10^{-6} erg, respectively. These values are too large in com-

TABLE III. Values for Eq. (6a).

ϵ, 10^{-6} erg	cot $[1.24\times10^{15}(V_0-\epsilon)^{\frac{1}{2}}a]$	$\frac{0.5V_0/\sqrt{\epsilon}-\sqrt{\epsilon}}{(V_0-\epsilon)^{\frac{1}{2}}}$
7.8	0.018	0.067
8.0	0.038	0.037
8.2	0.061	0.027

TABLE IV. Values for Eq. (6a).

ϵ, 10^{-6} erg	cot $[1.24\times10^{15}(V_0-\epsilon)^{\frac{1}{2}}a]$	$\frac{0.5V_0/\sqrt{\epsilon}-\sqrt{\epsilon}}{(V_0-\epsilon)^{\frac{1}{2}}}$
8.2	−0.020	0.206
8.3	−0.013	0.005
8.4	−0.004	−0.083

parison with the experimental value. The correct results will be obtained if we take $V_0=5.93$ Mev.

III. PROTON-NEUTRON SCATTERING

Let us denote by E the kinetic energy of a proton and a neutron in a coordinate system in which the center of gravity of the particles is at rest; this is equal to one-half of the kinetic energy of the incident neutron in a system at rest. The radial function u_l will satisfy the equation,

$$\frac{\hbar^2}{M}\left(\frac{d^2u_l}{dr^2}-\frac{l(l+1)}{r^2}u_l\right)+\{E-V\}u_l=0. \quad (8)$$

Asymptotically for large r, the solution of (8) is

$$u_l=c\sin(kr-\tfrac{1}{2}l\pi+\delta_l) \quad (9)$$

with

$$k^2=ME/\hbar^2.$$

Then the cross section $d\sigma$ i.e., the number of neutrons scattered per unit time through an angle between θ and $\theta+d\theta$, if there is one neutron crossing unit area per unit time in the incident beam,—is given by the well-known formula

$$d\sigma=\frac{\pi}{2k^2}|\textstyle\sum_l(2l+1)P_l(\theta)(e^{2i\delta_l}-1)|^2\sin\theta d\theta. \quad (10)$$

It has been shown[3] that if $(1/k)\gg a$, a being the range of the force, all phases δ_l will be small except δ_0. Then

$$d\sigma=2\pi k^{-2}\sin^2\delta_0\sin\theta d\theta,$$

and

$$\sigma=\int d\sigma=4\pi k^{-2}\sin^2\delta_0. \quad (11)$$

For the ground state of the deuteron, we have already shown that

$$\left(\frac{1}{u_0^-}\frac{du_0^-}{dr}\right)_{r=a}=\left(\frac{MV_0}{2\beta\hbar^2}-\beta\right)\equiv\alpha. \quad (12)$$

Now,[3] in the present case E is positive, we should have[3]

$$\left(\frac{1}{u_0^+}\frac{du_0^+}{dr}\right)_{r=a}=\alpha-\frac{M(E+\epsilon)}{\hbar^2u_0^+u_0^-}\int_0^a u_0^+u_0^-dr,$$

$$=\alpha-\frac{\eta aM(E+\epsilon)}{\hbar^2}\equiv A, \quad (13)$$

where u_0^+ and u_0^- are u_0 functions for $r<a$, corresponding to the positive and negative values of E respectively; i.e.,

$$u_0^+=B\sin kr,\quad u_0^-=D\{e^{\beta r}-e^{-\beta r}\},$$

thus

$$\eta a=\frac{1}{u_0^+u_0^-}\int_0^a u_0^+u_0^-dr=\frac{\beta\coth\beta a-k\cot ka}{\beta^2+k^2}. \quad (14)$$

For large r,

$$\left(\frac{1}{u_0^+}\frac{du_0^+}{dr}\right)_{r=a}=k\cot(ka+\delta_0). \quad (15)$$

In order to join the wave function smoothly at $r=a$, the expressions given by (13) and (15) must be equal, thus

$$k\cot(ka+\delta_0)=A. \quad (16)$$

[3] Cf. reference 2, page 115.

TABLE V. Numerical results.

E_l Mev	σ_0	σ	σ_{0B}	σ_B	σ (obs.)
2.15	0.85×10^{-24}	1.43×10^{-24}	1.2×10^{-24}	1.8×10^{-24}	$(0.5\text{–}0.8)10^{-24}$
1.05	1.03×10^{-24}	1.86×10^{-24}	1.6×10^{-24}	2.4×10^{-24}	$(1.1\text{–}1.5)10^{-24}$

Under the assumption that ka is small, we get

$$\cot \delta_0 = A/k, \qquad (17)$$

and by (11),

$$\sigma = 4\pi/(k^2+A^2). \qquad (18)$$

If η is small, $A \approx \alpha$, we have

$$\sigma_0 = 4\pi/(k^2+\alpha^2). \qquad (19)$$

The numerical results are given in Table V, in which we take $V = -Ae^{k/r}$, and the values of α are obtained from the Table I at the point of joint. Corresponding to (18) and (19), Bethe and Bacher have derived the formulae: $\sigma_{0B} = 4\pi\hbar^2/(\epsilon+E)$, $\sigma_B \doteq \frac{3}{2}\sigma_0$,[3] and the values calculated from these are given in the 4th and 5th columns of Table V for comparison.

It is seen that the present results are in better agreement with the experimental results than those given by Bethe and Bacher.

If we take $V = -(B/r)e^{k/r}$, the results do not differ appreciably from those given above.

In conclusion, the author wishes to express his thanks to Dr. K. C. Wang for suggesting this calculation and for helpful discussions.

PHYSICAL REVIEW VOLUME 68, NUMBERS 7 AND 8 OCTOBER 1 AND 15, 1945

Fourier Transforms of Retarded and Advanced Potentials

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(Received March 20, 1945)

The Fourier transforms of the retarded and advanced potentials of the electromagnetic field and of the wave fields of elementary particles are obtained with the help of the invariant functions of Jordan, Pauli, and Dirac, together with their generalizations. It is shown that the Fourier transforms of these potentials are closely related to those of the outward and inward moving waves given by Dirac for the scattering problems in quantum mechanics, and their connection is discussed. It is also shown that there exists a type of potentials which represents waves with frequencies of opposite signs propagating in opposite directions.

1. INTRODUCTION

JORDAN and Pauli[1] introduced into the quantum theory of electromagnetic field a function

$$\frac{\delta(x_0+|\mathbf{x}|)-\delta(x_0-|\mathbf{x}|)}{|\mathbf{x}|},$$

which plays a very important part in the commutation relations of the field variables. Following Dirac,[2] we shall denote it by $-\Delta$ and write

$$\Delta(\mathbf{X}) = \frac{\delta(x_0-|\mathbf{x}|)-\delta(x_0+|\mathbf{x}|)}{|\mathbf{x}|}, \qquad (1)$$

where $\mathbf{X}$ denotes the four-vector $(\mathbf{x}, x_0)$. This Δ function is relativistically invariant and satisfies the wave equation of electromagnetic field in vacuum

$$\Box\Delta(\mathbf{X}) = 0, \qquad (2)$$

with

$$\Box = \frac{\partial^2}{\partial x_0{}^2} - \nabla^2. \qquad (3)$$

It can be expressed as a triple Fourier integral of the form

$$\Delta(\mathbf{X}) = \frac{1}{2\pi^2}\int \frac{\sin kx_0}{k} \exp(\pm i\mathbf{k}\cdot\mathbf{x})d^3k, \qquad (4)$$

where $k = |\mathbf{k}|$, or as a quadruple Fourier integral of the form[3]

$$\Delta(\mathbf{X}) = \frac{i}{4\pi^2}\int \Delta(\mathbf{K}) \exp(-ik_0x_0) \times \exp(\pm i\mathbf{k}\cdot\mathbf{x})d^4k, \qquad (5)$$

where $\mathbf{K}$ denotes the four-vector $(\mathbf{k}, k_0)$.

Some functions of the same type but more general than that of Jordan and Pauli appeared first in Dirac's[4] positron theory, which are in the

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