the γ-system. Hence the hypothesis of Herzberg and Mundie and its conclusions should be abandoned.[10]

Te_2

In collaboration with R. Migeotte we have recently studied the ultraviolet part of the Te_2 spectrum.[11] The following observation concerning a predissociation effect is of interest. The ultraviolet end of the Te_2 spectrum consists of the $v''=0$ progression. Theoretically this progression should have a regular intensity distribution with a single maximum. This is what we observe in absorption, the maximum being found at $v'=15$. In emission, however, the bands $v'=14$ and $v'=16$ are abnormally weak (for the numbering of the Te_2 bands).[12] It seems that we are confronted with a predissociation phenomenon which can be accounted for as an accidental predissociation of the vibrational type analogous to a vibrational perturbation affecting a whole band.

The unresolved isotopic bands $v'=14$ and $v'=16$ are not only weakened, but they present in emission a very pronounced asymmetry. In connection with the observations of Olsson,[13] who states that the positions of the individual absorption bands are quite normal, this means that the weakening by accidental predissociation is different for different individual isotopic bands. No such asymmetry is observed in absorption.

Two regions of predissociation are known in the S_2 and Se_2 spectra. In S_2, predissociation occurs at the levels $v'\geq 10$ and $v'\geq 17$[5] whereas in Se_2 it occurs at the levels $v'\geq 10$ and $v'\geq 22$.[14] In the spectrum of Te_2 only one predissociation region, setting in at $v'=21$, has been reported.[15] Our new observation indicates that the Te_2 molecule also can predissociate at intermediate v' values. However, contrary to the other molecules of the same group, the predissociation of Te_2 at intermediate v' values is only possible through an intermediate state, by an accidental predissociation of the vibrational type. The close analogy existing between the spectra of the three molecules is thus strengthened, especially since considerable perturbations of the vibrational type are known for numerous excited levels of S_2 and Se_2.

[10] We have recently published a detailed investigation of the various ultraviolet band systems of NO (Bull. Soc. Roy. Sci., Liége, Belgium, pp. 40 and 49, 1945). Subsequently we have learned that three papers had been recently published on the NO molecule, one by Gerö, Schmid, and Szily (Physica **9**, 144 (1944)); the others by Gaydon (Proc. Phys. Soc. **56**, 95, and 160 (1944)). As far as predissociation is concerned, these authors reach the same conclusion as we do.

[11] R. Migeotte and B. Rosen, Bull. Soc. Roy. Sci., Liége (in press).

[12] R. Migeotte, Bull. Soc. Roy. Sci., Liége **13**, 48 (1942).

[13] E. Olsson, Thesis (Stockholm, 1938); Zeits. f. Physik **95**, 215 (1935).

[14] B. Rosen, Physica **6**, 205 (1939).

[15] E. Hirschlaff, Zeits. f. Physik **75**, 315 (1932).

PHYSICAL REVIEW VOLUME 68, NUMBERS 5 AND 6 SEPTEMBER 1 AND 15, 1945

Space Charge between Coaxial Cylinders

LEIGH PAGE AND NORMAN I. ADAMS, JR.
Sloane Physics Laboratory, Yale University, New Haven, Connecticut

(Received August 12, 1945)

New solutions of the space charge equation are obtained, which converge much more rapidly than Langmuir's solution in the important case where the radius of the outer electrode is large compared with that of the inner electrode.

THE theory of the limitation of a current by space charge in an evacuated region containing ions of one sign only was given first by Child[1] for parallel plane electrodes and later by Langmuir[2] and Langmuir and Blodgett[3] for the far more important case of coaxial cylindrical electrodes, as found in the usual construction of

[1] C. D. Child, Phys. Rev. **32**, 498 (1911).

[2] I. Langmuir, Phys. Rev. **2**, 450 (1913).

[3] I. Langmuir and K. B. Blodgett, Phys. Rev. **22**, 347 (1923).

the diode vacuum tube. In the first case the solution is obtained easily in closed form, but in the second a non-linear differential equation is encountered which can be solved only in series. Langmuir's solution is a power series in the logarithm of the ratio of the distance r from the common axis of the electrodes to the radius a of the inner electrode or cathode. While this series converges satisfactorily for small values of r/a, it converges very poorly in the much more important region where r/a is large, and special approximate methods had to be employed for values of r/a greater than 66. The object of the present paper is to develop a solution which converges the better the larger is the ratio r/a. In fact, five terms of the solution obtained give an accuracy of two or three parts in ten thousand for all values of r/a greater than 3.

If e/m is the ratio of charge to mass of the ions (electrons) in the tube, $V(r)$ the excess of the potential over that of the cathode, $\rho(r)$ the charge density, and j_l the current per unit legnth of the coaxial electrodes, the equations to be satisfied are, in Heaviside-Lorentz units,

$$\dot{r}^2 = -\frac{2e}{m}V, \tag{1}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = -\rho, \tag{2}$$

$$j_l = 2\pi\rho r\dot{r}, \tag{3}$$

subject to the boundary conditions

$$dV/dr = 0, \quad \rho = \infty, \tag{4}$$

at $r = a$. The boundary conditions imply saturation current and negligible initial velocities.

Eliminating ρ and $\dot{r}$, we find, if we put $\eta \equiv a/r$,

$$j_l = \frac{8\pi}{9}\left(\frac{2e}{m}\right)^{\frac{1}{2}} \frac{(-V)^{\frac{3}{2}}}{rg(\eta)}, \tag{5}$$

where $g(\eta)$ satisfies the non-linear differential equation

$$g\left[3\eta\frac{d}{d\eta}\left(\eta\frac{dg}{d\eta}\right) - 4\eta\frac{dg}{d\eta} + 2(g-1)\right] = \left[\eta\frac{dg}{d\eta}\right]^2, \quad 0 \leqslant \eta \leqslant 1. \tag{6}$$

The function g is identical[4] with Langmuir's β^2.

It is obvious that $g=1$ satisfies (6). Hence to obtain a solution which converges rapidly for large r or small η, it would be natural to try the power series

$$g = 1 + a_1\eta + a_2\eta^2 + \cdots.$$

However, substitution in the differential equation shows that a power series solution in η does not exist. Hence we change the independent variable from η to z, getting in place of (6)

$$g\left[3\eta^2\left(\frac{dz}{d\eta}\right)^2 \frac{d^2g}{dz^2} + 3\eta^2\frac{d^2z}{d\eta^2}\frac{dg}{dz} - \eta\frac{dz}{d\eta}\frac{dg}{dz} + 2(g-1)\right] = \left[\eta\frac{dz}{d\eta}\frac{dg}{dz}\right]^2,$$

and assume a power series solution in z, to wit:

$$g = 1 + a_1 z + a_2 z^2 + \cdots.$$

Substituting in the differential equation and equating the coefficients of a_1 on both sides, we find that a_1 is arbitrary provided

$$3\eta^2\frac{d^2z}{d\eta^2} - \eta\frac{dz}{d\eta} + 2z = 0,$$

the solution of which is

$$z = A_1\eta^{p_1} + A_2\eta^{p_2},$$

where

$$p_1 \equiv \frac{2+i(2)^{\frac{1}{2}}}{3}, \quad p_2 \equiv \frac{2-i(2)^{\frac{1}{2}}}{3}.$$

Now put

$$y \equiv \eta^{\frac{2}{3}}, \quad \beta \equiv \frac{(2)^{\frac{1}{2}}}{2}\log y + \alpha.$$

Then a power series solution in

$$z_1 \equiv cye^{i\beta}, \quad z_2 \equiv cye^{-i\beta},$$

exists, of the form

$$g = 1 + (z_1 + z_2) + (a_{11}z_1^2 + a_{12}z_1z_2 + a_{22}z_2^2) + (a_{111}z_1^3 + a_{112}z_1^2z_2 + a_{122}z_1z_2^2 + a_{222}z_2^3) + \cdots. \tag{7}$$

This is a *complete* solution, since it contains the two arbitrary constants c and α.

[4] The reason for this departure from an established notation is that our treatment leads to a differential equation in g, and it would be awkward and unusual to express the dependent variable as the square of a function.

To evaluate the coefficients in (7), it is convenient to change the independent variable in (6) from η to y. Then

$$g\left[3y\frac{d}{dy}\left(y\frac{dg}{dy}\right)-6y\frac{dg}{dy}+\frac{9}{2}(g-1)\right]=\left[y\frac{dg}{dy}\right]^2, \quad (8)$$

and we find for the first five terms of the solution[5]

$$\begin{aligned}g=1&+2cy\cos\beta\\&+2c^2y^2\left[\frac{5}{33}\cos 2\beta+\frac{2(2)^{\frac{1}{2}}}{33}\sin 2\beta+\frac{1}{3}\right]\\&+2c^3y^3\left[-\frac{17}{594}\cos 3\beta+\frac{19(2)^{\frac{1}{2}}}{2376}\sin 3\beta\right.\\&\left.+\frac{1}{396}\cos\beta+\frac{(2)^{\frac{1}{2}}}{264}\sin\beta\right]\\&+2c^4y^4\left[\frac{2386}{421,443}\cos 4\beta\right.\\&-\frac{7541(2)^{\frac{1}{2}}}{1,685,772}\sin 4\beta-\frac{91}{33,858}\cos 2\beta\\&\left.-\frac{82(2)^{\frac{1}{2}}}{16,929}\sin 2\beta+\frac{17}{11,286}\right]+\cdots. \quad (9)\end{aligned}$$

When we attempt to determine the constants c and α from the boundary conditions $g=0$ and $dg/dy=0$ at $y=1$, we find that this series does not converge rapidly enough for this value of y to ensure the desired accuracy of one part in a thousand from the five terms computed. Instead of calculating further terms we have developed a second series which converges very rapidly in the neighborhood of $y=1$, and then made use of the principle of analytic continuation to fit the series (9) to it. This second series is a power series in $x\equiv 1-y=1-\eta^{2/3}$.

Changing the independent variable in (8) from y to x we have

$$g\left[3(1-x)\frac{d}{dx}\left\{(1-x)\frac{dg}{dx}\right\}+6(1-x)\frac{dg}{dx}+\frac{9}{2}(g-1)\right]=\left[(1-x)\frac{dg}{dx}\right]^2,\quad 0\leqslant x\leqslant 1. \quad (10)$$

[5] After expressing the solution in trigonometrical form, the numerical coefficients were checked by substituting back in the differential equation.

In order to satisfy the boundary conditions at the cathode, the function g must be of the form

$$g=\frac{9}{4}x^2[1+b_1x+b_2x^2+\cdots].$$

We find the following recursion formula for the coefficients:

$$\begin{aligned}(3p+2)(p+1)b_p=&[6p^2-5.6p-2]b_{p-1}\\&-[3p^2-10.756,667p+10.41]b_{p-2}\\&+[(5b_3-8b_2+3b_1)p\\&\quad-(65b_3-68b_2+16.5b_1)]b_{p-3}\\&+[(6b_4-10b_3+4b_2)p\\&\quad-(102b_4-125b_3+36.5b_2)]b_{p-4}\\&+[(7b_5-12b_4+5b_3)p\\&\quad-(147b_5-198b_4+64.5b_3)]b_{p-5}\\&+[(8b_6-14b_5+6b_4)p\\&\quad-(200b_6-287b_5+100.5b_4)]b_{p-6}\\&\cdots\cdots\cdots\cdots\\&+[(\cdots)p-(\cdots)]b_1, \quad (11)\end{aligned}$$

where first differences are constant in the coefficient of p and second differences in the other

TABLE I.

r/a	x	g from (12)	y	g from (14)
1	0	0		
2	0.37004	0.2793		
3	0.51925	0.5171	0.48075	0.5170
4	0.60315	0.6671	0.39685	0.6670
5	0.65800	0.7666	0.34200	0.7667
6	0.69715	0.8363	0.30285	0.8363
7	0.72672	0.8870	0.27328	0.8870
8	0.75000	0.9254	0.25000	0.9254
9	0.76888	0.9550	0.23112	0.9549
10			0.21544	0.9782
11			0.20218	0.9970
12			0.19079	1.0123
20			0.13572	1.0716
40			0.08550	1.0947
80			0.05386	1.0846
160			0.03393	1.0634
320			0.02137	1.0422
640			0.01347	1.0253
1280			0.00848	1.0135
2560			0.00534	1.0060
5120			0.00337	1.0019
10240			0.00212	0.9998
20480			0.00134	0.9991
40960			0.00084	0.9990
81920			0.00053	0.9991

term. This gives for the solution

$$g=\frac{9}{4}x^2[1-0.200{,}000x-0.110{,}833x^2$$
$$-0.065{,}303x^3-0.042{,}006x^4$$
$$-0.028{,}876x^5-0.020{,}858x^6$$
$$-0.015{,}647x^7-0.012{,}091x^8$$
$$-0.009{,}568x^9-0.007{,}720x^{10}$$
$$-0.006{,}330x^{11}-0.005{,}261x^{12}$$
$$-0.00442x^{13}-0.00375x^{14}$$
$$-0.00321x^{15}-\cdots]. \quad (12)$$

This series is superior to that given by Langmuir in that the variable goes to unity instead of infinity as r/a increases without limit.

After a few trials it appeared that $x=2/3$, $y=1/3$ was the best point at which to fit the complete solution (9) to the solution (12) satisfying the boundary conditions. At this point

$$g=0.78220, \quad y\frac{dg}{dy}=-0.59784,$$

which gives

$$c=-0.9780, \quad \alpha=-22°.83,$$

to one part in a thousand. Consequently, if we put

$$\theta\equiv 93°.288 \log_{10}(1/y)+22°.83 \quad (13)$$

to avoid negative signs, the solution (9) satisfying the specified boundary conditions at the cathode is

$$g=1-[1.9560\cos\theta]y+[0.63767$$
$$+0.28985\cos 2\theta-0.16396\sin 2\theta]y^2$$
$$-[0.00472\cos\theta-0.01002\sin\theta$$
$$-0.05354\cos 3\theta-0.02116\sin 3\theta]y^3$$
$$+[0.00276-0.00492\cos 2\theta$$
$$+0.01253\sin 2\theta+0.01036\cos 4\theta$$
$$+0.01158\sin 4\theta]y^4-\cdots. \quad (14)$$

These five terms give an accuracy of better than one part in a thousand for all r/a greater than three. In fact, for r/a greater than ten the first four terms are sufficient, for r/a greater than fifty only three terms are needed, and for r/a greater than five hundred two terms suffice.

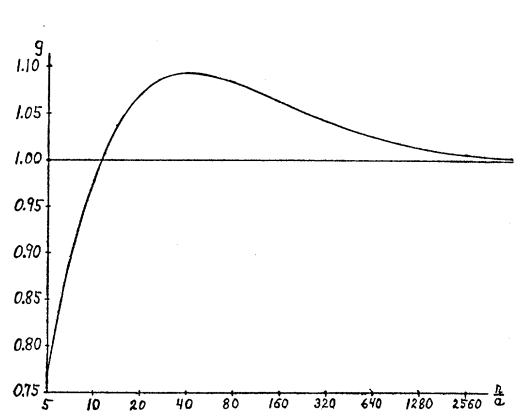

FIG. 1. Value of the function g.

We have calculated g for a number of values of r/a from both the "near" formula (12) and the "far" formula (14) for the purpose of showing the excellence of the fit obtained. The results are included in Table I. It will be noticed that in the overlap extending from $r/a=3$ to $r/a=9$ inclusive, where both series converge satisfactorily, discrepancies are nowhere greater than two parts in ten thousand. Values of g calculated from (14) for larger values of r/a are also contained in the table.

In Fig. 1 we have plotted the function g against the logarithm of r/a, to illustrate how the function behaves in the range $r=5a$ to $r=2560a$. Beyond the latter point the curve oscillates about the straight line $g=1$ with ever decreasing amplitude. The function g becomes unity first at $r=11.18a$, again at $r=9426a$, and next at $r=7{,}389{,}000a$. In fact, $g=1$ for

$$r=9426a[783.9]^n, \quad (15)$$

where n is any positive integer including zero.