High Frequency Vibrations of Thin Crystal Plates

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The boundary conditions of free vibration can be satisfied on the major surfaces of a plane-parallel plate if the displacement components are assumed to be products of trigonometric functions. In addition, the boundary conditions can be approximately satisfied on the minor surfaces when the plate is thin. The theory leads to a frequency equation

$$\nu=\tfrac{1}{2}(c/\rho)^{\frac{1}{2}}[(n/2b)^2+k(m/2a)^2]^{\frac{1}{2}}$$

which has been found empirically to satisfy observations. The theoretical values of the constant k are 3.7 and 1.8 for the AT and BT quartz plates, respectively, while the observed values are 3.9 and 1.7, respectively.

INTRODUCTION

FREE elastic vibrations of anisotropic plates were treated extensively by W. Voigt[1] who developed a general theory. Voigt's theory is limited to the case where the wave-length is large as compared with the thickness and it must fail when the wave-length is of the order of magnitude or smaller than the thickness. This latter case was not interesting to Voigt; because of their high frequency it was impractical to excite these vibrations. By piezoelectric excitation, it has been possible to excite vibrations of high frequency in thin plates. I. Koga[2] treated them theoretically by assuming the plate to be infinitely extended and gave a rigorous solution for this case. Experiments[3,4] showed, however, that while one resonant frequency and the calculated mode of motion were in good agreement, numerous other frequencies could be detected which were not foreseen by the theory of the infinite plate. The purpose of the present paper is to explain these deviations by presenting a theory of the thin plate whose lateral extension is large but not infinite.

The consideration will be limited to the case where the elastic constants c_{15}, c_{16}, c_{25}, c_{26}, c_{35}, c_{36}, c_{45}, and c_{46} are zero or negligibly small. According to Voigt,[1] the former is the case when the x-axis of the reference system (which we assume parallel to one long edge) coincides with a twofold axis of symmetry. This case includes those quartz plates whose one major edge is parallel to the crystallographic x-axis. Most of the experimental work was done on this type of quartz plate.

Standing sine-waves along one major edge will be first considered so that the boundary conditions on the major surface can be met rigorously. With these solutions, the boundary conditions on the smaller lateral surfaces can be satisfied approximately.

Standing Sine-Waves along the X-Direction

We consider a rectangular plane-parallel plate of edges $2a$, $2b$, and $2c$ where $2b$ is the thickness. The origin of coordinates is in the center and the x-, y-, and z-directions are parallel to the edges $2a$, $2b$, and $2c$, respectively. If, as we assume,

$$c_{15}=c_{16}=c_{25}=c_{26}=c_{35}=c_{36}=c_{45}=c_{46}=0,$$

the stress-strain relations are:*

$$\begin{aligned}
-X_x&=c_{11}x_x+c_{12}y_y+c_{13}z_z+c_{14}y_z\\
-Y_y&=c_{12}x_x+c_{22}y_y+c_{23}z_z+c_{24}y_z\\
-Z_z&=c_{13}x_x+c_{23}y_y+c_{33}z_z+c_{34}y_z\\
-Y_z&=c_{14}x_x+c_{24}y_y+c_{34}z_z+c_{44}y_z\\
-X_z&=c_{55}x_z+c_{56}x_y\\
-X_y&=c_{56}x_z+c_{66}x_y.
\end{aligned}\tag{1}$$

The boundary conditions for the free body are:

$$\begin{aligned}
X_x\cos(n,x)+X_y\cos(n,y)+X_z\cos(n,z)&=0,\\
X_y\cos(n,x)+Y_y\cos(n,y)+Y_z\cos(n,z)&=0,\\
X_z\cos(n,x)+Y_z\cos(n,y)+Z_z\cos(n,z)&=0,
\end{aligned}\tag{2}$$

[1] W. Voigt, *Lehrbuch der Kristallphysik* (B. G. Teubner, Leipzig, 1928).
[2] I. Koga, Physics **3**, 70 (1932).
[3] J. V. Atanasoff and P. J. Hart, Phys. Rev. **59**, 85 (1941).
[4] W. P. Mason, Bell Sys. Tech. J. **19**, 74 (1940).

* Voigt's notations are used, so that compressional stresses are positive.

which should be satisfied on all six faces. The dynamic equations are:

$$\rho\omega^2 u = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z},$$
$$\rho\omega^2 v = \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}, \quad (3)$$
$$\rho\omega^2 w = \frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z},$$

where $\omega = 2\pi\nu$ is the angular frequency and u, v, and w are the components of the displacement vector.

We desire at first to satisfy Eqs. (3) in the whole body and Eqs. (2) on the major surfaces $y = \pm b$ only. Furthermore, the displacement components will be assumed to be independent of z. Tentatively, the solution will be written as a linear combination of plane waves of the form

$$u = U \exp[i(\alpha x + py)],$$
$$v = V \exp[i(\alpha x + py)], \quad (4)$$
$$w = W \exp[i(\alpha x + py)],$$

where U, V, and W are constants.

If the strains are calculated in the usual way from Eqs. (4) and substituted into Eqs. (1), we obtain:

$$-X_x = i(c_{11}\alpha U + c_{12}pV + c_{14}pW) \times \exp[i(\alpha x + py)],$$
$$-Y_y = i(c_{12}\alpha U + c_{22}pV + c_{24}pW) \times \exp[i(\alpha x + py)],$$
$$-Z_z = i(c_{13}\alpha U + c_{23}pV + c_{34}pW) \times \exp[i(\alpha x + py)],$$
$$-Y_z = i(c_{14}\alpha U + c_{24}pV + c_{44}pW) \times \exp[i(\alpha x + py)], \quad (5)$$
$$-X_z = i(c_{55}\alpha W + c_{56}pU + c_{56}\alpha V) \times \exp[i(\alpha x + py)],$$
$$-X_y = i(c_{56}\alpha W + c_{66}pU + c_{66}\alpha V) \times \exp[i(\alpha x + py)].$$

Substituting into Eqs. (3), we get:

$$U(-\rho\omega^2 + c_{66}p^2 + c_{11}\alpha^2) + V\alpha p(c_{12} + c_{66}) + W\alpha p(c_{14} + c_{56}) = 0,$$
$$U\alpha p(c_{12} + c_{66}) + V(-\rho\omega^2 + c_{66}\alpha^2 + c_{22}p^2) + W(c_{56}\alpha^2 + c_{24}p^2) = 0, \quad (6)$$
$$U\alpha p(c_{14} + c_{56}) + V(c_{56}\alpha^2 + c_{24}p^2) + W(-\rho\omega^2 + c_{55}\alpha^2 + c_{44}p^2) = 0.$$

The compatability of Eqs. (6) requires that

$$\begin{vmatrix} c_{66}p^2 + c_{11}\alpha^2 - \rho\omega^2 & \alpha p(c_{12} + c_{66}) & \alpha p(c_{14} + c_{56}) \\ \alpha p(c_{12} + c_{66}) & c_{22}p^2 + c_{66}\alpha^2 - \rho\omega^2 & c_{56}\alpha^2 + c_{24}p^2 \\ \alpha p(c_{14} + c_{56}) & c_{56}\alpha^2 + c_{24}p^2 & c_{44}p^2 + c_{55}\alpha^2 - \rho\omega^2 \end{vmatrix} = 0. \quad (7)$$

Solving Eqs. (7) for p^2, we obtain three roots p_1^2, p_2^2, p_3^2, and from Eqs. (6) the corresponding ratios $U_1 : V_1 : W_1$; $U_2 : V_2 : W_2$ and $U_3 : V_3 : W_3$, where the p_i^2s and the ratios $U_i : V_i : W_i$ are functions of $\rho\omega^2$ and α^2.

The form of this function can be seen if we substitute $p/\alpha = q$ and divide each row of the determinant (7) by α^2:

$$\begin{vmatrix} c_{66}q^2 + c_{11} + \rho\omega^2/\alpha^2 & q(c_{12} + c_{66}) & q(c_{14} + c_{56}) \\ q(c_{12} + c_{66}) & c_{22}q^2 + c_{66} - \rho\omega^2/\alpha^2 & c_{56} + c_{24}q^2 \\ q(c_{14} + c_{56}) & c_{56} + c_{24}q^2 & c_{44}q^2 + c_{55} - \rho\omega^2/\alpha^2 \end{vmatrix} = 0. \quad (8)$$

From this, we see that the roots p_i^2 will have the form

$$p_i^2 = \alpha^2 f_i(\rho\omega^2/\alpha^2), \quad (9)$$

and the ratios $U_i : V_i : W_i$ are only functions of $\rho\omega^2/\alpha^2$. To each value p_i^2 belong two complex waves:

$$u = U_i \exp[i(\alpha x + p_i y)]$$
$$\cdots\cdots\cdots$$

and

$$u = U_i \exp[i(\alpha x - p_i y)].$$
$$\cdots\cdots\cdots$$

We form real linear combinations of these complex waves:

$$u = U_i \cos \alpha x \sin p_i y,\quad v = V_i \sin \alpha x \cos p_i y,\quad w = W_i \sin \alpha x \cos p_i y; \tag{10A}$$

or

$$u = U_i \cos \alpha x \cos p_i y,\quad v = V_i \sin \alpha x \sin p_i y,\quad w = W_i \sin \alpha x \sin p_i y; \tag{10B}$$

and the final solution will be a linear combination of the three functions (10A) or (10B):

$$u = \cos \alpha x \cdot \sum L_i U_i \sin p_i y,\quad v = \sin \alpha x \cdot \sum L_i V_i \cos p_i y,\quad w = \sin \alpha x \cdot \sum L_i W_i \cos p_i y; \tag{11A}$$

or

$$u = \cos \alpha x \cdot \sum L_i U_i \cos p_i y,\quad v = \sin \alpha x \cdot \sum L_i V_i \sin p_i y,\quad w = \sin \alpha x \cdot \sum L_i W_i \sin p_i y; \tag{11B}$$

where the $U_i \cdots$ and p_i are known functions of $\rho\omega^2$ and α^2, while the L_i are to be determined by the boundary conditions. The strains are now:

$$\begin{aligned} x_x &= -\alpha \sin \alpha x \cdot \sum L_i U_i \sin p_i y,\\ y_y &= -\sin \alpha x \cdot \sum p_i L_i V_i \sin p_i y,\\ z_y &= -\sin \alpha x \cdot \sum p_i L_i W_i \sin p_i y,\\ z_x &= \alpha \cos \alpha x \cdot \sum L_i W_i \cos p_i y,\\ x_y &= \cos \alpha x \cdot \sum L_i (U_i p_i + \alpha V_i) \cos p_i y, \end{aligned} \tag{12A}$$

for case A; or

$$\begin{aligned} x_x &= -\alpha \sin \alpha x \cdot \sum L_i U_i \cos p_i y,\\ y_y &= \sin \alpha x \cdot \sum p_i L_i V_i \cos p_i y,\\ z_y &= \sin \alpha x \cdot \sum p_i L_i W_i \cos p_i y,\\ z_x &= \alpha \cos \alpha x \cdot \sum L_i W_i \sin p_i y,\\ x_y &= \cos \alpha x \cdot \sum L_i (\alpha V_i - p_i U_i) \sin p_i y, \end{aligned} \tag{12B}$$

for case B.

The boundary conditions require that the stresses

$$Y_y = Y_z = X_y = 0 \quad \text{at} \quad y = \pm b.$$

Substituting the strains, Eqs. (12), into Eqs. (1), we obtain

$$\begin{aligned} &\sum L_i (c_{12}\alpha U_i + c_{22} p_i V_i + c_{24} p_i W_i) \sin p_i b = 0,\\ &\sum L_i (c_{14}\alpha U_i + c_{24} p_i V_i + c_{44} p_i W_i) \sin p_i b = 0,\\ &\sum L_i (c_{66} p_i U_i + c_{66}\alpha V_i + c_{56}\alpha W_i) \cos p_i b = 0, \end{aligned} \tag{13A}$$

for case A; or

$$\begin{aligned} &\sum L_i (-c_{12}\alpha U_i + c_{22} p_i V_i + c_{24} p_i W_i) \cos p_i b = 0,\\ &\sum L_i (-c_{14}\alpha U_i + c_{24} p_i V_i + c_{44} p_i W_i) \cos p_i b = 0,\\ &\sum L_i (-c_{66} p_i U_i + c_{66}\alpha V_i + c_{56}\alpha W_i) \sin p_i b = 0, \end{aligned} \tag{13B}$$

for case B.

The compatibility of Eqs. (13) demands that the corresponding determinant vanish.

In view of Eq. (9), we can divide each Eq. (13) by α and we then have only functions of $\rho\omega^2/\alpha^2$ in the parentheses. The transcendental functions $\sin p_i b$ and $\cos p_i b$ will have the form

$$\sin \alpha b f_i^{\frac{1}{2}}(\rho\omega^2/\alpha^2), \quad \cos \alpha b f_i^{\frac{1}{2}}(\rho\omega^2/\alpha^2).$$

Consequently, the determinant of Eqs. (13) will yield a transcendental equation which relates $\rho\omega^2/\alpha^2$ to αb. Solution of this equation yields the frequency as a function of the wave-length $1/\alpha$ and of the product αb under the form

$$\rho\omega^2 = \alpha^2 g(\alpha b).$$

When this is done, Eqs. (8) yield the ratio $L_1 : L_2 : L_3$ and the displacement components are known from Eqs. (11).

While this formal solution is impractical for numerical results, a numerical solution can be carried out by assuming some numerical value for $\rho\omega^2/\alpha^2$ and calculating the roots, q_i, of Eq. (8) and the corresponding ratios $U_i : V_i : W_i$.

Then Eqs. (13) yield a transcendental equation for αb, the only unknown in (13). Thus the whole frequency spectrum can be determined.

It is shown in Appendix I that Eqs. (7) and (13) include as a special case Timoshenko's theory of flexural vibrations in an isotropic plate.

Thickness Vibrations

We shall discuss Eqs. (7) and (13) for the case where α is very small, i.e., the displacement is a very slowly variable function of x. Physically, this can be expected to be true in a plate whose thickness is very small as compared to its lateral dimensions. If the displacement is independent of x as can be assumed in an infinitely extended plate, $\alpha=0$ and Eq. (7) becomes

$$\begin{vmatrix} c_{66}p^2-\rho\omega^2 & 0 & 0 \\ 0 & c_{22}p^2-\rho\omega^2 & c_{24}p^2 \\ 0 & c_{24}p^2 & c_{44}p^2-\rho\omega^2 \end{vmatrix}=0. \quad (14)$$

This determinant has the characteristic values

$$p_{01}^2=\rho\omega^2/c_{66}, \quad p_{02}^2=\rho\omega^2/c_2', \quad p_{03}^2=\rho\omega^2/c_3', \quad (15)$$

where

$$c_{2,3}'=\tfrac{1}{2}(c_{22}+c_{44}\pm[(c_{22}-c_{44})^2+4c_{24}^2]^{\frac{1}{2}}). \quad (16)$$

I. Koga[2] has shown that the boundary conditions on the surface of an infinitely extended plane-parallel plate are satisfied when

$$p_{0i}=n\pi/2b\equiv\beta \quad \begin{pmatrix} i=1,2,3 \\ n=1,2,3\cdots \end{pmatrix}. \quad (17)$$

This must also be a solution of Eqs. (13A) or (13B). Indeed, if

$$p_{01}=(\rho\omega^2/c_{66})^{\frac{1}{2}}=n\pi/2b \quad (n \text{ odd}), \quad (18)$$

then we use the functions (11A). From the determinant (14), it is evident that the corresponding ratio is

$$U_1:V_1:W_1=1:0:0.$$

Because of Eq. (18)

$$\cos p_{01}b=0.$$

Therefore, the Eqs. (13A) are satisfied by

$$L_1:L_2:L_3=1:0:0.$$

If, on the other hand,

$$p_{02}=n\pi/2b \quad \text{or} \quad p_{03}=n\pi/2b \quad (n \text{ odd}),$$

then we use the functions (10B). From Eqs. (6) and (14), it can be seen that in both cases

$$U_2=U_3=0.$$

Again,

$$\cos p_{02}b=0,$$

or

$$\cos p_{03}b=0;$$

and (13B) is satisfied with

$$L_1=L_3=0,$$

or

$$L_1=L_2=0,$$

respectively. Similar considerations show that for n even, Koga's solution satisfies Eqs. (13A or B) if $\alpha=0$.

We return to the case where α is not zero, but small. Equations (6) can be written as

$$(A+\alpha/pB+\alpha^2/p^2C)\mathbf{U}=\rho\omega^2/p^2\mathbf{U}, \quad (19)$$

where

$$A=\begin{pmatrix} c_{66} & 0 & 0 \\ 0 & c_{22} & c_{24} \\ 0 & c_{24} & c_{44} \end{pmatrix},$$

$$B=\begin{pmatrix} 0 & c_{12}+c_{66} & c_{14}+c_{56} \\ c_{12}+c_{66} & 0 & 0 \\ c_{14}+c_{56} & 0 & 0 \end{pmatrix}, \quad (20)$$

$$C=\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{66} & c_{56} \\ 0 & c_{56} & c_{55} \end{pmatrix}.$$

Putting

$$1/pB+\alpha/p^2C=D, \quad (21)$$

Eq. (19) becomes

$$(A+\alpha D)\mathbf{U}=\rho\omega^2/p^2\mathbf{U}. \quad (22)$$

Disregarding for the moment the fact that D still contains α and p, we obtain from perturbation theory the second-order solution of Eq. (22):

$$\mathbf{U}_k=\mathbf{U}_{0k}+\alpha\sum_l{}'\frac{D_{kl}\mathbf{U}_{0l}}{\rho\omega^2/p_{0k}^2-\rho\omega^2/p_{0l}^2}, \quad (23)$$

$$\frac{\rho\omega^2}{p_k^2}=\frac{\rho\omega^2}{p_{0k}^2}+\alpha D_{kk}+\alpha^2\sum_l{}'\frac{D_{kl}^2}{\rho\omega^2/p_{0k}^2-\rho\omega^2/p_{0l}^2}, \quad (24)$$

where

$$\begin{aligned} D_{kl}&=(D\mathbf{U}_{0k}\cdot\mathbf{U}_{0l})=(\mathbf{U}_{0k}\cdot\{1/PB+\alpha/p^2C\}\mathbf{U}_{0l}) \\ &=1/p(\mathbf{U}_{0k}\cdot B\mathbf{U}_{0l})+\alpha/p^2(\mathbf{U}_{0k}\cdot C\mathbf{U}_{0l}) \quad (25) \\ &=1/pB_{kl}+\alpha/p^2C_{kl}. \end{aligned}$$

In Eq. (25), $\mathbf{U}_{0k}$ are the normalized eigenvectors of the secular equation (14) which satisfy the following relations:

$$\begin{aligned} c_{22}V_{02}+c_{24}W_{02}&=c_2'V_{02}, \\ c_{24}V_{02}+c_{44}W_{02}&=c_2'W_{02}; \end{aligned} \quad (26a)$$

$$\begin{aligned} c_{22}V_{03}+c_{24}W_{03}&=c_3'V_{03}=-c_3'W_{02}, \\ c_{24}V_{03}+c_{44}W_{03}&=c_3'W_{03}=c_3'V_{02}. \end{aligned} \quad (26b)$$

The third member of Eqs. (26b) follow from the orthogonality of the eigenvectors. Further,

$$\begin{aligned} \mathbf{U}_{01}&=(1,0,0), \quad \mathbf{U}_{02}=(0,V_{02},W_{02}), \\ \mathbf{U}_{03}&=(0,-W_{02},V_{02}), \end{aligned} \quad (27)$$

with

$$V_{02}=\left[1+\left(\frac{c_{22}-c_2'}{c_{24}}\right)^2\right]^{-\frac{1}{2}}=W_{03},$$

$$W_{02}=-\frac{(c_{22}-c_2')}{c_{24}}\left[1+\left(\frac{c_{22}-c_2'}{c_{24}}\right)^2\right]^{-\frac{1}{2}}=-V_{03}; \tag{28}$$

and, of course,

$$V_{02}^2+W_{02}^2=V_{03}^2+W_{03}^2=1. \tag{29}$$

The expansions (23) and (24) will fail if the matrix A is accidentally degenerate or quasi-degenerate, i.e., if two of the values

$$c_{66},\quad c_2',\quad c_3'$$

are equal or nearly equal. In this case, the degeneracy should be first removed by the conventional methods.

It can be seen that $B_{kk}=(\mathbf{U}_{0k}\cdot B\mathbf{U}_{0k})$ vanishes. Indeed, the vectors $\mathbf{U}_{0k}$ have either no y- and z-components or no x-component. Application of the matrix B to these vectors produces a new vector which has no x-component or no y- and z-components, respectively, so that the scalar product of vector and transformed vector vanishes in either case. Therefore, by Eq. (25):

$$D_{kk}=\alpha/p^2(C\mathbf{U}_{0k}\cdot\mathbf{U}_{0k})=\alpha/p^2C_{kk}. \tag{30}$$

According to Eq. (25):

$$D_{kl}^2=1/p^2(\mathbf{U}_{0k}\cdot B\mathbf{U}_{0l})^2+\cdots=1/p^2B_{kl}^2 \qquad (k\neq l), \tag{31}$$

where the dots stand for terms of higher order. Equations (23) and (24) become

$$\mathbf{U}_k=\mathbf{U}_{0k}+\alpha/p_{0k}\cdot\sum{}'\frac{B_{kl}\mathbf{U}_{0l}}{\rho\omega^2/p_{0k}^2-\rho\omega^2/p_{0l}^2}, \tag{32}$$

$$\rho\omega^2/p_k^2=\rho\omega^2/p_{0k}^2+\alpha^2/p_{0k}^2\kappa_k \tag{33}$$

where

$$\kappa_k=C_{kk}+\sum{}'\frac{B_{kl}^2}{\rho\omega^2/p_{0k}^2-\rho\omega^2/p_{0l}^2}.$$

In Eqs. (32) and (33) we have, on the right-hand side, replaced p_k^2 by p_{0k}^2 because a correction would only add terms of higher order. As shown above, if $\alpha=0$,

$$\rho\omega^2/\beta^2=\begin{cases}c_{66}\\ c_2'\\ c_3'.\end{cases}$$

If α is small, the expansion

$$\rho\omega^2/\beta^2=(1+\epsilon)\begin{cases}c_{66}\\ c_2'\\ c_3'\end{cases} \tag{34}$$

can be assumed. ϵ must be determined as a function of α. Combining Eqs. (33), (34), and (15), we get

$$p_k^2=\frac{\rho\omega^2}{c+\alpha^2\kappa_k/p_{0k}^2}=\frac{\beta^2(1+\epsilon)c}{c+\alpha^2c\kappa_k/\rho\omega^2}$$
$$\cong\frac{\beta^2(1+\epsilon)c}{c+\alpha^2\kappa_k/\beta^2(1+\epsilon)}\cong\beta^2(1+\epsilon-r^2\kappa_k/c), \tag{35}$$

where c stands for either one of the three values given in Eq. (34), and the notation

$$\alpha/\beta=r, \tag{36}$$

is used.

In the following, *we shall treat explicitly the shear vibration whose zero-order approximation is given by Eq. (18)*. If

$$\rho\omega^2/\beta^2=(1+\epsilon)c_{66}, \tag{37}$$

then by Eq. (35)

$$p_1^2=\beta^2(1+\epsilon-r^2\kappa_1/c_{66}). \tag{38}$$

κ_1 is given by Eq. (33). In appendix II it is shown that explicitly

$$\kappa_1=c_{11}+\frac{2(c_{14}+c_{56})c_{24}(c_{12}+c_{66})-(c_{14}+c_{56})^2(c_{22}-c_{66})-(c_{12}+c_{66})^2(c_{44}-c_{66})}{(c_{22}-c_{66})(c_{44}-c_{66})-c_{24}^2} \tag{39}$$

Substituting the expansions (32) and (33) with (27) and (17) into (13A), we have

$$L_1c_{66}(p_{01}+\cdots)\cos p_1b+L_2\alpha\left(c_{66}\frac{B_{12}}{c_2'-c_{66}}+c_{66}V_{02}+c_{56}W_{02}+\cdots\right)\cos(p_{02}b+\cdots)$$
$$+L_3\alpha\left(c_{66}\frac{B_{13}}{c_3'-c_{66}}+c_{66}V_{03}+c_{56}W_{03}+\cdots\right)\cos(p_{03}b+\cdots)=0,$$

$$L_1\alpha\left(c_{14}+c_{24}\sum_{2,3}\frac{B_{1l}V_{0l}}{c_{66}-c_l'}+c_{44}\sum_{2,3}\frac{B_{1l}W_{0l}}{c_{66}-c_l'}\right)\sin p_{01}b+L_2(c_{24}V_{02}+c_{44}W_{02}+\cdots)$$
$$\times(p_{02}+\cdots)\sin(p_{02}b+\cdots)+L_3(c_{24}V_{03}+c_{44}W_{03})(p_{03}+\cdots)\sin(p_{03}b+\cdots)=0, \quad (40)$$
$$L_1\alpha\left(c_{12}+c_{22}\sum_{2,3}\frac{B_{1l}V_{0l}}{c_{66}-c_l'}+c_{24}\sum_{2,3}\frac{B_{1l}W_{0l}}{c_{66}-c_l'}\right)\sin p_1b+L_2(c_{22}V_{02}+c_{24}W_{02}+\cdots)$$
$$\times(p_{02}+\cdots)\sin(p_{02}b+\cdots)+L_3(c_{22}V_{03}+c_{24}W_{03}+\cdots)(p_{03}+\cdots)\sin(p_{03}b+\cdots)=0.$$

The dots stand for terms of higher order which can be omitted.

The terms of Eqs. (40) will be simplified one by one: By Eq. (38)

$$p_1\cong\beta(1+\tfrac{1}{2}[\epsilon-r^2\kappa_1/c_{66}]), \quad (41)$$

$$\cos p_1b=\cos\frac{n\pi}{2}(1+\tfrac{1}{2}[\epsilon-r^2\kappa_1/c_{66}])\cong\mp\frac{\pi n}{4}(\epsilon-r^2\kappa_1/c_{66}). \quad (42)$$

The upper sign refers to $n=1, 5, 9\cdots$ and the lower to $n=3, 7, 11\cdots$. The factors $\sin p_1b$ which appear with the first terms of the second and third Eqs. (40) are

$$\sin p_1b\cong\sin n\pi/2=\pm1. \quad (43)$$

The two signs refer again to the two cases mentioned above. Smaller terms are omitted here as they are in addition to unity. By Eqs. (20) and (27)

$$B_{12}=(\mathbf{U}_{01}\cdot B\mathbf{U}_{02})=(c_{12}+c_{66})V_{02}+(c_{14}+c_{56})W_{02},$$
$$B_{13}=(\mathbf{U}_{01}\cdot B\mathbf{U}_{03})=-(c_{12}+c_{66})W_{02}+(c_{14}+c_{56})V_{02}; \quad (44)$$

$$c_{66}\frac{B_{12}}{c_2'-c_{66}}+c_{66}V_{02}+c_{56}W_{02}=V_{02}\left(\frac{c_{66}(c_{12}+c_{66})}{c_2'-c_{66}}+c_{66}\right)+W_{02}\left(\frac{c_{66}(c_{14}+c_{56})}{c_2'-c_{66}}+c_{56}\right)$$
$$=\frac{1}{c_2'-c_{66}}[V_{02}c_{66}(c_{12}+c_2')+W_{02}(c_{14}c_{66}+c_2'c_{56})]; \quad (45)$$

$$c_{66}\frac{B_{13}}{c_3'-c_{66}}+c_{66}V_{03}+c_{56}W_{03}=c_{66}\frac{B_{13}}{c_3'-c_{66}}-c_{66}W_{02}+c_{56}V_{02}$$
$$=\frac{1}{c_3'-c_{66}}[V_{02}(c_{66}c_{14}+c_3'c_{56})-W_{02}c_{66}(c_{12}+c_3')]. \quad (46)$$

In view of Eqs. (26a) and (26b):

$$c_{14}+c_{24}\left(\frac{B_{12}V_{02}}{c_{66}-c_2'}+\frac{B_{13}V_{03}}{c_{66}-c_3'}\right)+c_{44}\left(\frac{B_{12}W_{02}}{c_{66}-c_2'}+\frac{B_{13}W_{03}}{c_{66}-c_3'}\right)=\frac{c_2'B_{12}W_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}W_{03}}{c_{66}-c_3'}+c_{14}. \quad (47)$$

The second and third terms of the second Eq. (40) become in view of (26a) and (26b)

$$L_2c_2'p_{02}W_{02}\sin p_{02}b+L_3c_3'p_{03}V_{02}\sin p_{03}b. \quad (48)$$

Also,

$$c_{12}+c_{22}\left(\frac{B_{12}V_{02}}{c_{66}-c_2'}+\frac{B_{13}V_{03}}{c_{66}-c_3'}\right)+c_{24}\left(\frac{B_{12}W_{02}}{c_{66}-c_2'}+\frac{B_{13}W_{03}}{c_{66}-c_3'}\right)=c_{12}+\frac{c_2'B_{12}V_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}V_{03}}{c_{66}-c_3'}. \quad (49)$$

The second and third terms of the third Eq. (40) become in view of (26a) and (26b):

$$L_2c_2'p_{02}\sin p_{02}b-L_3c_3'W_{02}p_{03}\sin p_{03}b.$$

According to Eqs. (15) and (18):

$$p_{01}=\beta,\quad p_{02}=\beta(c_{66}/c_2')^{\frac{1}{2}},\quad p_{03}=\beta(c_{66}/c_3')^{\frac{1}{2}}.$$

Equations (40) read now:

$$\begin{aligned}
&\mp L_1 c_{66}\beta\frac{n\pi}{4}(\epsilon - r^2\kappa/c_{66}) + L_2\frac{\alpha}{c_2'-c_{66}}[V_{02}c_{66}(c_{12}+c_2') + W_{02}(c_{14}c_{66}+c_2'c_{56})]\cos\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}\\
&\qquad + L_3\frac{\alpha}{c_3'-c_{66}}[V_{02}(c_{14}c_{66}+c_3'c_{56}) - W_{02}c_{66}(c_{12}+c_3')]\cos\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}} = 0,\\
&\pm L_1\alpha\left(c_{14}+\frac{c_2'B_{12}W_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}W_{03}}{c_{66}-c_3'}\right) + L_2c_2'\beta(c_{66}/c_2')^{\frac{1}{2}}W_{02}\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}\\
&\qquad + L_3c_3'\beta(c_{66}/c_3')^{\frac{1}{2}}V_{02}\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}} = 0,\\
&\pm L_1\alpha\left(c_{12}+\frac{c_2'B_{12}V_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}V_{03}}{c_{66}-c_3'}\right) + L_2c_2'\beta(c_{66}/c_2')^{\frac{1}{2}}V_{02}\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}\\
&\qquad - L_3c_3'W_{02}(c_{66}/c_3')^{\frac{1}{2}}\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}} = 0.
\end{aligned} \tag{50}$$

Consequently, the determinant

$$\begin{vmatrix}
\mp c_{66}\beta\frac{n\pi}{4}(\epsilon - r^2\kappa/c_{66}) & \frac{\alpha}{c_2'-c_{66}}[V_{02}c_{66}(c_{12}+c_2') + W_{02}(c_{14}c_{66}+c_2'c_{56})]\cos\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}} & \frac{\alpha\cos\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}}{c_3'-c_{66}}[V_{02}(c_{14}c_{66}+c_3'c_{56}) - W_{02}c_{66}(c_{12}+c_3')]\\
\pm\alpha\left(c_{14}+\frac{c_2'B_{12}W_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}W_{03}}{c_{66}-c_3'}\right) & c_2'\beta(c_{66}/c_2')^{\frac{1}{2}}W_{02}\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}} & c_3'\beta(c_{66}/c_3')^{\frac{1}{2}}V_{02}\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}\\
\pm\alpha\left(c_{12}+\frac{c_2'B_{12}V_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}V_{03}}{c_{66}-c_3'}\right) & c_2'\beta(c_{66}/c_2')^{\frac{1}{2}}V_{02}\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}} & -c_3'\beta W_{02}(c_{66}/c_3')^{\frac{1}{2}}\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}
\end{vmatrix} \tag{51}$$

must vanish. The determinant (51) is expanded in terms of the elements of the first row. The first co-factor is

$$\begin{aligned}
\Delta_{11} &= c_{66}(c_2'c_3')^{\frac{1}{2}}\beta^2\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}(-W_{02}^2 - V_{02}^2)\\
&= -c_{66}(c_2'c_3')^{\frac{1}{2}}\beta^2\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}},
\end{aligned} \tag{52}$$

in view of Eq. (29). The second co-factor is:

$$\begin{aligned}
\Delta_{12} &= \pm\alpha c_3'\beta(c_{66}/c_3')^{\frac{1}{2}}\left[\left(c_{14}+\frac{c_2'B_{12}W_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}W_{03}}{c_{66}-c_3'}\right)W_{02}\right.\\
&\qquad\left. + \left(c_{12}+\frac{c_2'B_{12}V_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}V_{03}}{c_{66}-c_3'}\right)V_{02}\right]\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}\\
&= \pm\alpha c_3'\beta(c_{66}/c_3')^{\frac{1}{2}}\left[\frac{c_2'B_{12}}{c_{66}-c_2'}(W_{02}^2+V_{02}^2) + \frac{c_3'B_{13}}{c_{66}-c_3'}(W_{02}W_{03}+V_{02}V_{03})\right.\\
&\qquad\left. + c_{14}W_{02} + c_{12}V_{02}\right]\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}.
\end{aligned} \tag{53}$$

The second term in the bracket vanishes because the quantity in parenthesis represents the scalar product of two orthogonal eigenvectors. We substitute the value of B_{12} from Eqs. (44) into (53):

$$\Delta_{12}=\pm\alpha\beta(c_{66}c_3')^{\frac{1}{2}}\left[V_{02}\left(c_{12}+\frac{c_2'(c_{12}+c_{66})}{c_{66}-c_2'}\right)+W_{02}\left(c_{14}+\frac{c_2'(c_{14}+c_{56})}{c_{66}-c_2'}\right)\right]\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}$$

$$=\pm\alpha\beta\frac{(c_{66}c_3')^{\frac{1}{2}}}{c_{66}-c_2'}[V_{02}c_{66}(c_{12}+c_2')+W_{02}(c_{14}c_{66}+c_2'c_{56})]\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}. \quad (54)$$

The third co-factor is:

$$\Delta_{13}=\pm\alpha\beta c_2'(c_{66}/c_2')^{\frac{1}{2}}\left[V_{02}\left(c_{14}+\frac{c_2'B_{12}W_{02}}{c_{66}-c_2'}+\frac{c_3'B_{13}V_{02}}{c_{66}-c_3'}\right)\right.$$

$$\left.-W_{02}\left(c_{12}+\frac{c_2'B_{12}V_{02}}{c_{66}-c_2'}-\frac{c_3'B_{13}W_{02}}{c_{66}-c_3'}\right)\right]\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}$$

$$=\pm\alpha\beta c_2'(c_{66}/c_2')^{\frac{1}{2}}\left[\frac{c_3'B_{13}}{c_{66}-c_3'}(V_{02}^2+W_{02}^2)+\frac{c_2'B_{12}}{c_{66}-c_2'}(W_{02}V_{02}-W_{02}V_{02})\right.$$

$$\left.+V_{02}c_{14}-W_{02}c_{12}\right]\sin\frac{n\pi}{2}(c_{66}/c_2')$$

$$=\pm\alpha\beta c_2'\frac{(c_{66}/c_2')^{\frac{1}{2}}}{c_{66}-c_3'}[V_{02}(c_3'c_{56}+c_{14}c_{66})-W_{02}c_{66}(c_{12}+c_3')]\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}, \quad (55)$$

because $V_{02}=W_{03}$ and $W_{02}=-V_{03}$. The determinant Eqs. (51) can now be written in view of Eqs. (52), (54), and (55):

$$\pm c_{66}^2\beta^3\frac{\pi n}{4}(\epsilon-r^2\kappa_1/c_{66})(c_2'c_3')^{\frac{1}{2}}\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}$$

$$\mp\frac{\alpha^2\beta}{(c_2'-c_{66})^2}(c_{66}c_3')^{\frac{1}{2}}[V_{02}c_{66}(c_{12}+c_2')+W_{02}(c_{14}c_{66}+c_2'c_{56})]^2\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}\cos\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}$$

$$\mp\frac{\alpha^2\beta}{(c_3'-c_{66})^2}(c_{66}c_2')^{\frac{1}{2}}[V_{02}(c_{14}c_{66}+c_3'c_{56})-W_{02}c_{66}(c_{12}+c_3')]^2\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}\cos\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}=0.$$

Solving for ϵ, we obtain $\epsilon=r^2k_n$, where

$$k_n=\frac{1}{c_{66}}\left[\kappa_1+\frac{4}{n\pi(c_{66})^{\frac{1}{2}}}\left\{\frac{\cot\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}}{(c_2')^{\frac{1}{2}}(c_{66}-c_2')^2}[V_{02}c_{66}(c_{12}+c_2')+W_{02}(c_{14}c_{66}+c_2'c_{56})]^2\right.\right.$$

$$\left.\left.+\frac{\cot\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}}{(c_3')^{\frac{1}{2}}(c_{66}-c_3')^2}[V_{02}(c_{14}c_{66}+c_3'c_{56})-W_{02}c_{66}(c_{12}+c_3')]^2\right\}\right]. \quad (56)$$

If c_{14} and c_{56} are very small:

$$k_n=\frac{1}{c_{66}}\left[c_{11}-\frac{(c_{12}+c_{66})^2(c_{44}-c_{66})}{(c_{22}-c_{66})(c_{44}-c_{66})-c_{24}^2}+\frac{c_{66}^2(c_{12}+c_2')^2}{1+\left(\frac{c_{22}-c_2'}{c_{24}}\right)^2}\frac{4}{n\pi(c_{66})^{\frac{1}{2}}}\frac{\cot\frac{\pi n}{2}(c_{66}/c_2')^{\frac{1}{2}}}{(c_2')^{\frac{1}{2}}(c_{66}-c_2')^2}\right.$$

$$\left.+\left(\frac{c_{22}-c_2'}{c_{24}}\right)^2\frac{c_{66}^2(c_{12}+c_3')^2}{1+\left(\frac{c_{22}-c_2'}{c_{24}}\right)^2}\frac{4}{n\pi(c_{66})^{\frac{1}{2}}}\frac{\cot\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}}{(c_3')^{\frac{1}{2}}(c_{66}-c_3')^2}\right]. \quad (57)$$

The fourth term vanishes with c_{24}. Indeed, by Eq. (26a), $W_{02}=0$ when $c_{24}=0$. Physically, the constant c_{24} guarantees that there shall be no purely longitudinal plane waves propagated along the y-direction, and that the displacement vectors $\mathbf{U}_{02}$ and $\mathbf{U}_{03}$ form angles with the wave vector. The fourth term in Eq. (57) is due to this fact. If c_{24} vanishes, Eq. (57) becomes

$$k_n=\frac{1}{c_{66}}\left[c_{11}-\frac{(c_{12}+c_{66})^2}{c_{22}-c_{66}}+\frac{4c_{66}^{\frac{3}{2}}(c_{12}+c_{22})^2\cot\frac{n\pi}{2}(c_{66}/c_{22})^{\frac{1}{2}}}{n\pi c_{22}^{\frac{1}{2}}(c_{66}-c_{22})^2}\right], \tag{58}$$

because then $c_2'=c_{22}$ and $V_{02}=1$. In the isotropic case,

$$c_{11}=c_{22}=2\mu+\lambda,\quad c_{12}=\lambda,\quad c_{66}=\mu,$$

while all other constants vanish, and Eq. (58) becomes

$$k_n=1+\frac{16}{n\pi}\left(\frac{\mu}{2\mu+\lambda}\right)^{\frac{1}{2}}\cot\frac{n\pi}{2}\left(\frac{\mu}{2\mu+\lambda}\right)^{\frac{1}{2}}. \tag{59}$$

We now return to the general case: The ratio $L_1:L_2:L_3$ is equal to the ratio of the corresponding co-factors:

$$L_1:L_2:L_3=\Delta_{11}:\Delta_{12}:\Delta_{13}$$

$$=1:\left\{\pm\frac{r}{\sin\frac{n\pi}{2}(c_{66}/c_2')^{\frac{1}{2}}}\,\frac{V_{02}c_{66}(c_{12}+c_2')+W_{02}(c_{14}c_{66}+c_2'c_{56})}{(c_2'-c_{66})(c_2'c_{66})^{\frac{1}{2}}}\right\}$$

$$:\left\{\pm\frac{r}{\sin\frac{n\pi}{2}(c_{66}/c_3')^{\frac{1}{2}}}\,\frac{V_{02}(c_3'c_{56}+c_{14}c_{66})-W_{02}c_{66}(c_{12}+c_3')}{(c_3'-c_{66})(c_{66}c_3')^{\frac{1}{2}}}\right\}. \tag{60}$$

If we substitute these values into Eq. (11A) and take into account Eqs. (32), (33), and (35), we obtain equations of the form

$$\begin{aligned} u&=\cos\alpha x\sin\beta y[1+\tfrac{1}{2}r^2(k-\kappa_1/c_{66})],\\ v&=r\sin\alpha x[A_1\cos\beta y+A_2\cos\beta y(c_{66}/c_2')^{\frac{1}{2}}+A_3\cos\beta y(c_{66}/c_3')^{\frac{1}{2}}],\\ w&=r\sin\alpha x[A_3\cos\beta y+A_4\cos\beta y(c_{66}/c_2')^{\frac{1}{2}}+A_5\cos\beta y(c_{66}/c_3')^{\frac{1}{2}}],\end{aligned} \tag{61}$$

where the constants $A_1\cdots A_5$ are only functions of the elastic constants. We do not need the explicit values, but it is simple to write them.

In the isotropic case, L_3, W_{01}, and W_{02} vanish. Therefore, we have

$$w=0,\quad \Delta_{11}:\Delta_{12}:\Delta_{13}=1:\left\{\pm\frac{r}{\sin\frac{n\pi}{2}\left(\frac{\mu}{\lambda+2\mu}\right)^{\frac{1}{2}}}\,\frac{\mu(\lambda+\mu)}{(\lambda+\mu)[\mu(\lambda+2\mu)]^{\frac{1}{2}}}\right\}:0,$$

$$u=\cos\alpha x\sin\beta y\left[1+r^2\frac{8}{n\pi}\left(\frac{\mu}{2\mu+\lambda}\right)^{\frac{1}{2}}\cot\frac{n\pi}{2}\left(\frac{\mu}{2\mu+\lambda}\right)^{\frac{1}{2}}\right], \tag{62}$$

$$v=r\sin\alpha x\left[-\cos\beta y\pm2\left(\frac{\mu}{\lambda+2\mu}\right)^{\frac{1}{2}}\frac{\cos\beta y\left(\frac{\mu}{\lambda+2\mu}\right)^{\frac{1}{2}}}{\sin\frac{n\pi}{2}\left(\frac{\mu}{\lambda+2\mu}\right)^{\frac{1}{2}}}\right].$$

The difference between Eqs. (61) and the solution for the infinite plate is:

(1) Instead of $u=\sin\beta y$ we have a slow variation of u with x.

(2) The derivative $\partial u/\partial y$ is not exactly zero at $y=\pm b$, because p_1 differs by a small quantity (proportional to r^2) from $\beta=n\pi/2b$.

(3) While for the infinite plate v and w vanish, we find small quantities proportional to r in the case of the finite plate. v and w do not in general vanish anywhere.

Approximate Solution for the Free Plate

The Eqs. (1) and (13A or B) are rigorously valid for sinusoidal standing waves in a plane-parallel plate. Consequently, the frequencies (56) and the modes (61) are correct solutions for this case, if higher orders of r are disregarded. For a free limited plate, however, the boundary conditions in Eqs. (2) should be satisfied on the lateral surfaces. The boundary conditions in Eqs. (2) on the faces $x=\pm a$ require that

$$\begin{aligned}-X_x&=c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}+c_{14}\frac{\partial w}{\partial y}\\&=\alpha\sin\alpha a\cdot F_1(y)=0,\\-X_y&=c_{56}\frac{\partial w}{\partial x}+c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\&=\beta\cos\alpha a\cdot F_2(y)=0,\\-X_z&=c_{55}\frac{\partial w}{\partial x}+c_{56}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\&=\beta\cos\alpha a\cdot F_3(y)=0,\end{aligned}\tag{63}$$

where F_1, F_2, and F_3 are functions of y which, in general, are of order of unity. It is evident from Eqs. (63) that it is impossible to satisfy rigorously the boundary conditions by a choice of α. If the second and third Eqs. (63) are to be satisfied, $\cos\alpha a$ must vanish, but then $\sin\alpha a$ cannot vanish simultaneously, and therefore the first Eq. (63) will not be satisfied. However, if we put

$$\cos\alpha a=0;\quad \alpha=m\pi/2a\quad(m=1,3,5\cdots)\tag{64}$$

the traction X_x is proportional to $1/a$ and tends toward zero as the lateral dimensions are increased, while the other traction components vanish completely. We shall consider Eqs. (64) as the definition of α. By moving the origin of the reference system along the x-axis, we can write instead of Eqs. (61):

$$\begin{aligned}u&=\sin\alpha x\sin y\beta\cdot[1+\tfrac{1}{2}r^2(k-\kappa/c_{66})],\\v&=r\cos\alpha x\cdot[\cdots],\\w&=r\cos\alpha x\cdot[\cdots].\end{aligned}\tag{61a}$$

The boundary conditions Eqs. (2) are now:

$$\begin{aligned}-X_x&=\alpha\cos\alpha a\cdot F_1(y)=0,\\-X_y&=-\beta\sin\alpha a\cdot F_2(y)=0,\\-X_z&=-\beta\sin\alpha a\cdot F_3(y)=0.\end{aligned}\tag{63a}$$

To satisfy the second and third Eqs. (63a), we have to assume

$$\sin\alpha a=0,\quad \alpha=m\pi/2a\quad(m=2,4,6\cdots).\tag{64a}$$

Taking the even and odd modes of (64) and (64a) together, we note that m can now take on any integer value.

If the extension in the z-direction, i.e., 2c, is not infinite, the boundary conditions

$$\begin{aligned}-Z_z&=c_{13}\frac{\partial u}{\partial x}+c_{23}\frac{\partial v}{\partial y}+c_{34}\frac{\partial w}{\partial y}=0,\\-Y_z&=c_{14}\frac{\partial u}{\partial x}+c_{24}\frac{\partial v}{\partial y}+c_{44}\frac{\partial w}{\partial y}=0,\\-X_z&=c_{55}\frac{\partial w}{\partial x}+c_{56}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0.\end{aligned}\tag{65}$$

on the faces $z=\pm c$ cannot be satisfied. However, if

$$c_{13}=c_{14}=c_{23}=c_{24}=c_{34}=c_{56}=0,$$

(e.g., in the isotropic case) w vanishes and the Eqs. (65) are satisfied exactly. Fortunately, these constants are often small so that we can expect a fair agreement between theory and experiment despite Eq. (65).

We shall consider Eq. (61) and (61a) with the definitions (64) and (64a) of α as the approximate solution of the problem for the free thin plate.

Comparison with Experiment

Combining Eqs. (37), (56), (64), and (64a), we obtain:

$$\rho\omega^2=\beta^2c_{66}(1+r^2k_n)$$

or

$$\nu=\tfrac{1}{2}(c_{66}/\rho)^{\frac{1}{2}}\left[\left(\frac{n}{2b}\right)^2+k_n\left(\frac{m}{2a}\right)^2\right]^{\frac{1}{2}}.\tag{66}$$

Observations on thin quartz plates of the socalled Y_Θ type were carried out by Sykes.[5] In this case, one major edge is parallel to the twofold x-axis, so that comparison with the present theory is possible. Sykes found an empirical formula for the shear type vibrations:

$$\nu=\tfrac{1}{2}(c_{66}/\rho)^{\frac{1}{2}}\left[\left(\frac{n}{2b}\right)^2+k\left(\frac{m}{2a}\right)^2+k'\left(\frac{p-1}{2c}\right)^2\right]^{\frac{1}{2}}. \quad (67)$$

When $p=1$, this empirical formula agrees with the theoretical Eq. (66). The numerical value of the empirical constant k is not given explicitly, but it can be calculated from Sykes' diagrams. Expansion of Eq. (66) yields, when $n=1$ and $a\gg b$:

$$\nu_m=\tfrac{1}{2}(c_{66}/\rho)^{\frac{1}{2}}\frac{1}{2b}\left(1+\frac{k}{2}m^2\frac{b^2}{a^2}\right),$$

and

$$\frac{\nu_m-\nu_1}{\nu_1}=\frac{k(b/a)^2(m^2-1)}{2[1+\frac{1}{2}k(b/a)^2]}\cong\tfrac{1}{2}k(b/a)^2(m^2-1);$$

or

$$k=\frac{2}{n^2-1}\left(\frac{a}{b}\right)^2\frac{\nu_m-\nu_1}{\nu_1}.$$

From Sykes' Fig. 6.15, we find, for $a/b=32$:

$$\nu_1=1662, \quad \nu_3=1688, \quad \nu_5=1736$$

which gives $k=4.0$ and 3.8, respectively. The average value is $k=3.9$ for the AT-cut plate. From Sykes' Fig. 6.17, we find, for $a/b=48$:

$$\nu_1=2552, \quad \nu_7=2600, \quad \nu_9=2628, \quad \nu_{11}=2658$$

which gives $k=1.8$, 1.7, and 1.6, respectively. The average value is $k=1.7$ for the BT-cut plate. With the elastic constants of quartz measured by Mason[6] and the transformation formulas listed by Sykes,[5] we obtain the following constants for the plates in units of 10^{10} dyn/cm^2 (Table I). From Eqs. (56), (39), (16), and (28), we calculate the following values for $k(n=1)$:

	AT	BT
k_{obs}	3.9	1.7
k_{calc}	3.7	1.8

The agreement is satisfactory. Theoretically, k should be independent of m, but depend on n. Sykes does not state whether this is the empirical meaning of k. However, comparison of Sykes' Figs. 6.15 and 6.16 referring to the cases $n=1$ and $n=3$ seems to show that k is definitely lower in the latter case.

TABLE I.

	AT	BT		AT	BT
c_{11}	86.05	86.05	c_{12}	−10.35	26.20
c_{22}	129.86	97.77	c_{13}	25.85	−10.70
c_{44}	39.06	40.85	c_{14}	3.55	0.13
c_{66}	29.34	68.91	c_{24}	−5.82	13.11
			c_{56}	−2.46	6.45

It appears from the empirical Eq. (67) that for the case $p=1$, the frequency is not affected by the dimension c. This confirms our assumption that for the type of vibration under consideration, the boundary conditions (65) can be disregarded.

In the lowest mode ($m=1$), where

$$\alpha=\pi/2a,$$

the displacement u is maximum in the center and decreases monotonically toward the edge $x=\pm a$. The displacement components v and w are small, but they have their maximum value at the edge. In a lycopodium powder experiment, we should expect the powder to be most vigorously moved at the center and only slightly at the edge, while no exact nodes appear. This is in agreement with experimental evidence.[7] The clamping of a crystal on the corners does not modify considerably its mode, as can be expected.

In the mode corresponding to $m=2$, i.e.,

$$\alpha=\pi/a,$$

the mode is given by Eqs. (61a). The displacement component u has the form

$$u=f(y)\sin \pi x/a,$$

which vanishes in the center while it has opposite signs in the left- and right-hand half of the plate. Observations by Sykes[6] (third figure in Fig. 6.5) are in agreement with this result.

Experiments show that a freely vibrating crystal of the kind considered here, if mounted between metal plates parallel to its major faces, excites ultrasonic vibrations of the air between the surface and the plate. This cannot be explained by the theory of the infinite plate

[5] R. A. Sykes, Bell. Sys. Tech. J. **23**, 52 (1944).
[6] W. P. Mason, Bell Sys. Tech. J. **22**, 178 (1943).
[7] G. M. Thurston, U. S. Patent 1,883,111, October 18, 1932.

because a shearing motion alone cannot cause compression or dilatation. The present theory shows that the component v is not zero and thus explains the existence of air waves.

Sykes' observations show that Eq. (67) ceases to be valid at certain ratios $a:b$ and $c:b$. With these ratios, two nearby frequencies appear, one lower and one higher than that given by Eq. (67). Sykes explains this fact qualitatively by coupling of the thickness vibration with other types of modes. We shall discuss the phenomenon of coupling from the theoretical viewpoint.

It was shown in a previous paper[8] how it is possible to represent the solution of an elastic vibration problem as a linear combination of "zero-order" or uncoupled modes. We can consider the mode given by Eq. (61) as a zero-order mode. It differs from an exact solution in that it does not exactly comply with the boundary conditions on the lateral faces. Other zero-order modes can be obtained by different solutions of Eqs. (7) and (13). We have only considered those solutions where α is small. It can be shown, however, that other types of solutions corresponding to about the same frequency

$$\nu=(c_{66}/\rho)^{\frac{1}{2}}\frac{1}{4b}$$

can be obtained when α is of the order of magnitude of p. One type corresponds to Timoshenko's flexural waves treated in Appendix I. These other modes will be equally zero-order modes because they will satisfy the boundary conditions on the lateral faces only approximately. Their frequency depends mainly on the dimensions a and c.[9] Consequently, a coupling in the sense defined in a previous paper[8] will exist between the thickness mode Eq. (61) and the "lateral" modes. As the coupling is caused by the stresses X_x (on $x=\pm a$) and Z_z, Y_z, and X_z (on $z=\pm c$), which are all small, the coupling itself will be small. As shown by the form of the secular determinant in the same paper, a weak coupling has a noticeable effect only when the frequencies of two zero-order modes are nearly equal. This explains why the exceptional regions, where the uncoupled frequency becomes incorrect, occur at the intersection of the curve representing the uncoupled thickness shear mode and one representing an uncoupled "lateral" mode.

[8] H. Ekstein, Phys. Rev. 66, 108 (1944).

[9] The proof is omitted here.

APPENDIX I

It will be shown that Eqs. (7) and (13) include, as a special case, Timoshenko's theory of flexural waves in an isotropic plate.[10] In the isotropic case,

$$c_{11}=c_{22}=2\mu+\lambda,\quad c_{44}=c_{55}=c_{66}=\mu,\quad c_{12}=\lambda,\qquad (1.1)$$

and all other constants appearing in Eqs. (7) and (13) are zero. The determinant (7) reduces to

$$\begin{vmatrix}\mu p^2+(\lambda+2\mu)\alpha^2-\rho\omega^2 & \alpha p(\lambda+\mu)\\ \alpha p(\lambda+\mu) & (\lambda+2\mu)p^2+\mu\alpha^2-\rho\omega^2\end{vmatrix}=0,\qquad (1.2)$$

with $W_1=W_2=0$. The third solution:

$$\mu p^2+\mu\alpha^2-\rho\omega^2=0,$$

and

$$U_3=V_3=0,$$

can be disregarded for this discussion, as will be seen. The two solutions of (1.2) are:

$$p_1^2=\frac{\rho\omega^2}{\lambda+2\mu}-\alpha^2,\qquad (1.3)$$

and

$$p_2^2=\rho\omega^2/\mu-\alpha^2.$$

If we substitute this into the first two Eqs. (6), the ratios

$$U_1:V_1=\alpha:p_1,\quad U_2:V_2=-p_2:\alpha\qquad (1.4)$$

are found. We choose the case (A) corresponding to Eqs. (10A) and (13A). Equations (13A) reduce to

$$\begin{aligned}&\sum L_i[\lambda\alpha U_i+(\lambda+2\mu)p_iV_i]\sin p_ib=0,\\ &\sum L_i\mu p_iW_i\sin p_ib=0,\\ &\sum L_i\mu(p_iU_i+\alpha V_i)\cos p_ib=0.\end{aligned}\qquad (1.5)$$

The second Eq. (1.5) can be satisfied identically by

$$L_3=0,$$

because $W_1=W_2=0$. We consider frequencies low enough so that:

$$\rho\omega^2/\mu<\alpha^2$$

and all the more:

$$\rho\omega^2/(\lambda+2\mu)<\alpha^2.$$

[10] S. P. Timoshenko, Phil. Mag. [6] 43, 125 (1922).

According to (1.3), p_1 and p_2 then are both imaginary. If we put

$$m=\alpha\left[1-\frac{\rho\omega^2}{\alpha^2(\lambda+2\mu)}\right]^{\frac{1}{2}},$$
$$n=\alpha\left[1-\frac{\rho\omega^2}{\alpha^2\mu}\right]^{\frac{1}{2}},\qquad(1.6)$$

so that m and n are positive real numbers, the Eqs. (1.5) become, in view of (1.3) and (1.4):

$$\begin{aligned}&L_1[\lambda\alpha^2-m^2(\lambda+2\mu)]\sinh mb]\\&\qquad+iL_2 2\mu n\alpha\sinh mb=0,\\&iL_1 2m\alpha\cosh mb+L_2(n^2+\alpha^2)\cosh nb=0.\end{aligned}\qquad(1.7)$$

By eliminating L_1 and L_2 from (1.7), we obtain Timoshenko's "frequency equation":

$$4\mu\alpha^2 mn\tanh nb=(\alpha^2+n^2)[(\lambda+2\mu)m^2-\lambda\alpha^2]\tanh mb.\qquad(1.8)$$

APPENDIX II

Instead of using Eq. (33) for the explicit value of κ_1, it is easier to find it by direct expansion of the determinant (8). If

$$p/\alpha=q.\quad(1/q\ll1)$$

we divide the second and third rows and columns by q:

$$\begin{vmatrix}q^2(-\rho\omega^2/p^2+c_{66}+c_{11}/q^2) & c_{12}+c_{66} & c_{14}+c_{56}\\ c_{12}+c_{66} & -\rho\omega^2/p^2+c_{22}+c_{66}/q^2 & c_{24}+c_{56}/q^2\\ c_{14}+c_{56} & c_{24}+c_{56}/q^2 & -\rho\omega^2/p^2+c_{44}+c_{55}/q^2\end{vmatrix}=0.\qquad(2.1)$$

In view of Eqs. (33) and (15):

$$\rho\omega^2/p_1^2=c_{66}+\alpha^2\kappa_1/p_{01}^2,$$

so that we can neglect the term with α^2 everywhere except in the first element of the first column of (2.1), finally omitting the terms c_{ik}/q^2 where they are in addition to c_{ik}, we obtain:

$$\begin{vmatrix}q^2(-\rho\omega^2/p_1^2+c_{66}+c_{11}/q^2) & c_{12}+c_{66} & c_{14}+c_{56}\\ c_{12}+c_{66} & c_{22}-c_{66} & c_{24}\\ c_{14}+c_{56} & c_{24} & c_{44}-c_{66}\end{vmatrix}=0.\qquad(2.2)$$

We expand (2.2) and get:

$$\begin{aligned}q^2(-\rho\omega^2/p_1^2+c_{66}+c_{11}/q^2)[(c_{22}-c_{66})(c_{44}-c_{66})-c_{24}^2]-(c_{12}+c_{66})[(c_{12}+c_{66})(c_{44}-c_{66})-c_{24}(c_{14}+c_{56})]\\+(c_{14}+c_{56})[(c_{12}+c_{66})c_{24}-(c_{22}-c_{66})(c_{14}+c_{56})]=0.\end{aligned}\qquad(2.3)$$

Solving (2.3) for $\rho\omega^2/p_1^2$:

$$\rho\omega^2/p_1^2=c_{66}+1/q^2\left[c_{11}+\frac{(c_{14}+c_{56})\{c_{24}(c_{12}+c_{66})-(c_{22}-c_{66})(c_{14}+c_{56})\}-(c_{12}+c_{66})\{(c_{12}+c_{66})(c_{44}-c_{66})-c_{24}(c_{14}+c_{56})\}}{(c_{22}-c_{66})(c_{44}-c_{66})-c_{24}^2}\right].\qquad(2.4)$$

In the expression: $1/q^2=\alpha^2/p_1^2$ we can, in view of Eq. (15), substitute p_{01}^2 for p_1^2 without adding correction terms of the second order. Comparing (2.4) with (33), we find that:

$$\kappa_1=c_{11}+\frac{2c_{24}(c_{14}+c_{56})(c_{12}+c_{66})-(c_{14}+c_{56})^2(c_{22}-c_{66})-(c_{12}+c_{66})^2(c_{44}-c_{66})}{(c_{22}-c_{66})(c_{44}-c_{66})-c_{24}^2}.\qquad(39)$$