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A.C. Network Analyzer Study of the Schrödinger Equation

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Utilizing the type of circuit described by Kron tests were carried out on the a.c. network analyzer. The tests reported include only one-dimensional circuits. Measurements were made of eigenvalues and eigenfunctions for the particular cases of the linear oscillator, the rectangular potential well, the double rectangular barrier, the single barrier, and the rigid rotator. These tests show the circuit representation to be valid and to be adaptable to solution on the network analyzer.

ALTERNATING CURRENT NETWORK ANALYZER

THE alternating current network analyzer[1] consists of a set of adjustable inductance, resistance, and capacitance units, each connected to a pair of flexible cords and plugs. Connections between units to form any desired network are made by inserting the plugs in adjacent jacks in a jack panel. As many units as desired can be connected to a common point. Alternating current electric power is supplied by a motor-generator set to individual generator units so that several different voltages, independently adjustable in both phase and magnitude, can be inserted in different parts of the network. A centrally located set of instruments (voltmeter, ammeter, wattmeter, and varmeter) can be connected to any unit or circuit by a set of key switches.

METHOD OF TEST

The circuit used is that of Fig. 3 of the companion paper,[2] repeated here as Fig. 1 for convenience. The circuit was extended, not from minus to plus infinity, but to the limit of the circuit elements available.

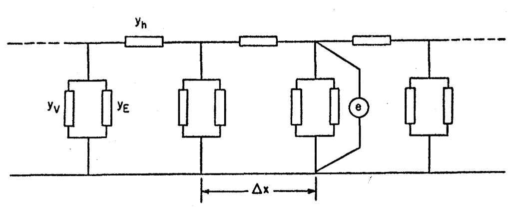

FIG. 1. Equivalent circuit used.

Discrete energy levels are characterized in the circuit usually by a standing voltage wave which attenuates to zero outside of some special region.

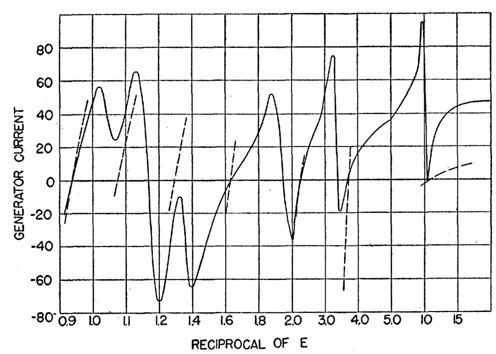

FIG. 2. Generator current as a function of energy levels of Fig. 3.

For this type of wave, the circuit should be terminated in an equivalent impedance. Since no attempt was made to do this, such levels should show an error which increases with the voltage remaining at the end of the line. On the other hand, the continuous energy levels in general exhibit a standing wave which is a space sinusoid of constant amplitude extending to infinity in at least one direction. Hence, at the end of a finite length of line, such a wave must have a loop if the line is open-circuited, a node if it is short-circuited, or other space phase for other types of termination. Since only one length of line and one type of termination (open circuit) were used in these tests, only particular levels and only one wave at each level were actually measured.

For the line elements, inductors and capacitors were used, the former being arbitrarily denoted

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[1] H. P. Kuehni and R. G. Lorraine, Trans. A.I.E.E. **57**, 67 (1938).
[2] G. Kron, Phys. Rev. **67**, 39 (1945).

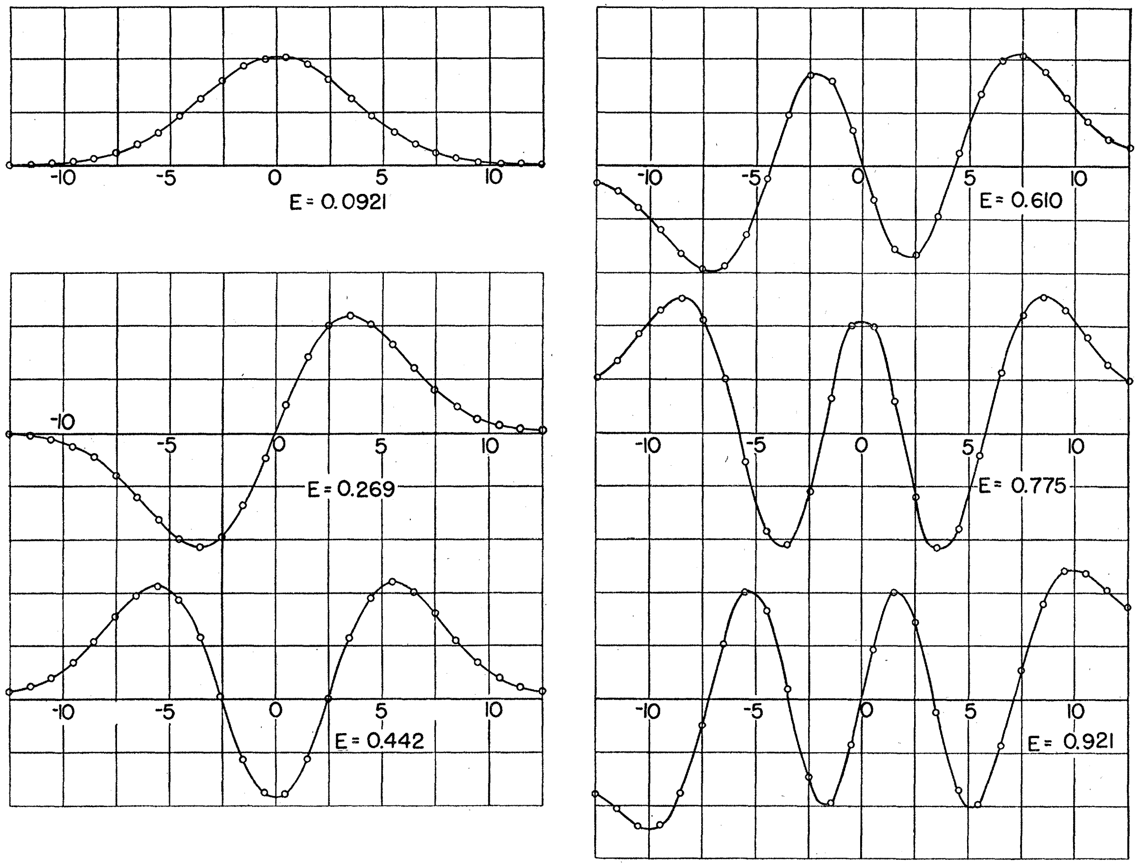

FIG. 3. Eigenfunctions of the linear harmonic oscillator.

as negative admittance units. Curves of generator reactive current *versus* energy level E (admittance y_E) were measured for several generator locations along the line; from these the eigenvalues (values of y_E for zero reactive current) could be determined and the corresponding eigenfunctions (voltage distributions) measured.

Since qualitative, rather than quantitative, results were the prime objective at this time, no special measures were taken to set admittance values closer than the nearest step on the analyzer units.

LINEAR HARMONIC OSCILLATOR

Taking Δx as unit distance,[3] we represent the linear oscillator[4] by $y_h = \hbar^2/2m = 1.10$ and by $y_V = V = 0.00704x^2$. The resulting generator reactive current as a function of energy level $y_E = E$ is shown in Fig. 2. Ideally, the current curve shown in Fig. 2 should always slope upward toward the right (increasing values of $1/E$), passing successively through zero, positive infinity, and negative infinity; the zero points then represent conditions in which the circuit should maintain its oscillations even without the external driving force. Practically, the presence of resistance in the network forces the current to pass through additional zero points in going from large plus values to large minus values; but these additional zeros are readily recognized because the curve has the negative slope in passing through them, and they can therefore be ignored. The solid line is for one generator position; the dotted lines are check curves for other generator positions. Those eigenvalues at which the solid curve tries to reach zero but does not succeed correspond to eigenfunctions such that this

[3] See Appendix for discussion of scale factors.
[4] V. Rojansky, *Introductory Quantum Mechanics* (Prentice-Hall, Inc., New York, 1942).

TABLE I.

Measured eigenvalues	0.0921; 0.269; 0.442; 0.610; 0.775; 0.921
Calculated eigenvalues	0.0880; 0.264; 0.440; 0.616; 0.792; 0.968

TABLE II.

Measured eigenvalues	0.0593; 0.216; 0.465; 0.800
Calculated eigenvalues	0.0552; 0.220; 0.486; 0.832

TABLE III.

Measured l	4.14; 5.12; 6.10; 7.13; 8.12; 9.13
Theoretical l	4 ; 5 ; 6 ; 7 ; 8 ; 9

generator position is near a node of the eigenfunction; it is not practical to supply the circuit losses from such a point.

Figure 3 shows the measured eigenfunctions up to an energy level almost as great as the value of V at the end of the line ($V=1.10$). The measured eigenvalues compare with the calculated values as in Table I. The maximum discrepancy is only about 5 percent. At the lowest value, the discrepancy is largely caused by the approximate representation of the small values of V; while at the highest value, the line termination is probably responsible. While the eigenfunctions have not been normalized for comparison, it can be seen that these too are in good qualitative agreement with the known forms for such functions.*

POTENTIAL WELL

The second case is that of a rectangular potential well. Taking again $\hbar^2/2m=1.10$ and Δx as unit distance, we represent the well by $V=0$ from -6 to $+6$ and by $V_0=1.10$ outside this region. One of the eigenfunctions and eigenvalues in the discrete energy spectrum $E<V_0$, is shown in Fig. 4. Comparison of the measured with the calculated eigenvalues is given in Table II. The lowest figure in this case was not quite at zero current but represented the maximum inductance of the board units; the highest again is subject to some correction because of improper termination. Nevertheless, the agreement is very satisfactory. Again, though not normalized, the eigenfunctions show good agreement with the calculated shapes.*

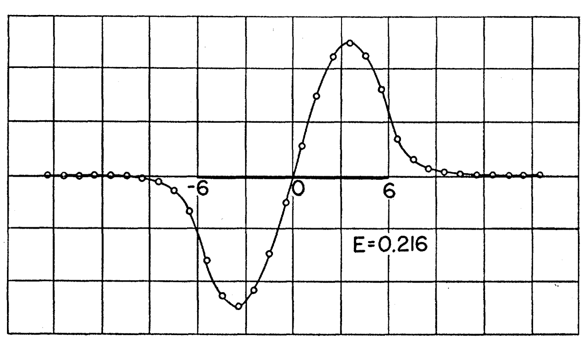

FIG. 4. An eigenfunction of the rectangular potential well in the discrete spectrum.

Figure 5 shows a measured eigenfunction in the continuous spectrum, $E>V_0$. This is, of course, only the particular function corresponding to a loop of the wave at each end of the existing line. For the same energy level, infinitely many other functions are theoretically possible; and a finite number of these, depending upon the number of circuit units per wave-length and the number of units available, could be obtained on the analyzer by simply shifting the nodes in space. Similar statements are true for any energy level $E>V_0$, of course. The figure is given here to show the performance of the circuit and the analyzer in the continuous spectrum region.

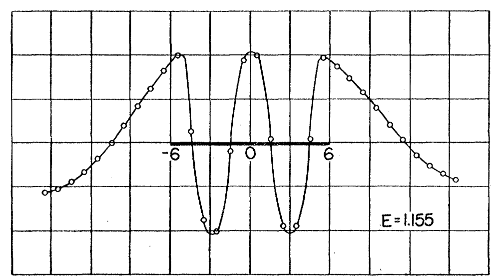

FIG. 5. An eigenfunction of the rectangular potential well in the continuous spectrum.

DOUBLE BARRIER

The double rectangular barrier was represented by $V_0=1.10$ between $+5$ and $+10$ and between -5 and -10, with $V=0$ elsewhere; Δx is again unit distance, and $\hbar^2/2m=1.10$. Figure 6 gives six measured eigenfunctions corresponding to nodes at the ends of the line. The three curves on the

* Reference 4, p. 24.

*Reference 4, p. 156.

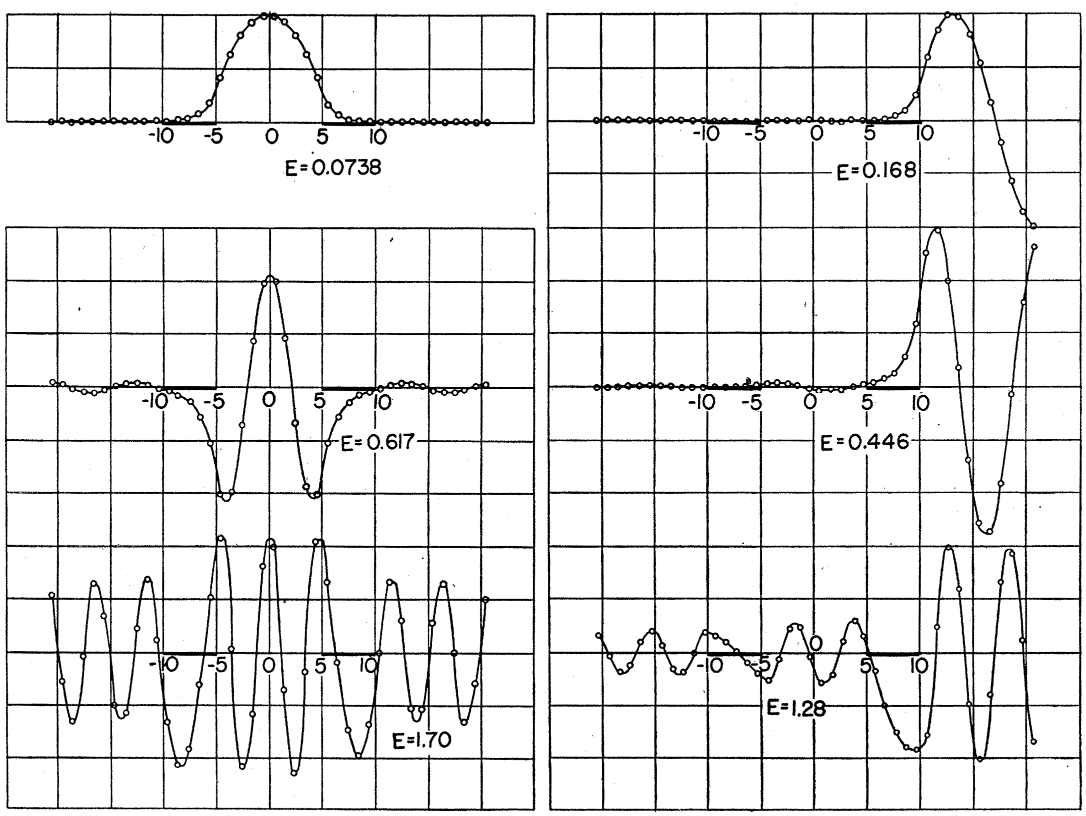

FIG. 6. Eigenfunctions for double rectangular potential barrier.

left represent virtual binding.* The three on the right apparently represent a penetration phenomenon, related to the next section. The function which is large on the outside of both barriers and small between was not obtained in this series of tests, because only one generator was used to supply resistance losses, and these are not readily transmitted through the low voltage region. The curves shown are again but individuals out of infinite families at every energy level.

RECTANGULAR BARRIER

The final test in this group was the single rectangular barrier represented by $V=0$ for $x<0$, $V_0=1.10$ between 0 and 3, and $V_1=0.40$ for $x>3$; as before, Δx is unity and $\hbar^2/2m=1.10$. Figure 7 shows some of the eigenfunctions measured. The three on the left represent penetration from the left, so to speak, at energy levels below V_1, between V_1 and V_0, and above V_0; the three curves on the right, penetration from the right at levels above V_1.

* Reference 4, p. 225.

RIGID ROTATOR

Legendre's equation, which arises in the problem of the rigid rotator,* can be written in the form

$$\frac{d}{dx}(1-x^2)\frac{d\theta}{dx}+\left[\frac{m^2}{1-x^2}-l(l+1)\right]\theta=0.$$

This is of the form of the Schrödinger equation, where $\hbar^2/2m$ is replaced by $(1-x^2)$, V by $m^2/(1-x^2)$ and E by $l(l+1)$; the range of x here is from -1 to $+1$. Tests on this equation were carried out only for $m=0$; one of the resulting measured θ functions with the corresponding value of l is shown in Fig. 8. Comparison with the theoretical values of l, in Table III, shows very good agreement in the range measured. The measured functions have not been normalized;

* Reference 4, p. 437.

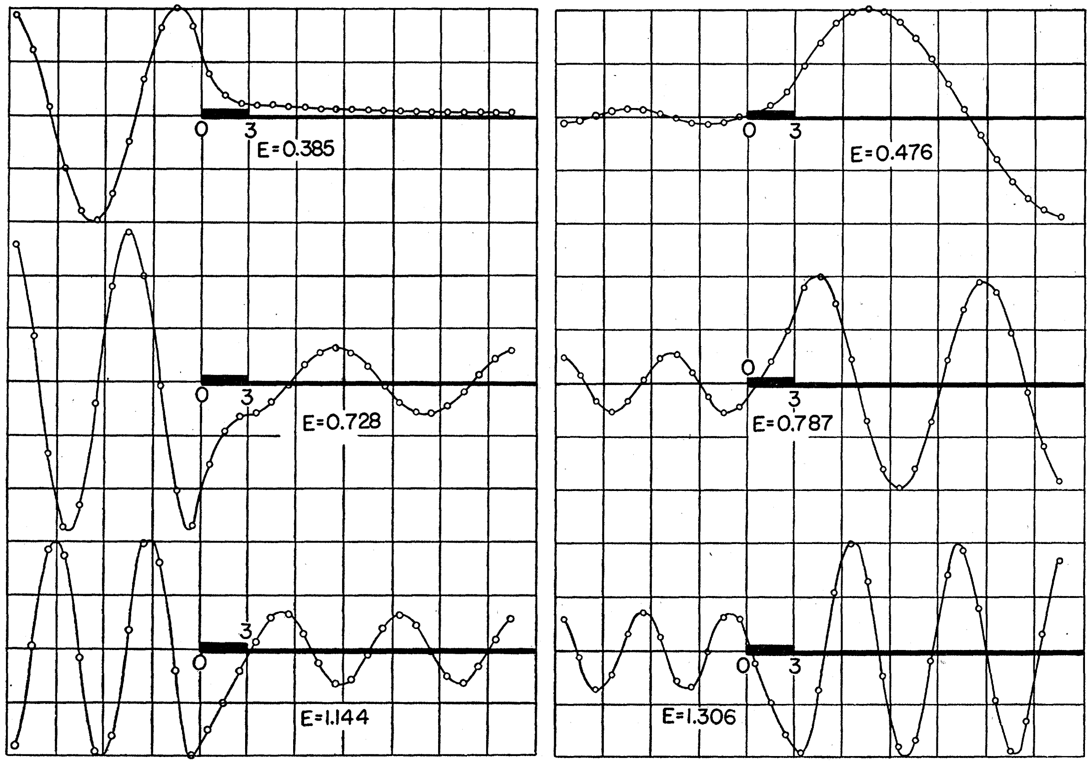

FIG. 7. Eigenfunctions for rectangular barrier between potential levels: Left of barrier $V=0$; barrier height $V_0=1.0$; right of barrier $V_1=0.40$.

but calculated end points, indicated by the dotted end extensions, line up very well with the measured curves.

SUMMARY

The tests reported here demonstrate the applicability of the electric circuit equivalent for determining eigenvalues and eigenfunctions of one-dimensional equations of the Schrödinger type. At the same time, they indicate that it is practical to use for this purpose existing a.c. network analyzers. Experience in these tests leads also to the conclusion that the equivalent circuit together with a small acquaintance with transmission line theory goes far in aiding the visualization of the characteristics to be expected of the solutions.

The significance of the equivalent-circuit method of studying characteristic values is that it is just as simple to study cases which are not readily solved analytically as it is to study the idealized cases given here.

ACKNOWLEDGMENT

The writers are indebted to C. Concordia, F. J. Maginniss, and H. A. Peterson of the Analytical Division, Central Station Engineering Divisions, General Electric Company, for helpful discussion and suggestions.

APPENDIX—SCALE FACTORS

The equation represented in most of these tests is Schrödinger's equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+[V(x)-E]\psi=0.$$

The equation represented by the circuit of Fig. 1 corresponds to

$$-\frac{d}{dx}y_h\frac{de}{dx}(\Delta x)^2+[y_V-y_E]e=0.$$

When the two equations represent the same system, they give the proportions

$$\frac{\hbar^2/2m}{y_h(\Delta x)^2}=\frac{V(x)}{y_V}=\frac{E}{y_E}=K,$$

where K is the scale factor between the energy level E and the admittance in the equivalent circuit. In Figs. 3–7, the unit of horizontal distance has been taken as Δx; the physical distance represented by this unit is determined by the equation

$$(\Delta x)^2=\frac{\hbar^2/2m}{Ky_h}=\frac{\hbar^2/2m}{V(x)}\cdot\frac{y_V}{y_h}.$$

In the case of the potential well, for example,

$$y_{V_0}=y_h,\quad (\Delta x)^2=\frac{\hbar^2/2m}{V_0}.$$

The half-width of the well is

$$a=6\Delta x$$

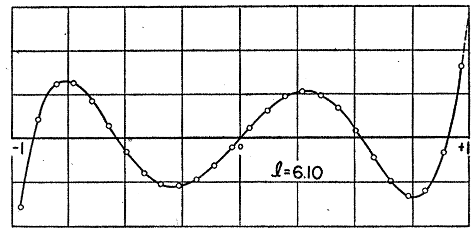

FIG. 8. An eigenfunction of the rigid rotator.

so that

$$(\Delta x)^2=\frac{a^2}{36}=\frac{\hbar^2/2m}{V_0},$$

and

$$\frac{a^2V_0}{\hbar^2/2m}=36,$$

which is the case of Exercise 3.* Other proportions for the well are obtained either by changing the number of Δx units in a or by changing the ratio of y_{V_0} to y_h.

* Reference 4, p. 155.