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On Accelerated Coordinate Systems in Classical and Relativistic Mechanics

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A simplified discussion is given of the problem of introducing linearly accelerated axes by a method applicable alike to classical and relativistic mechanics. The mathematical method is based on the Lie theory of the 4-dimensional conformal group which is presented in such a way that the transformations leading to accelerated axes are exhibited explicitly. The classical theory is shown to be a degenerate case of the relativistic theory, so that the transformations to accelerated axes have a geometrical interpretation in terms of inversive transformations in both theories.

1. INTRODUCTION

FROM time to time, modifications of the special theory of relativity to include transformations between relatively accelerated Euclidean coordinate systems have been discussed in the literature. The first steps in this direction seem to have been taken by Einstein[1] and Born.[2] More recently, the question has been investigated again by Milne[3] in connection with his theory of the expanding universe, while Page[4] has proposed a detailed theory of the kinematical side of the problem. The work of Milne and Page has been subjected to critical discussion by Bourgin,[5] and more particularly by Robertson[6] who has given a very penetrating treatment of the general mathematical theory, with particular reference to its implications in cosmological theories. Finally, Engstrom and Zorn[7] have pointed out independently the connection between Page's theory and the 4-dimensional conformal transformation group. Unfortunately, the widely divergent physical and mathematical viewpoints of the various authors make it difficult to arrive at a clear view of the physical implications of their discussions.

The present work originated in an independent attempt to develop a relativistic equivalent of the simple process in Newtonian mechanics whereby one introduces a system of coordinates moving with an accelerated particle. In this paper we shall give a direct solution of this problem by the simplest applicable means and in such a manner as to bring out the complete analogy between the classical and relativistic theories.

While the present discussion is non-quantum mechanical in character, it may be noted that the establishment of wave equations invariant under the conformal group has been discussed in the literature.[8]

2. STATEMENT OF THE PROBLEM

The starting point of our discussion will be the replacement of the ultimate problem of establishing transformations between accelerated coordinate systems by another which at first sight may perhaps appear to have only a trivial significance. However, its solution gives the key to the more general question.

Apart from the trivial case of uniform motion under no forces, the simplest situation in mechanical theory for which a clear-cut formulation can be given without involving difficulties in the discussion of force fields is that of the uniformly accelerated motion of a particle. This can be characterized in both classical and relativistic theories as a motion in which the acceleration is constant when measured in a system of coördinates in which the particle is instantaneously at rest. Such a set of coordinates is called a *rest-system;* a set of coordinates in which the particle is not only at rest but also has no acceleration will be designated as a *proper-system.*

[1] A. Einstein, Jahrbuch der Radioaktivität **4**, 411 (1907).
[2] M. Born, Ann. d. Physik **30**, 1 (1909). A brief discussion is also given by H. Bateman, Proc. Lond. Math. Soc. [2] **8**, 223 (1910).
[3] E. A. Milne, *Relativity, Gravitation and World-Structure* (Oxford University Press, New York, 1935).
[4] L. Page, Phys. Rev. **49**, 254 (1936).
[5] D. Bourgin, Phys. Rev. **50**, 864 (1936).
[6] H. P. Robertson, Phys. Rev. **49**, 755 (1936); Zeits. f. Astrophys. **7**, 153 (1933); Astrophys. J. **82**, 284 (1935), **83**, 187 (1936), **83**, 257 (1936).
[7] H. T. Engstrom and M. Zorn, Phys. Rev. **49**, 701 (1936).
[8] P. A. M. Dirac, Ann. Math. [2] **37**, 429 (1936); H. J. Bhabha, Proc. Camb. Phil. Soc. **32**, 622 (1936).

Since a particle at rest is but a special case of one in uniformly accelerated motion, the transformation from a rest-system to a proper-system has the property that it transforms one type of uniformly accelerated motion into another. This suggests the following formulation of the problem: *To determine those transformations for which the uniformly accelerated motion of a particle is transformed into another of the same type.*

In this form the problem can be attacked in the same manner in both classical and relativistic theories, and it has the further advantage of suggesting of itself the most direct method of solution. In the discussion of the coordinate transformations which arise, we shall proceed on the consideration that each such set of coordinates may be interpreted as associated with an "observer" to whom the motion of the particle as described by his coordinates is its true motion.

3. THE CLASSICAL ONE-DIMENSIONAL PROBLEM

In order to clarify the essentials of the method, we start with the simplest possible classical case. We consider a particle which in some appropriate initial coordinate system moves along a straight line with a constant acceleration g. Using x as its coordinate, the differential equation of its motion is

$$d^2x/dt^2 = g. \tag{1}$$

The differential equation characterizing *all* uniformly accelerated motions is therefore

$$d^3x/dt^3 = 0.$$

With the notation

$$v = dx/dt, \quad a = dv/dt, \quad b = da/dt,$$

this characteristic differential equation becomes

$$F_c(x, t, v, a, b) \equiv b = 0. \tag{2}$$

Our problem is now reduced to that of seeking all transformations of coordinates which leave this equation invariant. In each such new system, the particle will appear to move with constant acceleration, but the magnitude of the acceleration will vary with the system.

A study of the transformation properties of Eq. (2) shows that the only point-transformations which leave it invariant are those generated by the following set of operators:[9]

$$\begin{aligned} &X_1 \equiv -\partial_x, && X_2 \equiv -\partial_t, \\ &X_3 \equiv -t\partial_x, && X_4 \equiv -\tfrac{1}{2}t^2\partial_x, \\ &X_5 \equiv -xt\partial_x - \tfrac{1}{2}t^2\partial_t, && X_6 \equiv x\partial_x + t\partial_t, \\ &X_7 \equiv x\partial_x - t\partial_t, \end{aligned} \tag{3}$$

where, for simplification, $\partial_x \equiv \partial/\partial x$, $\partial_t \equiv \partial/\partial t$.

For convenience we give also the formulas for the general infinitesimal transformation, extended three times to cover the order of Eq. (2):

$$\begin{aligned} x' &= x - \alpha_1 - t\alpha_3 - \tfrac{1}{2}t^2\alpha_4 - xt\alpha_5 + x\alpha_6 + x\alpha_7, \\ t' &= t - \alpha_2 - \tfrac{1}{2}t^2\alpha_5 + t\alpha_6 - t\alpha_7, \\ v' &= v - \alpha_3 - t\alpha_4 - x\alpha_5 + 2v\alpha_7, \\ a' &= a - \alpha_4 - (v - ta)\alpha_5 - a\alpha_6 + 3a\alpha_7, \\ b' &= b + 2tb\alpha_5 - 2b\alpha_6 + 4b\alpha_7. \end{aligned} \tag{4}$$

Since these are extended point-transformations, the formulas for v, a, and b are consequences of those for x and t.

The invariance of Eq. (2) follows at once from the fact that the quantity $(b' - b)$ vanishes with b; i.e., vanishes when Eq. (2) is satisfied.

The requirements of classical theory now lead us to anticipate the possibility of finding a 4-parameter subgroup of transformations which will include one of the type sought to an accelerated coordinate system. The four parameters would be used, (a) to permit shifting the origin in the (x, t)-plane in order to give the particle the position $x = 0$ at a given instant $t = 0$ (2 parameters), (b) to transform to a rest-system with the particle at the origin (1 parameter), and finally, (c) to transform to the proper-system with the particle still at the origin (1 parameter). It is to be expected also that this can be achieved using only transformations for which time intervals are invariant.

A study of Eqs. (4) shows that this can just be accomplished by using the four transformations generated by X_1, X_2, X_3, X_4. It follows from the results to be obtained in the next section that these transformations form a 4-parameter group. The finite transformations of this group can be

[9] Throughout the paper the operators have been written in such a manner as to facilitate the physical interpretation of the resulting transformations.

calculated readily[10] and reduced to the form

$$x'=(x-x_0)-v_0(t-t_0)-\tfrac{1}{2}a_0(t-t_0)^2, \quad t'=t-t_0,$$

with the notation

$$x_0=\alpha_1+(3\alpha_2\alpha_3+\alpha_4\alpha_2^2)/6; \quad t_0=\alpha_2,$$
$$v_0=\alpha_3+\alpha_2\alpha_4/2; \quad a_0=\alpha_4.$$

In this form it is immediately recognizable as a change to moving, accelerated coordinates with superposed changes of origin for x and t. By a proper choice of x_0 and t_0, we bring the particle to the origin in the (x, t)-plane; by equating v_0 to the velocity of the particle, we obtain a rest-system; and by equating a_0 to the acceleration of the particle in the rest-system, we reduce it to a proper-system. For general values of a_0, we obtain a 1-parameter family of accelerated axis systems.

4. THE RELATIVISTIC ONE-DIMENSIONAL PROBLEM

In the special theory of relativity, the equation of uniformly accelerated motion, replacing Eq. (1), is

$$c^3a/(c^2-v^2)^{\frac{3}{2}}=g, \tag{5}$$

where g is the constant acceleration as measured in a rest-system. By a further differentiation we find the differential equation characteristic of all uniformly accelerated motions to be

$$F_r(x, t, v, a, b)\equiv b+3va^2/(c^2-v^2)=0. \tag{6}$$

The point-transformations which leave this equation invariant form a 6-parameter group built on the operators:[11]

$$\begin{aligned}
X_1&\equiv-\partial_x, \quad X_2\equiv-\partial_t,\\
X_3&\equiv-t\cdot\partial_x-(x/c^2)\cdot\partial_t,\\
X_4&\equiv-(c^2t^2+x^2)/2c^2\cdot\partial_x-(xt/c^2)\cdot\partial_t,\\
X_5&\equiv-xt\cdot\partial_x-(c^2t^2+x^2)/2c^2\cdot\partial_t,\\
X_6&\equiv x\partial_x+t\partial_t.
\end{aligned} \tag{7}$$

[10] The necessary theory of continuous groups can be found in the book of J. E. Campbell, *Theory of Continuous Groups* (Oxford University Press, New York, 1903) or that of G. Kowalewski, *Einführung in die Theorie der Kontinuierlichen Gruppen* (Akademische Verlagsgesellschaft, Leipzig, 1931). The method of finding the finite transformations of a group which is defined in terms of its infinitesimal elements is discussed in Campbell, p. 47.

[11] The full set of symmetry transformations of both Eqs. (2) and (6) form a 10-parameter family, of which the remaining ones are contact transformations, but not point-transformations.

TABLE I. Commutator table of the symmetry operators of relativistic one-dimensional uniformly accelerated motion.

X_1	0					
X_2	0	0				
X_3	X_2/c^2	X_1	0			
X_4	$-X_6/c^2$	X_3	X_5/c^2	0		
X_5	X_3	$-X_6$	X_4	0	0	
X_6	$-X_1$	$-X_2$	0	X_4	X_5	0
	X_1	X_2	X_3	X_4	X_5	X_6

For convenience of reference we tabulate the general infinitesimal transformations to the third orders of derivatives:

$$\begin{aligned}
x'&=x-\alpha_1-\alpha_3t-\alpha_4(c^2t^2+x^2)/2c^2-\alpha_5xt+\alpha_6x,\\
t'&=t-\alpha_2-\alpha_3x/c^2-\alpha_4xt/c^2\\
&\qquad-\alpha_5(c^2t^2+x^2)/2c^2+\alpha_6t,\\
v'&=v-\alpha_3(c^2-v^2)/c^2-\alpha_4t(c^2-v^2)/c^2\\
&\qquad-\alpha_5x(c^2-v^2)/c^2,\\
a'&=a+\alpha_33va/c^2+\alpha_4(xa+3tva+v^2-c^2)/c^2\\
&\qquad+\alpha_5[c^2ta+3xva-c^2v+v^3]/c^2-\alpha_6a,\\
b'&=b+\alpha_3(4vb+3a^2)/c^2\\
&\qquad+\alpha_4(2xb+4tvb+6va+3ta^2)/c^2\\
&\qquad+\alpha_5(2c^2tb+4xvb+6v^2a+3xa^2)/c^2-\alpha_62b.
\end{aligned} \tag{8}$$

It is readily verified that this transformation leaves Eq. (6) invariant.

Table I is the commutator table for the set of operators of Eq. (7).

Each entry in Table I gives the commutator of the operator at the side of its row with that at the bottom of its column. The operators X_1, X_2, X_3 generate the 3-parameter subgroup of translation and velocity transformations, while X_4, X_5, X_6 generate a second 3-parameter subgroup. Neither of these subgroups is normal within the whole group.

In order to obtain the operators of Section 3, we need only take the limit $c\rightarrow\infty$ in the operators of Eq. (7), in the general infinitesimal transformations of Eq. (8), and in Table I.[12]

To develop the "acceleration transformations" we now suppose that the starting coordinate system has been chosen as a rest-system in which the particle is brought to the origin in the (x, t)-plane. Equations (8) show that the subgroup of transformations generated by X_4, X_5, X_6 has the characteristic that the particle is left at the origin at rest.

The 1-parameter subgroup of transformations generated by X_4 alone gives a family in which the particle is left at the origin at rest, but for which its acceleration becomes $(g-\alpha_4)$. By equating α_4 to g, the proper-system of the particle is found. The finite transformations of this family are easily computed and found to give exactly the result found by Page[4] in his Eqs. (27) and (28), with $\alpha_4=\phi$.[13]

5. GEOMETRICAL INTERPRETATION

Each of the transformations used can be given a geometrical interpretation as an operation in the (x, t)-plane: (a) X_1 and X_2 generate translations of the origin, (b) X_3 has the usual interpretation given in the special theory of relativity as a real rotation in the complex (x, ict)-plane or as an imaginary rotation in the real plane, (c) in the real plane we define the transformation

$$T(x_0, t_0):\quad x'-x_0=\frac{R^2(x-x_0)}{(x-x_0)^2-c^2(t-t_0)^2};$$

$$t'-t_0=\frac{R^2(t-t_0)}{(x-x_0)^2-c^2(t-t_0)^2},$$

as an inversion in the hyperbola

$$(x-x_0)^2-c^2(t-t_0)^2=R^2.$$

If we let $S(x_0, t_0)$ be a translation of the origin through the distances x_0 and t_0 along the axes, then the infinitesimal transformations α_4X_4 and α_5X_5 are given by the following sequences

$$\alpha_4X_4:\quad S(-R^2\alpha_4/2c^2, 0)\cdot T(-R^2\alpha_4/2c^2, 0)\cdot T(0, 0),$$

$$\alpha_5X_5:\quad S(0, R^2\alpha_5/2c^2)\cdot T(0, R^2\alpha_5/2c^2)\cdot T(0, 0),$$

the order of the transformations being read from right to left. In the complex plane, these become combinations of inversions in circles combined with translations,[14] and (d) the transformations generated by X_6 are uniform stretchings and contractions of the (x, t)-plane.

From this we see that, even in the classical theory, the transformations to accelerated coordinate systems have a geometrical interpretation in terms of inversions. This result carries over unchanged into the 3-dimensional theory.

6. THREE-DIMENSIONAL MOTION

The characteristic differential equation for 3-dimensional uniformly accelerated motion is so complex that a direct study of its invariance would be very laborious. It is not difficult to set up the generalization of the operator system of Section 4 by direct evaluation, but it can be put on a more elegant basis by recognizing at once the connection with Lie's theory of the group of conformal transformations in 4 dimensions. The physical basis of the connection is that since a particle traveling with the velocity of light represents a special case of uniform acceleration in which further acceleration is impossible, the family of paths of light rays must therefore be invariant under the transformations. Since the differential equation of a light surface in the starting system is

$$(dx)^2+(dy)^2+(dz)^2-c^2(dt)^2=0,$$

this relation must also be satisfied after the transformation, which implies that

$$(dr')^2-c^2(dt')^2=\rho[(dr)^2-c^2(dt)^2],$$

[12] The operator X_T of Section 3 appears in the relativistic theory as a degenerate case of a contact transformation which becomes a point-transformation on taking the limit $c\rightarrow\infty$. The group used here is isomorphic to the symmetry group of the circles in the complex (x, ict)-plane. The passage from the relativistic to the classical theory appears in Lie's geometrical theory as the transition from the symmetry group of circles in the complex plane to that of the family of parabolas in the real plane forming the classical trajectories of possible uniformly accelerated particles. This geometrical theory is discussed in the books of Campbell and of Kowalewski mentioned in reference 10.

[13] The finite transformations of the 2-parameter subgroup based on X_4 and X_5 have been computed. The equations show three singular lines, of which two have the equations

$$(x\pm ct)(\alpha_4\pm\alpha_5c)+2c^2=0$$

and are the generalizations of those noted by Page. The third is the line

$$\alpha_5x-\alpha_4t=0$$

which transforms into itself according to the relation

$$(\alpha_5x'-\alpha_4t')\cdot(x^2-c^2t^2)=(\alpha_5x-\alpha_4t)\cdot(x'^2-c^2t'^2).$$

[14] A discussion of this 2-dimensional geometrical theory can be found in Chap. 19 of Campbell, reference 10.

TABLE II. Commutator table of the conformal group.

X_1	0														
X_2	0	0													
X_3	0	0	0												
X_4	0	0	0	0											
X_5	0	X_3	$-X_2$	0	0										
X_6	$-X_3$	0	X_1	0	$-X_7$	0									
X_7	X_2	$-X_1$	0	0	X_6	$-X_5$	0								
X_8	X_4/c^2	0	0	X_1	0	X_{10}	$-X_9$	0							
X_9	0	X_4/c^2	0	X_2	$-X_{10}$	0	X_8	X_7/c^2	0						
X_{10}	0	0	X_4/c^2	X_3	X_9	$-X_8$	0	$-X_6/c^2$	X_5/c^2	0					
X_{11}	$-X_{15}/c^2$	X_7/c^2	$-X_6/c^2$	X_8	0	X_{13}	$-X_{12}$	X_{14}/c^2	0	0	0				
X_{12}	$-X_7/c^2$	$-X_{15}/c^2$	X_5/c^2	X_9	$-X_{13}$	0	X_{11}	0	X_{14}/c^2	0	0	0			
X_{13}	X_6/c^2	$-X_5/c^2$	$-X_{15}/c^2$	X_{10}	X_{12}	$-X_{11}$	0	0	0	X_{14}/c^2	0	0	0		
X_{14}	X_8	X_9	X_{10}	$-X_{15}$	0	0	0	X_{11}	X_{12}	X_{13}	0	0	0	0	
X_{15}	$-X_1$	$-X_2$	$-X_3$	$-X_4$	0	0	0	0	0	0	X_{11}	X_{12}	X_{13}	X_{14}	0
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}

where ρ is an unspecified function of x, y, z, t. This relation may be used to characterize the group of transformations.[7,15]

We proceed at once to give the set of 15 operators which form the basis of the group of transformations:

$$X_1 \equiv -\partial_x, \quad X_2 \equiv -\partial_y, \quad X_3 \equiv -\partial_z, \quad X_4 \equiv -\partial_t,$$

$$X_5 \equiv -y\partial_z + z\partial_y, \quad X_6 \equiv -z\partial_x + x\partial_z,$$

$$X_7 \equiv -x\partial_y + y\partial_x, \quad X_8 = -t\partial_x - (x/c^2)\partial_t,$$

$$X_9 \equiv -t\partial_y - (y/c^2)\partial_t, \quad X_{10} \equiv -t\partial_z - (z/c^2)\partial_t,$$

$$X_{11} \equiv -(c^2t^2 + x^2 - y^2 - z^2)/2c^2 \cdot \partial_x - (xy/c^2)\partial_y - (xz/c^2)\partial_z - (xt/c^2)\partial_t,$$

$$X_{12} \equiv -(xy/c^2)\partial_x - (c^2t^2 - x^2 + y^2 - z^2)/2c^2 \cdot \partial_y - (yz/c^2)\partial_z - (yt/c^2)\partial_t,$$

$$X_{13} \equiv -(xz/c^2)\partial_x - (yz/c^2)\partial_y - (c^2t^2 - x^2 - y^2 + z^2)/2c^2 \cdot \partial_z - (zt/c^2)\partial_t,$$

$$X_{14} \equiv -xt\partial_x - yt\partial_y - zt\partial_z - (c^2t^2 + x^2 + y^2 + z^2)/2c^2 \cdot \partial_t,$$

$$X_{15} \equiv x\partial_x + y\partial_y + z\partial_z + t\partial_t.$$

The commutator relations of these operators are given in Table II.

X_1, X_2, X_3, X_4 generate the translations. X_5, X_6, X_7, generate the 3-dimensional rotations, while X_8, X_9, X_{10} generate the Lorentz transformations. X_{11}, X_{12}, X_{13} form the 3-dimensional acceleration sub-set, of which X_{14} gives the time-component. X_{15} again gives the homogeneous dilatations of the whole space.

The procedure for studying the transformation to accelerated axes proceeds by the same principles as in Section 4. By extending the transformations as far as the acceleration terms, one finds that if the particle is brought to the origin of coordinates in a rest-system, then it will be

[15] S. Lie, *Theorie der Transformationsgruppen* (Teubner, Leipzig, 1930). A very interesting, but difficult, discussion is given in a classic paper by H. Bateman, Proc. Lond. Math. Soc. [2] **8**, 223 (1910).

left in this condition under all transformations of the acceleration subgroup; i.e., under the 3-parameter family of transformations generated by X_{11}, X_{12}, X_{13}.[16]

On orienting the x-axis along the direction of the acceleration of the particle, the 1-parameter family of transformations which include its proper-system can be found. The transformation equations are:

$$x' = [x + \alpha_{11}(r^2 - c^2t^2)/2c^2]\cdot(t'/t),$$

$$y' = y\cdot(t'/t), \quad z' = z\cdot(t'/t),$$

$$t' = t/\{1 + (\alpha_{11}/c^2)[x + (\alpha_{11}/4c^2)(r^2 - c^2t^2)]\},$$

$$r^2 = x^2 + y^2 + z^2.$$

The passage to the classical theory is readily made, as in Section 4, merely by taking the limit $c \to \infty$ in all of the equations for the operators as well as in the commutator table. The general infinitesimal transformation may then be written in 3-dimensional vector form as

$$\delta\mathbf{r} \equiv \mathbf{r}' - \mathbf{r} = -\delta\mathbf{s} - \delta\boldsymbol{\omega}\times\mathbf{r} - t\delta\mathbf{v} - \tfrac{1}{2}t^2\delta\mathbf{a} - \mathbf{r}t\alpha_{14} + \mathbf{r}\alpha_{15},$$

$$\delta t \equiv t' - t = -\alpha_4 - \tfrac{1}{2}t^2\alpha_{14} + t\alpha_{15},$$

with the abbreviations

$$\delta\mathbf{s} = (\alpha_1\mathbf{i} + \alpha_2\mathbf{j} + \alpha_3\mathbf{k}),$$

$$\delta\boldsymbol{\omega} = (\alpha_5\mathbf{i} + \alpha_6\mathbf{j} + \alpha_7\mathbf{k}),$$

$$\delta\mathbf{v} = (\alpha_8\mathbf{i} + \alpha_9\mathbf{j} + \alpha_{10}\mathbf{k}),$$

$$\delta\mathbf{a} = (\alpha_{11}\mathbf{i} + \alpha_{12}\mathbf{j} + \alpha_{13}\mathbf{k}).$$

The operators X_{14} and X_{15} can be dropped, and the transformations to accelerated axes based on the remaining 13-parameter group. Time intervals are then invariant.

[16] The particle at the origin in a rest system is left at that position at rest under the 5-parameter group generated by X_{11}, X_{12}, X_{13}, X_{14}, X_{15}.

7. ELECTRODYNAMIC CONSIDERATIONS

The detailed proof of the invariance of the electromagnetic field equations and the derivation of the transformation relations of the field vectors and potentials have been given by Cunningham[17] and Bateman.[15] However, this only assures us that electromagnetic phenomena as observed by any of the family of observers related by transformations of the conformal group will appear to obey the field equations. Since the transformations are in general non-linear, plane waves do not transform into plane waves, and the existence of singular regions in the finite transformations makes it unclear to what extent retarded and advanced potentials may be intermixed.

We shall note here in conclusion that if one assumes that in its proper-system a point charge exhibits its usual Coulomb field, then Bateman's formulas can be used to find its field in other sets of axes. A first-order calculation of the field due to a charge e at the origin of coordinates and moving with infinitesimal velocity $\delta\mathbf{v}$ and acceleration $\delta\mathbf{a}$ yields the result

$$\mathbf{E} = e\mathbf{r}/r^3 - e(\mathbf{r}\cdot\delta\mathbf{a})\mathbf{r}/2r^3c^2 - e\delta\mathbf{a}/2rc^2,$$

$$\mathbf{H} = -e(\mathbf{r}\times\delta\mathbf{v})/r^3c.$$

which is in agreement with the usual formula to this order of approximation.

It is of interest to note also that if the potentials are computed, they contain contributions in α_{14} and α_{15}, even to first orders, but on passing to the field vectors these terms disappear, so that they introduce a type of gauge transformation.

[17] E. Cunningham, Proc. Lond. Math. Soc. [2] **8**, 77 (1910).