A Theorem of Larmor and Its Importance for Electrons in Magnetic Fields

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The importance of a well-known theorem, originally due to Larmor, is emphasized. It enables a definition of "momentum" and "moment of momentum" for electrons in a magnetic field, hence the possibility of writing the conservation of these quantities when the geometry of the structure is convenient. As typical examples of the method, two special cases are discussed: a plane electron beam and a cylindrical electron beam with longitudinal magnetic field. In both cases it is found that the space-charge density of the beam is entirely controlled by the magnetic field and that the maximum current is obtained for a suitable optimum magnetic field.

I. A GENERAL THEOREM ABOUT ELECTRONIC MOTIONS

THERE is a very important result about the mechanics of electrons, so important it may be granted the name of "theorem" and should be called after Larmor. The famous English physicist always attempted to reduce problems of electrons to the standard pattern of classical mechanics, with its famous canonical equations: principle of least action, Lagrange and Hamilton formulas. The difficulty was the "Lorentz force" of a magnetic field on a moving electron, a formula which does not seem to offer any similarity with any mechanical problem. Larmor first discovered the similarity with problems of rotating bodies and stressed the significance of the angular velocity

$$\omega_H = -\mu_0 \frac{e}{2m} H, \tag{1}$$

where μ_0 is the permeability, e is the charge of the electron, m is the mass, and H is the magnetic field. He also felt that this was not a special case, but only one instance of a more general law.[1] This took shape progressively, through contributions of H. A. Lorentz, and Schwarzschild, and is now found in every textbook on electromagnetism or quantum-theory. This theorem is also essential in the derivation of Dirac's equation for the spinning electron.

Let us sum up the results, and refer to textbooks for more details:[2] we shall use m.k.s. unit and the standard notations of J. A. Stratton.[3]

Let $\phi A_1 A_2 A_3$ be the components of the scalar and vector potentials in a system of rectangular coordinates $x_1x_2x_3$. The E and H components of the electric and magnetic field are (Stratton, p. 24)

$$E_k = -\partial\phi/\partial x_k - \partial A_k/\partial t, \quad k = 1\cdot 2\cdot 3 \tag{2}$$
$$\mu_0 H = \text{rot } A, \quad \mu_0 H_3 = \partial A_2/\partial x_1 - \partial A_1/\partial x_2.$$

We want to study the motion of an electron according to the well-known formula

$$m\ddot{x}_k = eE_k + e\mu_0[v \times H], \tag{3}$$

where the second term is the Lorentz force.

These laws of motion can be reduced to the standard Lagrange scheme if one uses as *Lagrange function*

$$L(\dot{x}_1\dot{x}_2\dot{x}_3, x_1x_2x_3, t) = \tfrac{1}{2}\,\text{m}v^2 - e\phi + e(v\cdot A), \tag{4}$$
$$(v\cdot A) = \dot{x}_1 A_1 + \dot{x}_2 A_2 + \dot{x}_3 A_3.$$

The proof is straight forward. First, we define the momentum of the electron by the standard Lagrange relation:

$$p_k = \partial L/\partial \dot{x}_k = m\dot{x}_k + eA_k, \tag{5}$$

and we emphasize the importance of this definition with its additional term containing the vector potential. Next, we write Lagrange's

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[1] J. Larmor, *Aether and Matter* (Cambridge University Press, Cambridge, England, 1900), Ch. VI.

[2] J. H. Van Vleck, *Theory of Electric and Magnetic Susceptibilities* (Clarendon Press, England, 1932), p. 19; L. Brillouin, *Atome de Bohr* (Presses Universitaires, Paris, 1931), pp. 98–100 and 107–112.

[3] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941).

equation of motion:

$$\frac{dp_k}{dt}=\dot{p}_k=\frac{\partial L}{\partial x_k}=-e\frac{\partial\phi}{\partial x_k}+e\left(\dot{x}_1\frac{\partial A_1}{\partial x_k}+\dot{x}_2\frac{\partial A_2}{\partial x_k}+\dot{x}_3\frac{\partial A_3}{\partial x_k}\right). \quad (6)$$

The important point is that d/dt means a derivative taken along the trajectory of the electron

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\dot{x}_1\frac{\partial}{\partial x_1}+\dot{x}_2\frac{\partial}{\partial x_2}+\dot{x}_3\frac{\partial}{\partial x_3}. \quad (7)$$

Hence, the translation of Eq. (6)

$$m\ddot{x}_k+e\frac{\partial A_k}{\partial t}+e\left(\dot{x}_1\frac{\partial A_k}{\partial x_1}+\dot{x}_2\frac{\partial A_k}{\partial x_2}+\dot{x}_3\frac{\partial A_k}{\partial x_3}\right)=-e\frac{\partial\phi}{\partial x_k}+e\left(\dot{x}_1\frac{\partial A_1}{\partial x_k}+\dot{x}_2\frac{\partial A_2}{\partial x_k}+\dot{x}_3\frac{\partial A_3}{\partial x_k}\right). \quad (8)$$

Taking $k=1$, we note that the first terms in each bracket cancel each other, and we are left with

$$m\ddot{x}_1=-e\frac{\partial A_1}{\partial t_1}-e\frac{\partial\phi}{\partial x_1}+e\dot{x}_2\left[\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2}\right]+e\dot{x}_3\left[\frac{\partial A_3}{\partial x_1}-\frac{\partial A_1}{\partial x_3}\right],$$

or, according to Eq. (2)

$$m\ddot{x}_1=eE_1+\mu_0e\dot{x}_2H_3-\mu_0e\dot{x}_3H_2, \quad (9)$$

which is exactly Eq. (3).

The Lagrange function appears in the principle of least action

$$\int\delta L dt=0 \quad (10)$$

as usual and can be used to build up the Hamilton function

$$\mathcal{H}(p_1p_2p_3x_1x_2x_3t)=\Sigma_kp_k\dot{x}_k-L, \quad (11)$$

or, according to Eqs. (4) and (5),

$$\mathcal{H}=\Sigma_k(m\dot{x}_k^2+e\dot{x}_kA_k)-\tfrac{1}{2}m\Sigma_k\dot{x}_k^2+e\phi-e\Sigma_k\dot{x}_kA_k, \quad (12)$$
$$\mathcal{H}=\tfrac{1}{2}\,\mathrm{m}v^2+e\Phi.$$

The terms in A cancel out, and in case of conservative systems (electric potential Φ independent of time), the Hamilton function represents the total energy as usual. The fact that it does not contain the vector potential any more corresponds to the result that Lorentz forces do not do any work. If Hamilton's function $\mathcal{H}$ is to be used in Hamilton's equations, it must be expressed as a function of the momenta p.

$$\mathcal{H}=\frac{1}{2m}\Sigma_k(p_k-eA_k)^2+e\varphi. \quad (13)$$

Hence the equation of motion

$$\dot{x}_k=\frac{\partial\mathcal{H}}{\partial p_k}=\frac{1}{m}(p_k-eA_k), \quad (14a)$$

$$\dot{p}_k=-\partial\mathcal{H}/\partial x_k. \quad (14b)$$

Equation (14a) is just the reverse of Eq. (5), and Eq. (14b) is identical with Eq. (6).

The same general scheme can be extended to relativistic mechanics for very fast electrons (L. B., *Atome de Bohr*, p. 107). It represents the safest way to attack problems of electron trajectories.

II. SOME EXAMPLES OF VECTOR POTENTIALS

In order to show how to use these general relations, we shall build the expressions for the vector potential corresponding to some examples of special importance.

A. Plane Problem

Assuming all fields independent of the coordinate x_1, we can use the following expression

$$A_1=-F(x_3)x_2,\quad A_2=0,\quad A_3=0, \quad (15)$$

which results in

$$\mu_0H_1=0,\quad \mu_0H_2=-\frac{dF}{dx_3}\cdot x_2,\quad \mu_0H_3=F(x_3). \quad (16)$$

The H_3 component of the magnetic field is constant in the x_1x_2 planes, but may depend upon x_3. These definitions satisfy the fundamental relation:

$$\operatorname{div} A+\mu_0\epsilon_0(\partial\phi/\partial t)=0 \text{ Lorentz}, \quad (17)$$

$\operatorname{div} H=0$ (see Stratton, Eq. (12), p. 24).

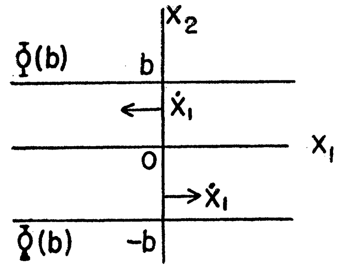

FIG. 1. Electrons are emitted from a filament along X_1. X_3 and H_3 are perpendicular to the X_1X_2 plane.

The Lorentz relation is obviously fulfilled when the scalar potential ϕ does not contain the time.

B. Cylindrical Symmetry

We want a field H depending only upon the distance r to the x_3 axis. Furthermore, the magnetic field lies along the axis (as is the case for the field of a coil centered on the x_3 axis, for instance).

Here we choose

$$A_1 = -\tfrac{1}{2}F(x_3)x_2, \quad A_2 = \tfrac{1}{2}F(x_3)x_1, \quad A_3 = 0, \quad (18)$$

hence

$$\mu_0 H_1 = -\tfrac{1}{2}\frac{dF}{dx_3}x_1, \quad \mu_0 H_2 = -\tfrac{1}{2}\frac{dF}{dx_3}x_2, \quad \mu_0 H_3 = F(x_3). \quad (19)$$

If the H_3 field is not constant, there is a radial component of the field,

$$\mu_0 H_r = -\tfrac{1}{2}\frac{dF}{dx_3}r. \quad (20)$$

The relations (17) are again automatically fulfilled.

III. CONSERVATION PRINCIPLES IN ELECTRON MECHANICS

The general theorem discussed in Section I leads directly to some important conservation principles:

Conservation of Energy

$$\tfrac{1}{2}\,\mathrm{mv}^2 + e\varphi = \mathrm{Const}\ (=0), \quad (21)$$

when Φ does not depend upon time. The sum of kinetic and potential energy remains constant. In case of electrons emitted without velocity by a cathode at potential zero, the constant is zero. This first result is well known.

Conservation of Momentum

In addition to this, we may have conservation of some components of momentum or of the moment of momentum. This will depend upon the symmetry of the structure. These additional conditions are very important to keep in mind and were too often overlooked by many authors, who proposed solutions which obviously did not satisfy these relations.

A. Plane Problem

Electrons are emitted by a filament located along the x_1 axis ($x_2 = x_3 = 0$) and the magnetic field on this cathode is H_{3_0} (Fig. 1). Electrons move between two plane electrodes, located at

$$x_2 = \pm \bar{b}.$$

In Fig. 1, the cathode is supposed to be behind the plane of the drawing, and we are looking at a cross section of the electron beam between the $\pm b$ electrode at potential $\varphi(\bar{b})$. These electrodes are supposed to extend both ways to infinity.

In such a structure, there is no electric force acting on the electron in the x_1 direction, hence

$$p_1 = \mathrm{const.}, \quad (22)$$

a condition which means conservation of the No. 1 component of the momentum, as, defined in Eq. (5). Now, on the cathode

$$p_{1_0} = m\dot{x}_{1_0} + eA_{1_0} = 0 - e\mu_0 H_{30} x_2 = 0, \quad (23)$$

since the initial velocity is zero, and the vector potential is given by Eq. (15). After the electrons travelled a distance x_3, we must still find $p_1 = 0$, but it now means

$$p_1 = m\dot{x}_1 - e\mu_0 H_3 x_2 = 0,$$

hence

$$\dot{x}_1 = (e/m)\mu_0 H_3(x_3) x_2 = -2\omega_H x_2, \quad (24)$$

where ω_{H} is defined by Eqs. (1) as Larmor's angular velocity corresponding to $H_3(x_3)$.

We obtained the relation Eq. (24) from our general theorem, but it can also be proved directly. The equation of motion in the x_1

direction according to Eq. (9) and Eq. (16) is

$$m\ddot{x}_1 = e\mu_0(\dot{x}_2H_3 - \dot{x}_3H_2) = e\mu_0\left[\dot{x}_2H_3 + x_2\frac{dH_3}{dx_3}\dot{x}_3\right], \quad (25)$$

but

$$\dot{x}_3 dH_3/dx_3 = dH_3/dt,$$

where the d/dt derivative is taken along the electron's trajectory. Hence

$$\ddot{x}_1 = (e/m)\mu_0(d/dt)(x_2H_3), \quad (26)$$

which results in Eq. (24).

The same results are obtained for a cathode of any arbitrary shape located in a region where the magnetic field H_{30} vanishes, since these conditions make again p_{10} zero in Eq. (23).

B. Cylindrical Problem

Electrons are emitted by a point cathode at $x_1 = x_2 = x_3 = 0$ where the magnetic field is H_{30}. If the magnetic field H_{30} is constant (zero for instance) in the region of the cathode, the cathode may be a filament of finite length extending along the x_3 axis. These electrons move afterwards inside a cylindrical anode of radius b, centered on the x_3 axis (Fig. 2).

Here the symmetry calls for the conservation of the moment of momentum about the x_3 axis, and the constant value of the moment of momentum is zero, according to the conditions of emission by the cathode:

$$p_2x_1 - p_1x_2 = 0, \quad (27)$$

hence

$$m(\dot{x}_2x_1 - \dot{x}_1x_2) + e(A_2x_1 - A_1x_2) = 0,$$

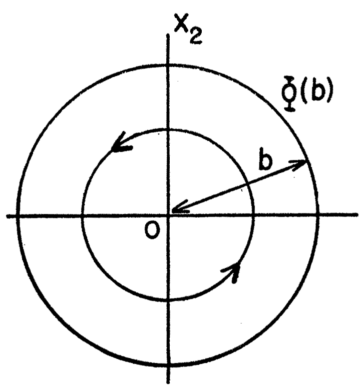

FIG. 2. Arrangement of a cathode along the X_3 axis. X_3 and H_3 are perpendicular to the X_1X_2 plane.

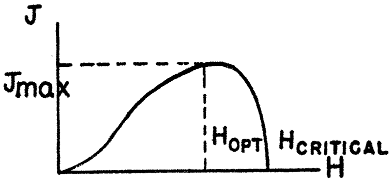

FIG. 3. Current *versus* magnetic field.

where the vector potential is given by Eq. (18):

$$\dot{x}_2x_1 - \dot{x}_1x_2 = -\tfrac{1}{2}(e/m)\mu_0H_3(x_3)(x_1^2 + x_2^2), \quad (28)$$

or in cylindrical coordinates r, θ, x_3

$$r^2\dot{\theta} = \omega_H r^2, \quad \dot{\theta} = \omega_H. \quad (29)$$

We can generalize this result if we assume the *cathode* to be a circular ring of *radius a* (or a circular cylinder of radius a extending about x_3 axis in a region of constant magnetic field H_{30}).

In this case, the constant moment of momentum is not zero but

$$p_2x_1 - p_1x_2 = m(\dot{x}_{20}x_{10} - \dot{x}_{10}x_{20}) + e(A_{20}x_{10} - A_{10}x_{20}) = 0 + \frac{e}{2m}\mu_0H_{30}(x_{10}^2 + x_{20}^2)$$

$$= \frac{e}{2m}\mu_0H_{30}a^2 = -\omega_{H0}a^2, \quad (30)$$

since our electrons are emitted on a radius a without velocity. The relation (29) is now replaced by

$$\dot{\theta} = \omega_H - \omega_{H0}(a^2/r^2). \quad (31)$$

In a *cylindrical magnetron*, the cathode is in the same constant magnetic field as the anode, and we obtain

$$\dot{\theta} = \omega_H[1 - (a^2/r^2)], \quad (32)$$

a relation which can be proved directly[4] as in the preceding case. These last formulas (31) and (32) yield infinite angular evlocity on the axis, which means that conditions near the axis should be discussed carefully if the structure does not prevent electrons from reaching that region.

IV. PLANE ELECTRON BEAM IN STEADY MOTION

Let us consider the problem A of the preceding section and look for the conditions corresponding

[4] L. Brillouin, Phys. Rev. **60**, 385 (1941), Eq. (1), (2), (3), (15).

to a steady motion of the beam along the x_3 axis. We assume that no current flows to the plane electrodes $x_2=\pm b$ and we use Eq. (24). The $\dot{x}_3$ component may still depend upon x_2:

$$\dot{x}_1=-2\omega_H x_2, \quad \dot{x}_2=0, \quad \dot{x}_3=f(x_2), \tag{33}$$

hence, by Eq. (21)

$$-\frac{2e}{m}\phi=\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2=4\omega_H^2x_2^2+f^2. \tag{34}$$

In order to keep $\dot{x}_2$ zero, we must insure a compensation between electric and Lorentz forces in the x_2 direction.

$$0=-e\frac{\partial\phi}{\partial x_2}+e\mu_0[v\times H]_2=4m\omega_H^2x_2 +mf\frac{df}{dx_2}-e\mu_0\dot{x}_1H_3=mf\frac{df}{dx_2}.$$

The field H is given by Eq. (16), ω_H was defined in Eq. (1), and $\dot{x}_1$ results from Eq. (33). The term compensation proves that

$$\dot{x}_3=f(x_2)=v_3, \text{ a } \textit{constant}. \tag{35}$$

Hence the *whole beam is moving with a constant* velocity v_3 along the x_3 axis, and the potential distribution (34) is simply

$$\phi=-\frac{m}{e}(2\omega_H^2x_2^2+\tfrac{1}{2}v_3^2). \tag{36}$$

This yields a *constant space charge density*

$$\partial^2\phi/\partial x_2^2=-\rho/\epsilon_0, \quad \rho=4\epsilon_0\omega_H^2(m/e), \tag{37}$$

and the current I per unit length in the x_1 direction is

$$I=2\bar{b}\rho\dot{x}_3=8b(m/e)\epsilon_0\omega_H^2\dot{x}_3. \tag{38}$$

What is physically measured is this current I and the voltage $\phi(b)$ of the plates. Let us rewrite it this way:

$$\begin{aligned} U&=-(2e/m)\phi(b)=\alpha\omega_H^2+\dot{x}_3^2, \ \alpha=4\bar{b}^2, \\ J&=(2e/m\epsilon_0)I=(1/\beta)\omega_H^2\dot{x}_3, \quad \beta=1/16\bar{b}. \end{aligned} \tag{39}$$

We introduce the α, β coefficients in order to enable us to use the following discussion in other examples, (cylindrical structures) where similar relations will be found.

The problem to be discussed is about the maximum possible current in the beam when the voltage ϕ is given. Eliminating $\dot{x}_3$, we have

$$U=\alpha\omega_H^2+(\beta^2/\omega_H^4)J^2, \quad \beta^2J^2=U\omega_H^4-\alpha\omega_H^6, \tag{40}$$

where ω_H measures the magnetic field intensity and $\alpha\beta$ are geometrical factors. If we vary the magnetic field, we obtain

$$\text{Zero current } J=0 \text{ for } \begin{cases}\omega_H=0, & \rho=0 \\ \omega_H=\omega_c, & \dot{x}_3=0,\end{cases} \tag{41}$$

where the *critical magnetic field* H_c corresponds to

$$U=\alpha\omega_c^2. \tag{42}$$

The maximum current is found for a certain optimum magnetic field (see Fig. 3).

$$\partial J/\partial\omega_H=0, \quad H_{\text{opt}}=(\tfrac{2}{3})^{\frac{1}{2}}H_{\text{critical}}, \quad \omega_{\text{opt}}=(\tfrac{2}{3})^{\frac{1}{2}}\omega_c \tag{43}$$

$$\alpha\omega_{\text{opt}}^2=\tfrac{2}{3}U, \quad J_{\max}=\frac{1}{\alpha\beta\sqrt{2}}(\tfrac{2}{3}U)^{\frac{3}{2}},$$

or

$$I_{\max}=-\frac{\epsilon_0}{\alpha\beta}\left(\frac{-e}{m}\right)^{\frac{1}{2}}(\tfrac{2}{3}\phi)^{\frac{3}{2}}.$$

The point is that this maximum current can be obtained only for a certain definite value of the magnetic field. It is proportional to the power 3/2 of the voltage, as in Langmuir's formula.

V. STEADY CYLINDRICAL BEAMS

We will find similar results for the cylindrical structure discussed in section III-B, where we found [Eq. (21), (29)]

$$U=-\frac{2e}{m}\varphi=\dot{r}^2+r^2\dot{\theta}^2+v_3^2=r^2\omega_H^2+v_3^2, \tag{44}$$

since Eq. (29) gave $\dot{\theta}=\omega_H$ and $\dot{r}$ must be zero in a steady beam with no radial motion. The longitudinal velocity v_3 might depend upon r and be different on successive cylindrical layers. A reasoning, very similar to the one used in the plane problem Eq. (33) and (35), will prove that v_3 must be a constant.

The radial electric field E_r and the Lorentz force must compensate the centrifugal force in order to give no radial acceleration and to maintain $\dot{r}$ zero. This means

$$eE_r+\mu_0eHr\dot{\theta}+mr\dot{\theta}^2=0, \tag{45}$$

but

$$\dot{\theta}=\omega_H=-\tfrac{1}{2}\mu_0(e/m)H,$$

and

$$E_r=-\partial\phi/\partial r=(m/2e)(\partial U/\partial r)$$
$$=m/e[r\omega_H{}^2+v_3(\partial v_3/\partial r)]. \quad (46)$$

Term compensation proves that $\partial v_3/\partial r$ must be zero which makes v_3 a constant.

We may now use Poisson's relation

$$(1/r)(\partial/\partial r)(rE_r)=\rho/\epsilon_0=2(m/e)\omega_H{}^2 \quad (47)$$

and we obtain a constant space-charge density ρ equal to one-half the space-charge density in the plane problem. Let us call $\phi(b)$ the voltage of the anode;

$$U=-(2e/m)\varphi(b)=\alpha\omega_H{}^2+v_3{}^2, \quad \alpha=b^2. \quad (48)$$

The total current I is

$$I=\pi b^2\rho v_3=2\pi\epsilon_0 b^2(m/e)\omega_H{}^2 v_3,$$
$$J=(2e/m\epsilon_0)I=(1/\beta)\omega_H{}^2 v_3, \quad \beta=1/4\pi\bar{b}^2. \quad (49)$$

Hence, the problem is reduced to the same scheme as in the preceding section [Eq. (39)] only with different values for the geometrical coefficients α, β. The results are again given by Eq. (43) and Fig. 3.

This applies for a beam filling completely the pipe of radius b.

We may also discuss the case of a beam of radius b surrounded by a larger pipe of radius $R>b$.

The equations inside the beam are the same as before, and outside the beam a logarithmic potential is obtained. This term must insure potential and field continuity on the beam's surface $r=\bar{b}$; hence

$$U=-(2e/m)\varphi=\omega_H{}^2\bar{b}^2[1+2\log(r/\bar{b})]+v_3{}^2. \quad (50)$$

Equation (49) for the current J is unchanged, and the coefficient α in Eq. (48) becomes

$$\alpha=\bar{b}^2[1+2\log(R/b)]. \quad (51)$$

This is the only modification in the result.

VI. ELECTRON BEAM WITH INITIAL ROTATION

The electron beam may enter the magnetic field with an initial rotation, or it may be generated inside the magnetic field from a cylindrical cathode of radius a, as assumed in Eq. (31) of Section III, B. In such cases, the angular velocity will be given by

$$\dot{\theta}=\omega_H[1+(A/r^2)], \quad (52)$$

Eq. (31) corresponding to the case

$$A=-a^2(\omega_{H_0}/\omega_H),$$

but A may eventually take positive values. Rewriting Eq. (44) to Eq. (46) of preceding section, we obtain

$$U=r^2\dot{\theta}^2+v_3{}^2=\omega_H{}^2\left(r^2+2A+\frac{A^2}{r^2}\right)+v_3{}^2,$$
$$E_r=\frac{m}{2e}\frac{\partial U}{\partial r}=\frac{m}{e}\omega_H{}^2\left(r-\frac{A^2}{r^3}\right)+\frac{m}{e}v_3\frac{\partial v_3}{\partial r}, \quad (53)$$

but E_r must satisfy a condition similar to Eq. (45)

$$E_r=2\omega_H\frac{m}{e}r\dot{\theta}-\frac{m}{e}r\dot{\theta}^2=\frac{m}{e}\omega_H{}^2 r\left(1-\frac{A^2}{r^4}\right), \quad (54)$$

which again results in the condition that v_3 must be a constant and yields

$$\rho=2\epsilon_0(m/e)\omega_H{}^2[1+(A^2/r^4)], \quad (55)$$

instead of Eq. (47).

$$U=-(2e/m)\phi=\omega_H{}^2[r+(A/r)]^2+v_3{}^2. \quad (56)$$

The formulas become infinite at $r=0$, hence the experimental device should prevent electrons from reaching the axis. This means that we shall deal with a hollow cylindrical beam extending from $r=a$ to $r=b$, instead of a solid cylindrical beam $(0\leqslant r\leqslant b)$, as in the preceding sections. This beam can be obtained experimentally in a coaxial cable with electrodes of radii $R_1\leqslant a$ and $R_2\geqslant b$. Eventually we may do without the central electrode if the potential be constant and the electric field zero inside the beam $(r<a)$.

In order to simplify the formulas, we shall assume that the radii of the electrodes are

$$R_1=a, \quad R_2=b. \quad (57)$$

Otherwise, we would obtain the same field and space-charge distributions inside the beam $(a\leqslant r\leqslant b)$ to be completed by convenient logarithmic potentials in $D_1\log r+D_2$ in the charge free regions $R_1 \leqslant r \leqslant a$ or $b \leqslant r \leqslant R_2$. The solution could be worked out for such problems if needed.

Using formula (56) we obtain the potentials on radius a and radius b

$$U(a) = \omega_H^2[a + (A/a)]^2 + v_3^2, \quad U(b) = \omega_H^2[b + (A/b)]^2 + v_3^2, \tag{58}$$

and the current

$$I = v_3 \int_a^b \rho 2\pi r dr = 4\pi \frac{m\epsilon_0}{e} \omega_H^2 v_3 \int_a^b \left(1 + \frac{A^2}{r^4}\right) r dr$$
$$= (2\pi m\epsilon_0/e)\omega_H^2 v_3 | r^2 - (A^2/r^2) |_a^b, \tag{59}$$

hence

$$J = (2e/m\epsilon_0)I$$
$$= 4\pi\omega_H^2 v_3[b^2 - (A^2/b^2) - a^2 + (A^2/a^2)]. \tag{60}$$

The field on the inner cylinder is

$$(m/2e)(\partial U/\partial r)_{r=a} = (m/e)\omega_H^2[a - (A^2/a^3)]. \tag{61}$$

Let us first assume a hollow electron beam with no central electrode. This is obtained by taking a zero field on a radius a. Hence,

$$A = \pm a^2, \tag{62}$$

and consequently,

$$U(b) = \omega_H^2[b \pm (a^2/b)]^2 + v_3^2, \quad J = 4\pi\omega_H^2 v_3[b - (a^4/b^2)]. \tag{63}$$

The relations are reduced to the standard type (39), (40) if we take

$$\alpha = [b \pm (a^2/b)]^2, \quad \beta = [b^2 - (a^4/b^2)]^{-1}(1/4\pi), \tag{64}$$

which solves the problem. It should be noted here that the $\pm$ sign introduces a new feature in the solution, namely, the respective orientation of the initial rotation in the beam and of the Larmor rotation. The case of the magnetron corresponds to the minus sign.

To discuss the problem of an electron beam with initial rotation ($A \neq 0$) filling the free space in a coaxial cable, we assume that *both electrodes* are at the *same potential.*

$$U(a) = U(b), \quad A = \pm ab. \tag{65}$$

Hence,

$$U = \omega_H^2(a \pm b)^2 + v_3^2,$$

and

$$J = 4\pi\omega_H^2 v_3 2(b^2 - a^2), \tag{66}$$

which reduces to the standard type with

$$\alpha = (a \pm b)^2, \quad \beta = 1/8\pi(b^2 - a^2). \tag{67}$$

The corresponding space-charge ρ and angular velocity $\dot{\theta}$ are given by Eq. (52) and Eq. (55). In the case of the minus sign

$$A = -ab, \quad \dot{\theta} = \omega_H[1 - (ab/r^2)], \tag{68}$$

one must notice that the angular velocity is positive on the outside of the beam and negative on the inside with a non-rotating layer at $r = (ab)^{\frac{1}{2}}$.

The general conclusion is that, in such problems, Larmor's theorem introduces an additional condition of conservation for either the momentum or the moment of momentum, which has been too often overlooked. If taken into account Larmor's theorem shows that the space-charge density is entirely conditioned and controlled by the magnetic field being proportional to H^2 [Eqs. (37) and (45)]. One should, therefore, be very cautious not to introduce into a discussion separate assumptions about space-charge and magnetic field. It is specially advised never to speak of the behavior of a device under an infinite or arbitrary large magnetic field since this would also mean infinite space-charge density which is a very troublesome factor.