Theoretical Calculations on Extensive Atmospheric Cosmic-Ray Showers

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Large cosmic-ray showers in air, investigated with ionization chambers and coincidence counters, have been explained hitherto as originating from primary electrons of very high energy. In recent experiments at an altitude of 3100 meters Lewis measured the frequency of coincident bursts in two unshielded ionization chambers. The theoretical cumulative frequency $H(P, D)$ of such coincidences is calculated as a function of the electron density P and the separation D of the chambers for this altitude and for sea level. The cumulative size-frequency distribution $H(P)$ for bursts in a single chamber is also computed. All calculations are based on the cascade theory of showers and the theory of the multiple scattering of electrons. The theoretical frequencies compared with those of experiments show (1) a much smaller absolute value, (2) a much slower drop as the chambers are separated, (3) a different form for the cumulative size-frequency distributions, and (4) a smaller increase with altitude. It is pointed out that the large number of narrow showers of high energy observed must originate much nearer to the chamber than the top of the atmosphere if they are to be explained by the cascade theory. It is concluded that the assumption of primary electrons is of little help in explaining the experimental results.

INTRODUCTION

LARGE cosmic-ray showers in air have been explained hitherto as originating from primary electrons of very high energy. The quantitative development of this theory has seemed to be in general agreement with experiments in the two cases it has been carried out. Euler[1] has computed the frequency of sea level bursts in a single unshielded ionization chamber as a function of the size of the burst, and his results are in rough agreement with the experiments of Carmichael[2] and the more recent ones of Lapp.[3] Hilberry[4] has indicated that the results of his experiments on counter coincidences at various altitudes are in agreement with the theory at least for the higher altitudes (2000–4000 meters). None of these experiments, however, gives a complete picture of the shower since experiments with a single chamber do not reveal the extent of the shower and those with counters do not reveal the density in the shower. In recent experiments at Echo Lake, Colorado, Lewis[5] used two unshielded ionization chambers and measured the frequency of coincident bursts as a function of the size of the bursts and the separation of the chambers. These data of Lewis' provide a more critical test for the theory.

To compute theoretically the results of these experiments, a form of the primary electron spectrum at the top of the atmosphere is assumed and the cascade theory of the multiplication in a shower is applied together with the theory of the multiple scattering of electrons. The frequency $H(P)$ of bursts of electron density greater than P in a single unshielded chamber and the frequency $H(P, D)$ of coincident bursts of electron density greater than P in each of two chambers separated by a distance D are calculated for showers detected at Echo Lake (altitude above sea level: 3100 m.; pressure: 52 cm Hg; depth from the top of the atmosphere: 709 g/cm^2 or 16.5 radiation units.) Electron densities from 500/m^2 to 2000/m^2 and chamber separations up to 10 meters are considered. The calculated results, are compared with the experimental results of Lewis.[6] The functions $H(P)$ and $H(P, D)$ are also calculated for showers detected at sea level, and the altitude effect is

[1] H. Euler, Zeits. f. Physik **116**, 73 (1940).
[2] H. Carmichael and D. Chang-Ning Chou, Nature **144**, 325 (1939).
[3] R. Lapp, Phys. Rev. **64**, 129 (1943).
[4] N. Hilberry, Phys. Rev. **60**, 1 (1941).
[5] L. G. Lewis and E. W. Lewis, Phys. Rev. **65**, 63A (1944). The complete work of Lewis is published elsewhere in this issue. The writer wishes to thank Mr. Lewis for giving him the experimental data cited in this paper in advance of publication.

[6] In comparing theory and experiment, it is assumed that the observed bursts are solely due to electrons.

discussed. The function $H(P, D)$ has not as yet been determined experimentally for sea level.

CALCULATIONS FOR SHOWERS DETECTED AT ECHO LAKE

1. Basic Assumptions

The theoretical computation follows the general method used by Euler,[1] but different forms of the functions representing the spatial distribution and energy distribution of the shower electrons are employed. In addition, the zenith angle effect neglected by Euler is considered. Since the calculations are made for large electron densities and small separations of the chambers, the results depend chiefly on the high density region of the showers and on shower electrons above the critical energy in air (about 10^8 ev). This means that the approximations necessary in treating low energy ($<10^8$ ev) shower electrons do not have a great effect on the results.

From the cascade theory the average number of electrons of energy between E and $E+dE$ at a depth of t radiation units from the top of the atmosphere in a shower initiated by a primary electron of energy E_0 can be given by

$$\pi(E_0, E, t)dE = k(t)K(E_0/E, t)\mu(E)E_0/E^2 dE. \quad (1)$$

For $t=16.5$, the value[7] of $k(t)$ is 0.034. The function $K(E_0/E, t)$ is equal to unity for that depth t at which $\pi(E_0, E, t)$ is a maximum, but is considerably less for significant departures from the maximum. For $t=16.5$, the function $K(E_0/E)$ was calculated for $E>10^8$ ev and is plotted against $\log_{10}(E_0/E)$ in Fig. 1. For $E<10^8$ ev the cascade theory has not been worked out sufficiently to give $K(E_0/E)$; as an approximation, therefore, $K(E_0/E)$ has been replaced by a factor $K'(E_0)$ depending only on the primary energy E_0. This factor has been determined so that

$$\int_0^{10^8 \text{ ev}} \pi(E_0, E, t)dE$$

is equal to the total number of electrons with energy $E<10^8$ ev in a shower due to a primary electron of energy E_0 as given by the cascade theory. The factor $K'(E_0)$ actually turns out to be very nearly the same as the value of $K(E_0/E)$ in Fig. 1 corresponding to $E=10^8$ ev. The function $\mu(E)$ accounts for the ionization loss of the shower electrons and approaches unity for $E \gg 10^8$ ev. Rossi and Greisen[7] give the form of μ for $E>10^8$ ev, while Richards[8] gives μ down to $E=4\times10^6$ ev.[9]

In treating the spatial distribution of the shower electrons it is assumed that the only significant cause of the lateral spread of the shower is the multiple scattering of electrons in the shower. It is also assumed that the distribution of shower electrons of one particular energy E along an axis perpendicular to the axis of the shower is given by a Gaussian function.[10] Then

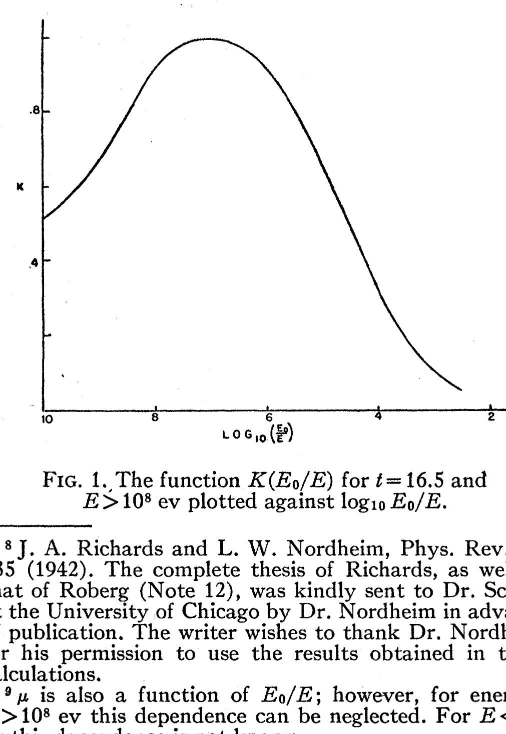

FIG. 1. The function $K(E_0/E)$ for $t=16.5$ and $E>10^8$ ev plotted against $\log_{10} E_0/E$.

[7] Calculations were made from the cascade theory as presented by Rossi and Greisen, Rev. Mod. Phys. **13**, 240 (1941).

[8] J. A. Richards and L. W. Nordheim, Phys. Rev. **61**, 735 (1942). The complete thesis of Richards, as well as that of Roberg (Note 12), was kindly sent to Dr. Schein at the University of Chicago by Dr. Nordheim in advance of publication. The writer wishes to thank Dr. Nordheim for his permission to use the results obtained in these calculations.

[9] μ is also a function of E_0/E; however, for energies $E>10^8$ ev this dependence can be neglected. For $E<10^8$ ev this dependence is not known.

[10] In a simple problem of the multiple scattering of a beam of particles (for example, mesotrons) of a single energy it is easily shown that the resultant spatial distribution is given by a Gaussian function. In the case of the shower electrons, however, the multiple scattering equation must be combined with the diffusion equations. The combined equations [given by Landau, J. Phys. USSR **2**, 237 (1940)] have never been solved. It has been assumed previously (e.g., by Euler) and is also assumed here that the spatial distribution for shower electrons of one particular energy E is not affected critically by the diffusion part of the combined equations and may be given to a good approximation by a Gaussian function.

the probability that an electron of energy E is at a distance between R' and $R'+dR'$ from the axis of the shower in a plane perpendicular to the axis can be given by

$$\varphi(R', E)2\pi R'dR' = (1/2\pi)\, E^2/(0.8\alpha E_s)^2 \times\exp\left(-\frac{R'^2E^2}{2(0.8\alpha E_s)^2}\right)2\pi R'dR', \quad (2)$$

where the distance R' is given in radiation units. The value of the exponent in the Gaussian function is chosen so that the root mean square radial spread for electrons of energy E is equal to $0.8\alpha(E_s/E)\sqrt{2}$ radiation units.[11] The value of $0.8\alpha(E_s/E)\sqrt{2}$ is given by Roberg,[12] and is strictly valid only for energies E corresponding to the maximum of $\pi(E_0, E, t)$ as a function of t (for which $K(E_0/E)=1$). Roberg finds that α is a function of the electron energy E, approaching unity for $E\gg 10^8$ ev. E_s represents a characteristic energy for the scattering of electrons, which has been used in previous calculations of multiple scattering and has a value of 1.5×10^7 ev, independent of material. The dependence of the scattering on E_0/E, which is only important when $\pi(E_0, E, t)$ is far from its maximum, is not known and therefore must be neglected in the calculations. In the present case, however, in which $t=16.5$ and densities between 500/m^2 and 2000/m^2 are being considered, the values of $\pi(E_0, E, t)$ used are chiefly those near the maximum. The function $\varphi(R', E)$ is normalized so that

$$\int_0^\infty \varphi(R', E)2\pi R'dR' = 1.$$

It is convenient to introduce as the unit of energy the critical energy in air, about 10^8 ev, and as the unit of length the root mean square lateral spread of shower electrons of the critical energy. This unit of length is equal to 60.6 meters at Echo Lake and is introduced by the equation[13]

$$R' = 0.8E_s r = 0.139r. \quad (3)$$

Introducing this unit into Eq. (2) one obtains

$$\varphi(r, E)2\pi rdr = (1/2\pi)\, E^2/\alpha^2 \exp\left(-\frac{r^2E^2}{2\alpha^2}\right)2\pi rdr. \quad (2')$$

The primary spectrum assumed here for the high energy electrons entering the earth's atmosphere is the same as that which is used for primaries of lower energies. The number of primary electrons of energy greater than E_0 (in critical energy units) incident at the top of the atmosphere per unit time and per unit area is given by

$$N(E_0) = H_1E_0^{-\gamma}, \quad \gamma = 1.8 \quad H_1 = 6\times 10^9/\text{hr. m}^2; \quad (4)$$

$$n(E_0) = h_1E_0^{-\gamma}, \quad \gamma = 1.8, \quad h_1 = 2.2\times 10^{13}/\text{hr. }(60.6\text{ m})^2. \quad (4')$$

This spectrum is the same as that used by Euler[1] and is very similar to that arrived at by Hilberry.[4]

2. Electron Density Function

The average electron density $\rho(r, E_0)$ recorded by an ionization chamber the center of which is at a distance r from the axis of a shower due to a primary electron of energy E_0 can be given by[14]

$$\rho(r, E_0) = \int_\beta^\infty \varphi(r, E)\pi(E_0, E)dE = \frac{kE_0}{2\pi}\int_\beta^\infty K(E_0/E)\mu(E)/\alpha^2(E) \times\exp\left(-\frac{r^2E^2}{2\alpha^2}\right)dE. \quad (5)$$

The density per square meter is given by

$$P(R, E_0) = \frac{\rho(r, E_0)}{(60.6)^2} = 10^{-3.57}\rho(r, E_0), \quad (5')$$

[11] The value used for the actual length of the radiation unit is that of a radiation unit at the place of observation (436 m at Echo Lake). Because of the fact that the actual length becomes greater with elevation in the atmosphere above the place of observation, one should use a larger value in all the calculations. Since, however, the correct value has not been calculated so far, the value of the radiation length at the place of observation has been used.

[12] Roberg, Phys. Rev. **61**, 735 (1942); **62**, 304 (1942). (See note 7).

[13] Quantities expressed in terms of this unit of length are given in small letters, those expressed in radiation units are given in primed capital letters, and those expressed in meters are given in unprimed capital letters.

[14] The depth t will no longer be given as a variable since the calculations of this part all refer to Echo Lake ($t=16.5$).

TABLE I. $\text{Log}_{10} P(R, E_0)$. Electron density as a function of primary energy E_0 and distance R from the axis of the shower.

Log E_0 (in ev)	R (in meters) 0	0.3	0.6	1.2	1.8	2.4	3	4.2	6	12	30	60	85	106
14.5	3.09	2.73	2.51	2.29	2.15	2.04	1.96							
15.2	4.03	3.60	3.35	3.10	2.94	2.83	2.73	2.60	2.45	2.14	1.70	1.28	.98	
15.7		4.18	3.92	3.65	3.48	3.37	3.27	3.13	2.97	2.64	2.20	1.77	1.47	1.22
16.2			4.45	4.17	4.00	3.88		3.62	3.45	3.12	2.67	2.24	1.94	1.69
16.7										3.57	3.10	2.67	2.37	2.12

where $R=60.6r$. It is assumed that no multiplication takes place in the walls of the chamber, but that there is a lower limit β to the energy E of electrons which can penetrate the walls. The value used for the energy β is 4 Mev; however, using $\beta=0$ does not change the values of the density function significantly except for $r>1$ ($R>60$ m).

Equation (5) is based on the assumption that the density recorded by the chamber is equal to the density at the center of the chamber. This assumption is approximately true except in the relatively rare instances that the axis of the shower is within the chamber (r less than the radius of the chamber). For such values of r and for values of $E(>2\times10^{10}$ ev) for which the root mean square radial spread is of the same order of magnitude as the radius of the chamber, the integrand in the integral of Eq. (5) must be replaced by the density of these high energy shower electrons averaged over the chamber. This consideration enters the calculations only in the case $r=0$ and turns out to have very little effect on the final calculated burst frequencies. For the calculation of $\rho(0, E_0)$ the radius was taken to be about 17.5 cm so that the results would correspond to the experiments of Lewis.[5]

The explicit form of the electron density $\rho(r, E_0)$ as a function of r may be seen readily for primary energies E_0 of about 10^{16} ev and for values of r between 0.02 and 0.10 (1.2 to 6 m) since in this case the energies E contributing mainly to the integral of Eq. (5) have values between 10^8 and 10^{10} ev for which the functions $K(E_0/E)$, $\mu(E)$, and $\alpha(E)$ are all close to unity. Using the approximation $K=\mu=\alpha=1$ in Eq. (5), we have

$$\rho(r, E_0)\simeq1.25kE_0/2\pi r. \tag{6}$$

For very small values of r, higher values of E and correspondingly lower values of the function $K(E_0/E)$ contribute significantly to the integral, causing the electron density $\rho(r, E_0)$ to rise less rapidly than the function $(1/r)$. For large values of r, electrons of energy E less than 10^8 ev become relatively more important; this causes the electron density $\rho(r, E_0)$ to fall more rapidly than $(1/r)$.

The values of the electron density $\rho(r, E_0)$ for various distances r from the axis of the shower and various primary electron energies E_0 have been found by numerical integration of Eq. (5). The values of the $\log_{10} P(R, E_0)$ are given in Table I for values of R between 0 and 106 m and values of E_0 between $10^{14.5}$ and $10^{16.7}$ ev. For two values of the primary energy E_0 representative of those important in this problem, $10^{15.2}$ and $10^{16.2}$ ev, the density $P(R, E_0)$ is plotted as a function of R on a double logarithmic graph in Fig. 2. The broken curve in the figure represents the lower curve placed a cycle higher so that the form of the function may be compared for the two values of the primary energy. It can be seen that the electron density as a function of R is almost independent of the primary energy E_0, although for the larger value of E_0 the density is relatively greater near the axis of the shower and less far from the axis than for the smaller value.

To a first approximation, therefore, the density $\rho(r, E_0)$ may be represented for primary energies E_0 between 10^{15} and $10^{16.5}$ ev and for all values of r by

$$\rho(r, E_0)=E_0q(r)/r. \tag{6'}$$

The electron density function of Eq. (6′) with $q(r)$ set equal to an exponential function of the form e^{-r} is practically the same as that of Euler.[1] Euler's unit of length, however, is much smaller, since his calculation of the root mean square ra-

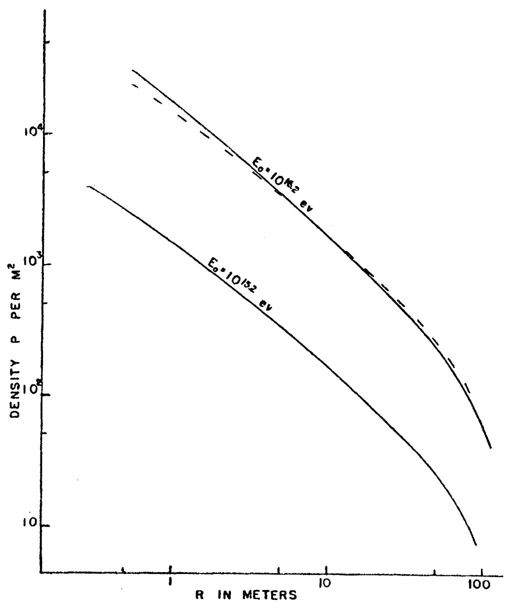

FIG. 2. Electron density $P(R, E_0)$ at Echo Lake plotted as a function of the distance R from the axis of the shower for primary energies E_0 of $10^{15.2}$ and $10^{16.2}$ ev. The broken curve shows the lower curve placed a cycle higher.

dial spread of shower electrons gives a value only one-third of that calculated by Roberg.[12] A less important difference is that the function $q(r)$ in the present calculation falls off somewhat less rapidly than an exponential for $r<1.5$ ($R<90$ m). This difference is most likely due to Euler's failure to consider the factor $\alpha(E)$.

The approximate percentage contributions of shower electrons of different energy ranges E to the value of $P(R, E_0)$ are shown in Table II for different distances R. For $R<12$ meters this is given for a shower of primary energy E_0 equal to $10^{15.7}$ ev, while for $R\gtrsim 12$ meters it is given for the particular case in which the function $K(E_0/E)$ is always 1. It was found that to a good approximation the values in Table II can be considered as correct for all values of E_0 between $10^{14.5}$ and $10^{16.5}$ ev.

3. Frequency of Single and Coincident Bursts

From the values of $\rho(r, E_0)$ which have been calculated the theoretical frequency of bursts in a single chamber and of coincident bursts in two chambers can be obtained. Inverting $\rho(r, E_0)$ we find $E_0(\rho, r)$, which is the minimum energy of a primary electron in order that on the average the shower it produces has an electron density greater than ρ at a distance r from the axis. Then, introducing the primary spectrum (Eq. (4)), the number of showers per unit time and per unit area having a density greater than ρ at a distance r from their axes is given by

$$n(E_0)=h_1E_0^{-\gamma}(\rho, r). \tag{7}$$

Considering all values of r, the frequency of all bursts of density greater than ρ in the chamber is given by

$$H'(\rho)=2\pi h_1\int_0^\infty E_0^{-\gamma}(\rho, r)rdr. \tag{8}$$

Similarly from a simple geometrical construction[1] the frequency of simultaneous bursts of densities greater than ρ in each of two chambers separated by a distance d is given by

$$H'(\rho, d)=2\pi h_1\int_{d/2}^\infty E_0^{-\gamma}(\rho, r)\frac{\text{arc cos }(d/2r)}{\pi/2}rdr. \tag{9}$$

Notice that $H'(\rho, 0)=H'(\rho)$. It is also of interest to consider the median values $\bar{r}(\rho, d)$ and $\bar{E}_0(\rho, d)$ defined by:

$$\tfrac{1}{2}H'(\rho, d)$$
$$=2\pi h_1\int_{d/2}^{\bar{r}} E_0^{-\gamma}(\rho, r)\frac{\text{arc cos }(d/2r)}{\pi/2}rdr \tag{10}$$

$$\bar{E}_0(\rho, d)=E_0(\rho, \bar{r}). \tag{10'}$$

Equations (8) and (9) would give the actual frequencies only if all the primary electrons were directed close to the vertical. However, it seems probable that the directions of these extremely high energy electrons are distributed uniformly over a solid angle of 2π. The electrons entering at a large angle θ to the vertical must traverse a much greater thickness t of the atmosphere and

TABLE II. Approximate percentage contribution of electrons of energy ranges E to $P(R, E_0)$ for various values of R.

	R (in meters)												
E (in ev)	0.6	1.2	1.8	2.4	3.0	4.2	6	12	18	30	60	85	106
$<10^8$	1	2	3	4	5	7	10	23	35	54	84	95	99
10^8–10^9	9	17	25	30	37	49	70	74	65	46	16	5	1
10^9–10^{10}	62	78	72	66	58	44	20	3					
$>10^{10}$	28	3	0.2										

therefore will produce a lower electron density $\rho(r, E_0)$ and a lower coincidence frequency $H'(\rho, d)$ than those coming from the vertical. The exact calculation of such a zenith angle effect, which would require the computation of $H'(\rho, d, \theta)$ for primary electrons entering at various angles θ, has not been carried out here in view of the fact that it would not affect the nature of the results. As a first approximation to take the zenith angle effect into account, however, the function $H'(\rho, d)$ has been multiplied by a factor $z(\rho, d)$ depending only on the median energy $\bar{E}_0(\rho, d)$.

To find this factor the energy $E_0'(\theta, E_0)$ defined as the minimum primary energy of an electron entering at an angle θ necessary to produce the same total number of electrons at the level of observation as a primary of energy E_0 entering vertically, is calculated from the cascade theory. If we define $z(E_0)$ by

$$z(E_0)=\int_0^{\pi/2}\frac{n(E_0')}{n(E_0)}\sin\theta d\theta, \quad (11)$$

then the factor $z(\rho, d)$ can be given by

$$z(\rho, d)=z(\bar{E}_0). \quad (11')$$

The value of z varies from 0.28 for $\bar{E}_0=10^{15.5}$ ev to 0.42 for $\bar{E}_0=10^{16.5}$ ev. If we introduce this factor, the frequency of coincident bursts can be given by

$$H(\rho, d)=2\pi h_1 z(\rho, d)\int_{d/2}^{\infty}E_0^{-\gamma}(\rho, r) \times\frac{\text{arc cos }(d/2r)}{\pi/2}rdr. \quad (9')$$

The frequency of bursts in a single chamber follows directly:

$$H(\rho)=H(\rho, 0)=2\pi h_1 z(\rho, 0)\int_0^{\infty}E_0^{-\gamma}(\rho, r)rdr. \quad (8')$$

TABLE III. Calculated frequency per hour $H(P, D)$ of coincident bursts at Echo Lake.

Separation D (in meters)	$500/\text{m}^2$	Density P $1000/\text{m}^2$	$2000/\text{m}^2$
0	0.064	0.0198	0.0060
0.6	0.0605	0.0183	0.00545
1.2	0.058	0.0172	0.0051
2.4	0.053	0.0156	0.0046
4.8	0.0455	0.0131	0.00385
9.6	0.036	0.0101	0.00285

The exact calculation of $z(\rho, d)$ might change the absolute values of the calculated frequencies by 20 percent, but this would not be a significant change in the results. The most important change that might be effected by the

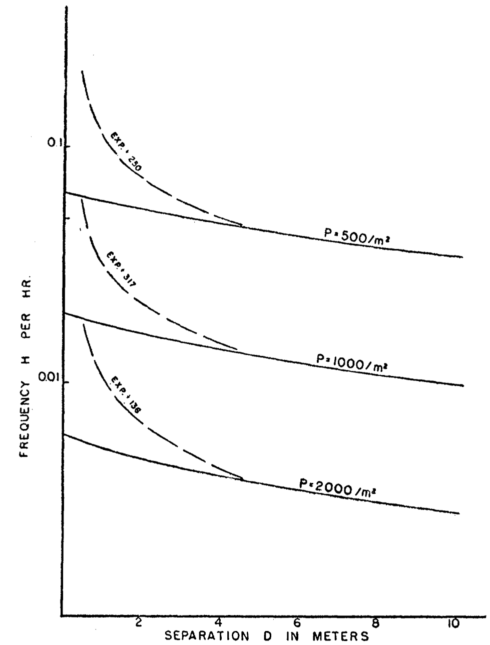

FIG. 3. Calculated coincidence frequency per hr $H(P, D)$ for Echo Lake as a function of D for densities P greater than $500/\text{m}^2$, $1000/\text{m}^2$, and $2000/\text{m}^2$. The broken curves show the corresponding experimental frequencies divided by factors of 250, 317, and 136, respectively, for values of D between 0.6 m and 4.5 m.

exact calculation is in the form of $H(\rho, d)$ as a function of d, but it is more likely that the effect would be to decrease rather than to increase the slope of this curve.

To get an approximate picture of $H(\rho)$ as a function of ρ one can use the approximation for $\rho(r, E_0)$ given in Eq. (6'). Solving for E_0 one obtains

$$E_0(\rho, r)=\frac{\rho r}{q(r)} \quad (12)$$

and, substituting this value of $E_0(\rho, r)$ into Eq. (8'),

$$H(\rho)=2\pi h_1 z(\rho, 0)\rho^{-\gamma}\int_0^{\infty}[q(r)]^{\gamma}r^{-\gamma+1}dr. \quad (13)$$

For values of ρ corresponding to densities from $500/\text{m}^2$ to $2000/\text{m}^2$, 70 percent of the integral of

Eq. (13) is contributed by values of $E_0(\rho, r)$ between 10^{15} and $10^{16.5}$ ev, for which the approximations used in obtaining Eq. (6′) are reasonably good. Neglecting the variation of $z(\rho, 0)$ with ρ for this range of densities, we find

$$H(\rho)=\text{const } \rho^{-\gamma}. \tag{14}$$

This means that $H(\rho)$ is a power function of ρ with the same exponent γ as figures in the primary spectrum $N(E_0)$. If the variation of $z(\rho, 0)$ with ρ is introduced, the approximate form of $H(\rho)$ can still be given by Eq. (14), but the absolute value of the exponent must be decreased slightly (from 1.8 to about 1.7).

Table III and the corresponding Figs. 3 and 4 show the result of the exact evaluation of Eqs. (8′) and (9′).[15] Figure 3 shows the frequency $H(P, D)$ of coincidences between two ionization

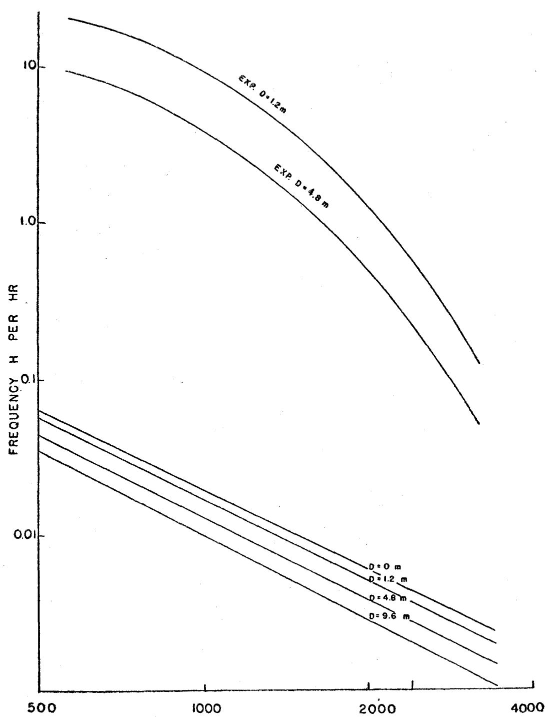

FIG. 4. Calculated coincidence frequency per hr $H(P, D)$ for Echo Lake as a function of P for separations D of 1.2, 4.8, and 9.6 m. The curve labeled $D=0$ is the cumulative size-frequency curve $H(P)$ for bursts in a single chamber. The two top curves are the corresponding experimental functions for $D=1.2$ m and $D=4.8$ m.

[15] The functions $H(P, D)$ and $H(P)$ are obtained from $H(\rho, d)$ and $H(\rho)$, respectively, by the relations: $P=\rho/3700$ and $D=60.6d$.

chambers as a function of the distance D between them up to 10 meters for densities P greater than 500/m², 1000/m², and 2000/m². In Fig. 4 the frequency $H(P, D)$ of coincidences is plotted on a double-logarithmic graph as a function of the density P for separations D of 1.2, 4.8, and 9.6 meters; the cumulative size-frequency curve $H(P)$ for bursts in a single chamber ($D=0$) is also shown. It is seen that each of the theoretical curves of Fig. 4 approximately satisfies Eq. (14) and thus reflects the primary energy spectrum $N(E_0)$. Table III gives all the calculated values of $H(P, D)$.

Table IV shows the percentage contribution of different ranges of r and of the corresponding ranges of the minimum energy $E_0(\rho, r)$ to the function $H'(\rho)$ (Eq. 8) for densities ρ corresponding to 500/m² and 2000/m². The corresponding values of $\bar{R}$ and $\bar{E}_0$ are also given. Because $H'(\rho)$ does not take into account the zenith angle effect, Table IV gives only an approximation to the contributions to the total burst frequency $H(P)$ of different primary energies E_0 and of different distances R of the chamber from the axis of the shower. Table IVa shows the approximate percentage contributions to the function $H'(\rho)$ from different energy ranges E for the same densities. It therefore gives an approximation to the average energy distribution of electrons in bursts of densities greater than 500/m² and 2000/m², respectively.

From Table IV it is seen that the frequencies depend very little on values of $R>60$ m ($r>1$), which has the consequence that the lower limit β to admitted energies E has little effect on the results. Similarly values of $R<20$ cm contribute very little, so that the results are almost independent of the size of the chamber for those chambers generally used. Table IVa indicates a 20 percent contribution to the results from values of $E<10^8$ ev, for which the calculations are only approximate. It should be noted that the contribution of these low energies is greater for the lower densities and also for the greater separations of the chambers, so that these results have to be considered less accurate. Table IV also indicates that for the 2000/m² density there is at least a 7 percent contribution to the calculated results from primary energies greater than 10^{17} ev. Experiments in which

TABLE IV. Percentage contribution of ranges of R and of the corresponding minimum energies $E_0(P, R)$ to the integral $H'(P)$ (Eq. (8)).

Density P	E_0	$10^{14.2}$–10^{15}	10^{15}–$10^{15.5}$	$10^{15.5}$–10^{16}	10^{16}–$10^{16.5}$	$10^{16.5}$–10^{17}	$>10^{17}$ ev	
500/m^2	R	0–1.8	1.8–7.0	7.0–20	20–43	43–73	>73 m	$\bar{R}=7.5$ m
	%	18	29	26	18	7.5	1.5	$\bar{E}_0=10^{15.55}$ ev
2000/m^2	R	0–0.4	0.4–1.7	1.7–5.4	5.4–15	15–36	>36 m	$\bar{R}=5.4$ m
	%	7	17	25	27	17	7	$\bar{E}_0=10^{16}$ ev

larger densities are measured and larger separations of the chambers are used than in those experiments carried out hitherto would be required to determine whether any showers of such enormous energies exist.

If the unit of length, that is, the root mean square radial spread of shower electrons is divided by a factor σ, it can be shown that the approximate effect is to multiply the burst frequency $H(P)$ by a factor of $\sigma^{2(\gamma-1)}$. Thus, the effect of using Euler's value for the radial spread would be to multiply $H(P)$ by a factor of $3^{1.6}$, or 6.

4. Comparison with Experiment

The experimental observations of Lewis[5] at Echo Lake, which are also shown in Figs. 3 and 4,[16] differ from the results of the present calculations in three striking ways. (a) The experiments show a much sharper decrease in the frequency of coincident bursts as the separation of the two chambers is increased up to 3 m. This is seen from the broken curves of Fig. 3, which represent the experimental frequencies reduced in magnitude so that the forms of the experimental and theoretical curves may be compared. The experimental curves are not shown for values of D between 5 and 10 m because the difference between the forms of the experimental and the theoretical curves for these values cannot be considered significant. (b) The experimental cumulative size-frequency distributions for different separations D of the chambers are represented by similar curves of an exponential type in contrast to the theoretical power-law curves. Figure 4 shows two such curves for $D=1.2$ m and $D=4.8$ m plotted on a double-logarithmic graph. (c) The absolute values of the observed single and coincident burst frequencies are much greater than the corresponding theoretical values. For example, for densities greater than 1000/m^2 the observed values are greater by a factor of about 550 in the case of the two chambers separated by a distance of 1 m and by a factor of about 300 in the case of 5 m separation. As the density increases above 1000/m^2, however, the corresponding theoretical and observed cumulative frequencies approach each other in value (Fig. 4).

TABLE IVa. Approximate contribution of ranges of values of electron energy E to $H'(P)$.

E (in ev)	$<10^8$	10^8–10^9	$>10^9$
Density P			
500/m^2	23	43	34
2000/m^2	15	35	50

[16] The experimental curves shown in Figs. 3 and 4 do not pass through actual experimental points, but were obtained by graphical interpolation of the experimental data. The experimental results shown for densities P less than 800/m^2 were obtained by extrapolation.

CALCULATIONS FOR SHOWERS DETECTED AT SEA LEVEL

A similar calculation of the frequencies $H(P)$ and $H(P, D)$ has been carried out for sea level ($t=24$) for electron densities P of 500/m^2 and 2000/m^2 and separations D up to 14 meters. This calculation is less accurate than that for Echo Lake because of the fact that for these densities at sea level there is a much larger contribution from shower electrons of energies E less than 10^8 ev. It was pointed out before that in this case special approximations have to be made. This also means that the results are more dependent on the lower limit β to admitted energies. In addition to this, for the primary energies which contribute most to the results for the above-mentioned densities (E_0 about 10^{16} ev), $\pi(E_0, E, t)$ (Eq. (1)) must be used farther from its maximum as a function of t, so that Eq. (2) for the spatial distribution of the shower electrons is less valid. From Eq. (3), for $t=24$, the unit of length is 42 meters.

Table V gives the calculated values of $H(P)$ and $H(P, D)$. The results, particularly for the lower density, must be considered as first approximations. The general form of the cumulative size-frequency distribution $H(P)$ can be obtained by the method used before (Eqs. (12–14)). It is found that for primary energies around 10^{16} ev the approximation

$$\rho(r) = (E_0)^{1.2} q'(r)/r, \qquad (15)$$

should be used in place of Eq. (6′). This gives

$$H(\rho) = \text{const } \rho^{-\gamma/1.2} = \text{const } \rho^{-1.5}. \qquad (16)$$

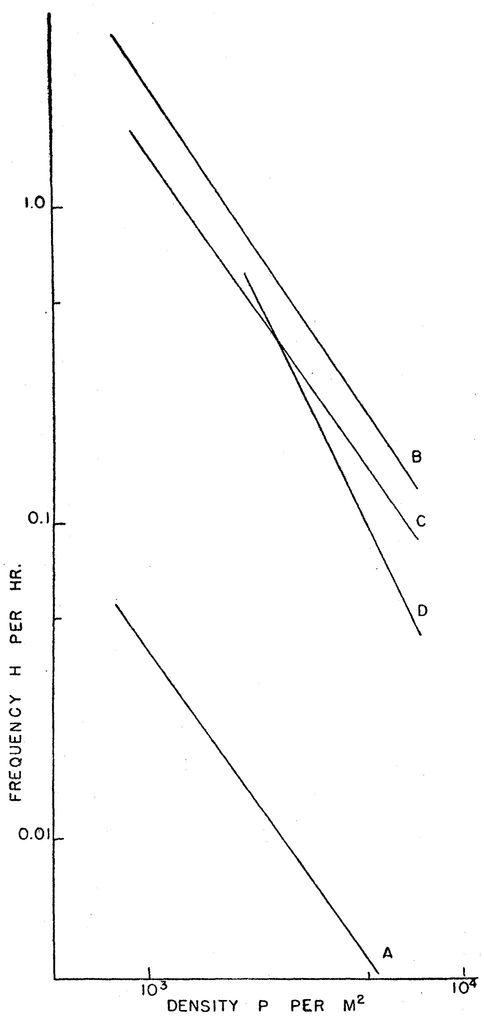

FIG. 5. Cumulative size-frequency function $H(P)$ for bursts in a single chamber at sea level. Curve A shows the results of the present calculations, curve B represents the corresponding theoretical results of Euler, while curves C and D show the experimental results of Carmichael and Lapp, respectively.

Introducing $z(\rho, 0)$ to account for the zenith angle effect results in a change of the exponent of Eq. (16) from 1.5 to 1.4. Curve A in Fig. 5 shows the calculated form of the function $H(P)$, curve B represents the corresponding theoretical results of Euler,[1] and curves C and D show the experimental results of Carmichael[2] and Lapp,[3] respectively. All the curves are approximated as straight lines, and only a representative portion of each is shown. The main reasons for the large discrepancy between the theoretical results obtained here and those obtained by Euler are the small value used by Euler for the mean square radial spread and the fact that he neglected completely the zenith angle effect. It is seen from Fig. 5 that even at sea level the experimental burst frequencies are about forty times greater than the theoretical ones. Furthermore the exponent in Lapp's experimental cumulative frequency function is greater than 2 in contrast to the theoretical Eq. (16).

Of particular interest is the remarkable difference which exists between the calculated and experimental altitude effects for bursts in an unshielded ionization chamber. The observed increase in frequency from sea level to Echo Lake (3100 meters) appears to be as much as 400[17] for densities greater than $2000/m^2$, which is much larger than that observed in counter experiments on extensive atmospheric showers. According to the theory, the bursts discussed here are on the average due to higher primary energies than the energies responsible for the counter observations and therefore should show an even smaller altitude effect. The calculations give an increase of frequency with altitude of 5.7 for densities greater than $1000/m^2$ and 4 for densities greater than $2000/m^2$.

CONCLUSION

The relatively large discrepancies which have been noted between the theoretical results and those of experiment do not seem to be explainable in any simple manner. Changing the form of the lateral distribution of shower electrons from the

[17] This may be calculated approximately by using the experimental results of Lapp and Carmichael at sea level and those of Lewis at Echo Lake. Unpublished sea-level data obtained at the University of Chicago with the same ionization chamber as that used by Lewis also give this result.

assumed Gaussian function or changing the form of the assumed primary electron spectrum (Eq. (4)) could, at best, it seems, resolve only part of these discrepancies. The sharp drop in frequency of coincidences observed by Lewis as the two chambers were separated indicates the presence at an altitude of 3100 meters of relatively narrow showers of high particle density which must originate fairly close to (a few radiation units above) the chambers if they are to be explained by the cascade theory. Since, for a given electron density in the chamber the total energy contained in such showers can be much less than that contained in showers close to their maximum development, it is reasonable that the narrow showers are observed more frequently even though the probability of any one shower's hitting the chambers is much less. The presence of these narrow showers, therefore, might explain in part the much larger absolute values of the experimental frequencies in comparison to the theoretical ones. The method by which the high energy ($>5\times10^{11}$ ev) electron or photon necessary to originate such a shower can be brought to the level at which such showers start is not known. Any explanation must also account for the large altitude effect for the bursts. It would be of great interest to know whether this altitude effect for bursts in an unshielded ionization chamber continues to be so large as the altitude increases above 3100 meters.

One might think now that the results could be explained by considering two types of showers, the narrow showers of an unexplained origin and the showers from primary electrons of very high energy. The introduction of such narrow showers, however, fails to explain the discrepancy that still seems to exist between the theoretical frequencies calculated above and the frequencies observed by Lewis for coincident bursts between two chambers separated by distances of 5 to 10 meters. This discrepancy exists both in the absolute values and in the form of the cumulative size vs. frequency distribution. These coincident bursts might be explained by showers originating higher than the narrow showers but well below the top of the atmosphere. The total energy in such a shower for a given density need not be so great as in a shower due to primaries, and consequently the frequency of these showers might be greater. It should be noted that in any case for such coincident bursts the energy of the electron initiating the shower must be greater than 10^{14} ev, if the shower is of the cascade variety. Only for the largest densities and separations of the chambers in the experimental results does it appear that a significant fraction of the observations can be explained by showers due to primary electrons; these, however, could be just as well explained by showers initiated by secondary electrons produced near the top of the atmosphere.

TABLE V. Calculated coincidence frequency per hour $H(P, D)$ at sea level.

Separation D (in meters)	Density P 500/m²	2000/m²
0	0.0105	0.00148
.85	0.0100	0.00141
3.4	0.0090	0.00126
6.8	0.0079	0.00108
13.6	0.0063	0.00087

It may therefore be concluded that the assumption of primary electrons is of little help in explaining the observations on extensive atmospheric showers detected at an altitude of 3100 meters. The experiments carried out at sea level are too incomplete to allow conclusions to be drawn from them.

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